Beam Excitation of a Non-Axisymmetric Structure:

Output-Coupling, Higher-Moments, Resonant and Non-Resonant Operation, Common Mode

David H. Whittum

In this note we continue the work of ARDB TN 152, and consider output coupling and convolution with real beam profiles, as well as other intrinsic features of a non-axisymmetric, high-*Q* pickup.

This is the third note in a series on the E158 BPM system.

Outline

1. Introduction

In a previous note,[1] we set down a multipole description for modes of non-axisymmetric structures, in analogy with that used for linac magnets. Here we apply these results to determine voltage waveforms emitted by a beam-excited cavity, taking a Gaussian beam as our model. The motivation for this work is the E158 experiment,[2] as well as our general interest in the features of planar structures. The working example used for illustration is the "*x*-cavity" of the "linac-style" beam-position monitor cavity triplet.[3]

2. External Coupling

Monopole Mode

This subject is developed at length in the notes on linacs;[4] we give an abbreviated description here. The cavity excitation may be described by a single waveform, V_c, a coefficient of the cavity modal electric field, typically chosen to be the accelerating voltage waveform. The waveguide and the cavity are joined at some plane, where there is a continuity condition on the electric field that may be expressed as $V_c = V_F + V_R$, where V_F, V_R are forward and reverse voltages in the waveguide, *i.e.*, coefficients, in the waveguide mode expansion, of the transverse modal electric field.[5] The strength of the coupling of between the cavity and the waveguide is quantified by $1/Q_e$, the reciprocal external Q.This adds with $1/Q_w$ to determine the loaded Q

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_w}.$$

In this connection one often refers to the coupling parameter

$$\beta = \frac{Q_w}{Q_e}.$$

In terms of these quantities, the usual monopole-mode, beam-cavity-waveguide system is described by a driven simple-harmonic oscillator equation,

$$\left\{\frac{d^2}{dt^2} + \frac{\omega_0}{Q_L}\frac{d}{dt} + \omega_0^2\right\}V_c = 2\frac{\omega_0}{Q_e}\frac{dV_F}{dt} + \frac{1}{2}\omega_0\left[\frac{R}{Q}\right]\frac{dI_b}{dt}.$$

This equation comes with the picture of Fig. 1.

---

[1] "Mode Classification in Non-Axisymmetric Structures", ARDB Technical Note 152

[2] David H. Whittum, Roger A. Erickson, and Mike B. Woods, "Overview of Cavity Beam Position Monitors for Precision Møller Scattering Measurements", ARDB Technical Note 134.

[3] *The Two-Mile Accelerator*, R. B. Neal, ed., (W. A. Benjamin, New York, 1968) pp. 500-516.

[4] "Introduction to Electrodynamics for Microwave Linear Accelerators", D. Whittum, *Proceedings of the Joint US-CERN-Japan School, RF Engineering for Particle Accelerators* (World Scientific, to be published). This article can be found at the ARDB Tech. Note web site.

[5] In the case of a beam-excited cavity monitor, V_R is the waveform being launched "upstairs" through the waveguide; in the case of an accelerator structure, V_F is the signal arriving from the klystron.

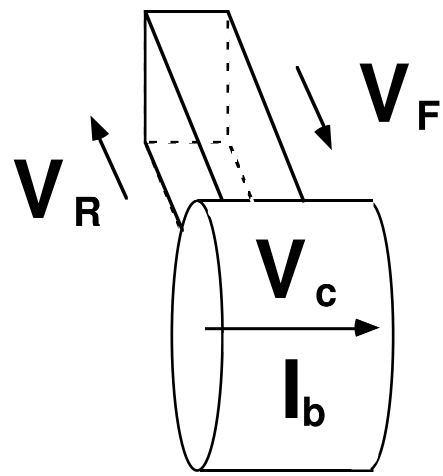

FIGURE 1. Sketch of the dynamic variables for a single cavity mode coupled to a waveguide fundamental mode, and a beam.

It is natural in waveguide to refer voltages to the characteristic impedance of the guide, Z_c. corresponding to forward and reverse voltages V^+, V^- that differ from the foregoing by a normalization factor, n, ("turns ratio" or "transformer ratio") such that $V_F = nV^+, \quad V_R = nV^-$. While this might seem to be merely a matter of normalization convention, it is not; a cavity driven on resonance can, in steady state, develop large fields, much larger than in the waveguide, *i.e.*, n can be large. Stored energy is determined from

$$U_c = \frac{|V_c|^2}{\omega_0[R/Q]},$$

(the definition of [*R/Q*]), and energy conservation determines that

$$n = \left(\frac{Q_e\left[\frac{R}{Q}\right]}{2Z_c}\right)^{1/2} = \left(\frac{R}{2Z_c\beta}\right)^{1/2}.$$

where the unloaded shunt impedance is $R = Q_w[R/Q]$. Net power flowing down the guide into the cavity is

$$P_w = \frac{|V_F|^2 - |V_R|^2}{2Z_c n^2} = \beta\frac{|V_F|^2 - |V_R|^2}{R}.$$

Dipole Mode

The description of dipole mode excitation in a cavity is the same, in the abstract, as the monopole mode case; the choice of variable describing the cavity excitation must be different, since the cavity voltage on-axis is zero. Our beam-cavity system takes the form,

$$\left\{\frac{d^2}{dt^2}+\frac{\omega_0}{Q_L}\frac{d}{dt}+\omega_0^2\right\}V_c' = 2\frac{\omega_0}{Q_e}\frac{dV_F'}{dt}+\frac{1}{2}\omega_0\left[\frac{R_\perp}{Q}\right]\frac{d}{dt}I_b x_b,$$

expressed in terms of the on-axis voltage *gradient*, $V_c' = \partial V_c / \partial x$. We consider an x-dipole mode for definiteness. The continuity condition takes the form $V_c' = V_F' + V_R'$, and the conversion to waveguide impedance is $V_F' = n'V^+$, $V_R' = n'V^-$. Stored energy is

$$U_c = \frac{|V_c'|^2}{\omega_0[R_\perp / Q]},$$

(the definition of $[R_\perp / Q]$), and energy conservation determines that

$$n' = \left(\frac{Q_e\left[\frac{R_\perp}{Q}\right]}{2Z_c}\right)^{1/2} = \left(\frac{R_\perp}{2Z_c\beta}\right)^{1/2},$$

where the unloaded transverse shunt impedance is $R_\perp = Q_w[R_\perp / Q]$. Net power flowing down the guide into the cavity is

$$P_w = \frac{|V_F'|^2 - |V_R'|^2}{2Z_c n'^2} = \beta\frac{|V_F'|^2 - |V_R'|^2}{R_\perp}.$$

Impedance Transformation

Where one cavity is coupling several modes, it is helpful to be be able to transform to a common impedance. So for example, where one cavity (such as our linac x-cavity) is functioning with both a dipole and a monopole mode, one would appreciate a prescription for adding induced voltages on the connecting waveguide. This is straightforward given the relation of the quantities V_R and V_R' to the waveguide voltage coefficient V_1^- normalized with respect to the waveguide mode characteristic impedance Z_c. Thus we expect the contribution due to a monopole mode to take the form $V_1^- = V_R / n$ and that due to a dipole mode $V_1^- = V_R' / n'$. In general it need not be the case that the reference plane for both of these modes is the same (the plane of the detuned short), and thus there may be a relative phase between the two mode contributions. So if we refer both voltages to the normalization employed for dipole mode quantities, the monopole cavity voltage must be multiplied by a factor $n'' = n' / n$, or

$$n'' = \left(\frac{Q_{ed}[R_\perp / Q]_d}{Q_{em}[R / Q]_m}\right)^{1/2}.$$

Thus for the case of the linac x-cavity analyzed in TN152, with $[R_\perp / Q]_d \approx 17.8\,\Omega\,\mathrm{cm}^{-2}$ and $[R_\perp / Q]_m \approx 196\,\Omega$, $n'' \approx 0.30\,\mathrm{cm}^{-1}$, assuming comparable external Q's.

Arbitrary Mode

As we have seen, there is a good deal more variety to modes than simply the on-axis voltage or voltage gradient. Asymmetric structures may have modes with higher multipole moments. The essential point to make is that where I_b is the driving term for a pure monopole mode, and $I_b x_b$ for a pure dipole mode, in the general case the driving term is proportional to $I_b w(\vec{r}_{\perp b})$. Let us first described the problem in a general, but somewhat abstract fashion, and then consider specifically the two examples of immediate interest: monopole mode with a quadrupole component, and dipole mode with a sextupole component.

In the formal treatment of cavity perturbation by beam, wall-losses and connecting guide, (as reviewed in Appendix A) we find that our cavity amplitude satisifies, in the frequency domain,

$$\left(\omega^2 - \omega_\lambda^2\right)\tilde{e}_\lambda \approx j\frac{\omega\omega_\lambda}{Q_w}\tilde{e}_\lambda - j\omega\frac{1}{\varepsilon_0}V_{1\lambda}I_1 - j\omega\frac{1}{\varepsilon_0}\tilde{J}_\lambda .$$

Here $V_{1\lambda}$ is an overlap integral of the transverse modal electric field associated with # 1 mode of the waveguide (the fundamental mode) and the transverse modal field associated with the cavity mode of interest, mode λ (solved for under conditions of an open-circuit at the chosen reference plane). The quantity I_1 is the coefficient of the transverse modal magnetic field in the waveguide, and may be related to the forward and reverse modal voltage coefficients according to $Z_{c1}I_1 = V_1^+ - V_1^-$, with normalization referred to the chosen waveguide mode characteristic impedance Z_{c1}. Continuity of the transverse electric field at the reference plane requires that $V_1^+ + V_1^- \approx V_{1\lambda}\tilde{e}_\lambda$. The quantity $\tilde{J}_\lambda = \tilde{I}_b(\omega)w_\lambda^*(\vec{r}_{\perp b})$ is the overlap integral of the beam current density and the mode electric field,

$$\tilde{w}(\vec{r}_\perp) = \int dz\, E_{z\lambda}(\vec{r}_\perp, z)e^{j\omega z/V} .$$

Our beam-cavity equation then takes the form

$$\left(\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right)\tilde{e}_\lambda \approx -j\omega\frac{1}{\varepsilon_0 Z_{c1}}V_{1\lambda}\left(V_1^+ - V_1^-\right) - j\omega\frac{1}{\varepsilon_0}\tilde{I}_b(\omega)w_\lambda^*(\vec{r}_{\perp b}).$$

Let us put in a still more transparent form. We decide to refer to the cavity excitation in terms of "cavity voltage", defined with respect to a choice of reference trajectory, $\tilde{V}_c = \tilde{e}_\lambda\tilde{w}_\lambda(\vec{r}_{\perp 0})$. Our continuity condition then takes the form $\left(V_1^+ + V_1^-\right)\tilde{w}_\lambda(\vec{r}_{\perp 0}) \approx V_{1\lambda}V_c$, and for simplicity we choose to write this as $V_c = V_F + V_R$, where $V_F = nV_1^+$, $V_R = nV_1^-$, and we introduce a transformer ratio $n = \tilde{w}_\lambda(\vec{r}_{\perp 0})/V_{1\lambda}$.[6] With these definitions our system reduces to

$$\left(\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right)\tilde{V}_c \approx -j\omega\frac{V_{1\lambda}^2}{\varepsilon_0 Z_{c1}}\left(\tilde{V}_F - \tilde{V}_R\right) - j\omega\frac{1}{\varepsilon_0}\tilde{I}_b(\omega)w_\lambda^*(\vec{r}_{\perp b})\tilde{w}_\lambda(\vec{r}_{\perp 0}),$$

and we are lead to introduce a beam-coupling strength,

[6] We are free, by choice of modal basis and coordinate origin to insure that *n* is real, although it is not necessary. To be more precise, we could add that *n* is function of frequency, and we are referring to *n* evaluated at the mode resonant frequency; this is consistent with the perturbative treatment, where we have started from a lossless cavity with open-circuit boundary conditions at a choice of reference plane in the connecting guide, and added wall-losses, beam, and external drive as small perturbations.

$$\left[\frac{R}{Q}\right]_\lambda = 2\frac{\left|\tilde{w}_\lambda(\vec{r}_{\perp 0})\right|^2}{\varepsilon_0\omega_\lambda}\ .$$

Note that this definition coincides with $[R/Q]_\lambda = |V_c|^2/\omega_\lambda U$, given our normalization of cavity modal fields to unit integral. We are also lead to introduce a dimensionless waveguide-coupling strength $1/Q_{e\lambda} = V_{1\lambda}^2/\varepsilon_0\omega_\lambda Z_{c1}$. In terms of these, we have

$$\left(\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right)\tilde{V}_c \approx -\frac{j\omega\omega_\lambda}{Q_e}\left(\tilde{V}_F - \tilde{V}_R\right) - \frac{1}{2}j\omega\omega_\lambda\left[\frac{R}{Q}\right]_\lambda \tilde{I}_b(\omega)\frac{w_\lambda^*(\vec{r}_{\perp b})}{w_\lambda^*(\vec{r}_{\perp 0})}.$$

Making use of the continuity condition, $V_c = V_F + V_R$, we may express this as

$$\left(\omega^2 - j\frac{\omega\omega_\lambda}{Q_L} - \omega_\lambda^2\right)\tilde{V}_c \approx -\frac{2j\omega\omega_\lambda}{Q_e}\tilde{V}_F - \frac{1}{2}j\omega\omega_\lambda\left[\frac{R}{Q}\right]_\lambda \tilde{I}_b(\omega)\frac{w_\lambda^*(\vec{r}_{\perp b})}{w_\lambda^*(\vec{r}_{\perp 0})},$$

where we introduce the loaded Q, $1/Q_L = 1/Q_w + 1/Q_e$. The steady-state power flowing into the cavity may be computed, noticing that

$$2n^2 Z_c = 2Z_c\frac{\left|\tilde{w}_\lambda(\vec{r}_{\perp 0})\right|^2}{\left|V_{1\lambda}\right|^2} = 2Z_c\frac{\frac{1}{2}[R/Q]_\lambda\varepsilon_0\omega_\lambda}{\varepsilon_0\omega_\lambda Z_{c1}/Q_{e\lambda}} = Q_{e\lambda}[R/Q]_\lambda,$$

so that

$$P = \frac{\left|V_1^+\right|^2 - \left|V_1^-\right|^2}{2Z_c} = \frac{\left|V_F\right|^2 - \left|V_R\right|^2}{2n^2 Z_c} = \frac{\left|V_F\right|^2 - \left|V_R\right|^2}{Q_{e\lambda}[R/Q]_\lambda} = \beta\frac{\left|V_F\right|^2 - \left|V_R\right|^2}{R_\lambda},$$

where $\beta = Q_{w\lambda}/Q_{e\lambda}$ is the coupling parameter and $R_\lambda = Q_{w\lambda}[R/Q]_\lambda$ is the unloaded shunt impedance.

In the time-domain we have

$$\left(\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_L}\frac{d}{dt} + \omega_\lambda^2\right)V_c \approx 2\frac{\omega_\lambda}{Q_e}\frac{dV_F}{dt} + \frac{1}{2}\omega_\lambda\left[\frac{R}{Q}\right]_\lambda\frac{d}{dt}I_b(t)X(\vec{r}_{\perp b}),$$

where

$$X(\vec{r}_{\perp b}) = \frac{w_\lambda(\vec{r}_{\perp b})}{w_\lambda(\vec{r}_{\perp 0})}.$$

This affords a complete description of the excitation of an isolated resonant mode by beam and waveguide, provided of course one has chosen the axis for the definition of cavity voltage such that $w_\lambda(\vec{r}_{\perp 0})$ is non-zero. Notice that the system variables are still those depicted in Fig. 1, with the exception that $I_b X$ is the beam drive term, in the general case, rather than I_b.

In TN152 we showed that the synchronous integral w satisfies the Laplace equation $\nabla_\perp^2 w = 0$, for a highly relativistic beam, and thus is amenable to a moment expansion according to

$$\tilde{w} = -\Im \sum_{n=0}^{\infty} \left(a_n - jb_n\right) \sum_{k=0}^{n} x^{n-k} y^k j^k \binom{n}{k},$$

with "*n*-choose-*k*", the binomial expansion coefficients,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Explicitly, and with the standard nomenclature attached,

$$\begin{aligned} w = \underset{\textit{monopole}}{b_0} + \underset{\substack{\textit{horizontal}\\ \textit{dipole}}}{b_1 x} + \underset{\textit{normal quad}}{b_2\left(x^2 - y^2\right)} + \underset{\textit{normal sextupole}}{b_3\left(x^3 - 3xy^2\right)} + \underset{\textit{normal octupole}}{b_4\left(x^4 - 6x^2y^2 + y^4\right)} + \underset{\textit{normal decapole}}{b_5\left(x^5 - 10x^3y^2 + 5xy^4\right)} + \ldots \\ - \underset{\substack{\textit{vertical}\\ \textit{dipole}}}{a_1 y} - \underset{\textit{skew quad}}{2a_2 xy} - \underset{\textit{skew sextupole}}{a_3\left(3x^2y - y^3\right)} - \underset{\textit{skew octupole}}{4a_4 xy\left(x^2 - y^2\right)} - \underset{\textit{skew decapole}}{a_5\left(y^5 + 5x^4y - 10x^2y^3\right)} + \ldots\ . \end{aligned}$$

Thus the pure monopole mode corresponds to $w = b_0$ and $\left[R/Q\right] = 2b_0^2/\varepsilon_0\omega_\lambda$ and these relations are indepedent of the choice of coordinate axis. A pure *x*-dipole mode corresponds to $w = b_1 x$, and in this case refers to the particular choice of the *x*-coordinate origin, as a monopole component is present for any other choice of origin. Applying our general formalism, with choice of integration axis $x = x_0$, we see that $\left[R/Q\right] = 2b_1^2 x_0^2/\varepsilon_0\omega_\lambda$. We can make the connection with conventional notation by identifying,

$$V_c' = \lim_{x_0 \to 0} \frac{V_c}{x_0} \equiv \frac{\partial V_c}{\partial x}, \ \left[\frac{R_\perp}{Q}\right] = \lim_{x_0 \to 0} \frac{1}{x_0^2}\left[\frac{R}{Q}\right].$$

Thus $\left[R_\perp/Q\right] = 2b_1^2/\varepsilon_0\omega_\lambda$. One can check the units for these quantities, noting that $1/\varepsilon_0 = Z_0 c$, and, with our modal normaliation, w has units of $m^{-1/2}$. ($Z_0 \approx 376.7\,\Omega$ and c is the speed of light.)

With the analysis of this section in-hand, we are in a position to determine the cavity emitted signal, V_F due to interaction of any particular mode with a beam. We turn next to apply this model to a beam of finite-extent, a realistic beam.

2. Excitation by a Tri-Gaussian Beam

Let us apply these results to the case of a non-pencil beam, employing a tri-Gaussian as a simple stand-in for more realistic beam profiles. The beam current density takes the form

$$J_b\left(x, y, \zeta\right) = \frac{I_b(\zeta)}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{\left(x - x_b\right)^2}{2\sigma_x^2} - \frac{\left(y - y_b\right)^2}{2\sigma_y^2}\right),$$

with current waveform

$$I_b(t) = \frac{Q_b}{(2\pi)^{1/2}\sigma_t} \exp\left(-\frac{(t - t_b)^2}{2\sigma_t^2} \right),$$

and beam arrival time at t=0 is t_b. This beam we view as a superposition of "pencil beams", and the induced cavity voltage is accordingly a superposition of the induced voltage from each. Our beam-cavity equation then takes the form

$$\left(\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_L}\frac{d}{dt} + \omega_\lambda^2 \right) V_c \approx 2\frac{\omega_\lambda}{Q_e}\frac{dV_F}{dt} + \frac{1}{2}\omega_\lambda \left[\frac{R}{Q} \right]_\lambda \frac{d}{dt} I_b(t) \langle X \rangle ,$$

where $\langle \mathrm{K} \rangle$ denotes an average over the beam cross-section. In this section, we will evaluate the average $\langle X \rangle = \langle w \rangle / w(\vec{r}_{\perp 0})$, for a general w, and integrate this equation for a Gaussian current profile. First, to compute $\langle w \rangle$ we need to compute the various beam moments.

Beam Weighted Multipoles

To compute $\langle x^m \rangle$, we express it as

$$\langle x^m \rangle = \frac{1}{(2\pi)^{1/2}\sigma_x} \int_{-\infty}^{+\infty} dx\, x^m \exp\left(-\frac{(x - x_b)^2}{2\sigma_x^2} \right) = \frac{1}{(2\pi)^{1/2}\sigma_x} \int_{-\infty}^{+\infty} d\xi \left(x_b + \xi \right)^m \exp\left(-\frac{\xi^2}{2\sigma_x^2} \right),$$

changing integration variables according to $x = x_b + \xi$. We then apply the binomial theorem to expand the integral, for example,

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2, \\
(a+b)^3 &= a^3 + 3a^2 b + 3ab^2 + b^3, \\
(a+b)^4 &= a^4 + 4ab^3 + 6a^2 b^2 + 4a^3 b + b^4, \\
(a+b)^5 &= a^5 + 5ab^4 + 10a^2 b^3 + 10a^3 b^2 + 5a^4 b + b^5.
\end{aligned}$$

And we make use of the integrals,

$$\begin{aligned}
\frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{+\infty} d\xi \exp\left(-\frac{\xi^2}{2\sigma^2} \right) &= 1, \\
\frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{+\infty} d\xi\, \xi^2 \exp\left(-\frac{\xi^2}{2\sigma^2} \right) &= \sigma^2, \\
\frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{+\infty} d\xi\, \xi^4 \exp\left(-\frac{\xi^2}{2\sigma^2} \right) &= 3\sigma^4.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle x \rangle &= \langle x_b + \xi \rangle = x_b, \\
\langle x^2 \rangle &= \langle x_b^2 + 2x_b\xi + \xi^2 \rangle = x_b^2 + \sigma_x^2,
\end{aligned}$$

$$\left\langle x^3 \right\rangle = \left\langle x_b^3 + 3x_b^2\xi + 3x_b\xi^2 + \xi^3 \right\rangle = x_b^3 + 3\sigma_x^2 x_b,$$
$$\left\langle x^4 \right\rangle = \left\langle x_b^4 + 4x_b\xi^3 + 6x_b^2\xi^2 + 4x_b^3\xi \ + \xi^4 \right\rangle = x_b^4 + 6\sigma_x^2 x_b^2 + 3\sigma_x^4,$$
$$\left\langle x^5 \right\rangle = \left\langle x_b^5 + 5x_b\xi^4 + 10x_b^2\xi^3 + 10x_b^3\xi^2 + 5x_b^4\xi \ + \xi^5 \right\rangle = x_b^5 + 15x_b\sigma_x^4 + 10x_b^3\sigma_x^2,$$

and similarly for y. In this way we obtain the averaged integral $\langle \tilde{w} \rangle = \langle \tilde{w} \rangle_{0,1,2} + \langle \tilde{w} \rangle_3 + \langle \tilde{w} \rangle_4 + \langle \tilde{w} \rangle_5 + \mathrm{K}$, where terms through 2nd order are

$$\langle \tilde{w} \rangle_{0,1,2} = \underbrace{b_0}_{\text{monopole}} \underbrace{- a_1 y_b}_{\text{vertical dipole}} + \underbrace{b_1 x_b}_{\text{horizontal dipole}} \underbrace{- 2a_2 x_b y_b}_{\text{skew quad}} + \underbrace{b_2\left\{x_b^2 + \sigma_x^2 - y_b^2 - \sigma_y^2\right\}}_{\text{normal quad}},$$

sextupole terms are

$$\begin{aligned}\langle \tilde{w} \rangle_3 &= \underbrace{-a_3\left\{3\left(x_b^2 + \sigma_x^2\right)y_b - \left(y_b^3 + 3\sigma_y^2 y_b\right)\right\}}_{\text{skew sextupole}} + \underbrace{b_3\left\{\left(x_b^3 + 3\sigma_x^2 x_b\right) - 3x_b\left(y_b^2 + \sigma_y^2\right)\right\}}_{\text{normal sextupole}} \\ &= -a_3 y_b\left\{\left(3x_b^2 - y_b^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)\right\} + b_3 x_b\left\{\left(x_b^2 - 3y_b^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)\right\}\end{aligned},$$

octupole terms are

$$\begin{aligned}\langle \tilde{w} \rangle_4 &= \underbrace{-4a_4\left\{\left(x_b^3 + 3\sigma_x^2 x_b\right)y_b - \left(y_b^3 + 3\sigma_y^2 y_b\right)x_b\right\}}_{\text{skew octupole}} \\ &\quad \underbrace{+ b_4\left\{\left(x_b^4 + 6\sigma_x^2 x_b^2 + 3\sigma_x^4\right) - 6\left(x_b^2 + \sigma_x^2\right)\left(y_b^2 + \sigma_y^2\right) + \left(y_b^4 + 6\sigma_y^2 y_b^2 + 3\sigma_y^4\right)\right\}}_{\text{normal octupole}} \\ &= -4a_4 y_b x_b\left\{\left(x_b^2 - y_b^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)\right\} \\ &\quad \underbrace{+ b_4\left\{\left(x_b^4 - 6x_b^2 y_b^2 + y_b^4\right) + 6\left(x_b^2 - y_b^2\right)\left(\sigma_x^2 - \sigma_y^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)^2\right\}}_{\text{normal octupole}},\end{aligned}$$

and decapole terms are

$$\begin{aligned}\langle \tilde{w} \rangle_5 &= \underbrace{-a_5\left\{\left(y_b^5 + 15y_b\sigma_y^4 + 10y_b^3\sigma_y^2\right) + 5\left(x_b^4 + 6\sigma_x^2 x_b^2 + 3\sigma_x^4\right)y_b - 10\left(x_b^2 + \sigma_x^2\right)\left(y_b^3 + 3\sigma_y^2 y_b\right)\right\}}_{\text{skew decapole}} \\ &\quad \underbrace{+ b_5\left\{\left(x_b^5 + 15x_b\sigma_x^4 + 10x_b^3\sigma_x^2\right) - 10\left(y_b^2 + \sigma_y^2\right)\left(x_b^3 + 3\sigma_x^2 x_b\right) + 5x_b\left(y_b^4 + 6\sigma_y^2 y_b^2 + 3\sigma_y^4\right)\right\}}_{\text{normal decapole}}, \\ &= -a_5 y_b\left\{\left(y_b^4 + 15\sigma_y^4 + 10y_b^2\sigma_y^2\right) + 5\left(x_b^4 + 6\sigma_x^2 x_b^2 + 3\sigma_x^4\right) - 10\left(x_b^2 + \sigma_x^2\right)\left(y_b^2 + 3\sigma_y^2\right)\right\} \\ &\quad + b_5 x_b\left\{\left(x_b^4 + 15\sigma_x^4 + 10x_b^2\sigma_x^2\right) - 10\left(y_b^2 + \sigma_y^2\right)\left(x_b^2 + 3\sigma_x^2\right) + 5\left(y_b^4 + 6\sigma_y^2 y_b^2 + 3\sigma_y^4\right)\right\} \\ &= -a_5 y_b\left\{\left(y_b^4 - 10x_b^2 y_b^2 + 5x_b^4\right) + 15\left(\sigma_x^2 - \sigma_y^2\right)^2 - 10\left(y_b^2 - 3x_b^2\right)\left(\sigma_x^2 - \sigma_y^2\right)\right\} \\ &\quad + b_5 x_b\left\{\left(x_b^4 - 10x_b^2 y_b^2 + 5y_b^4\right) + 15\left(\sigma_x^2 - \sigma_y^2\right)^2 + 10\left(x_b^2 - 3y_b^2\right)\left(\sigma_x^2 - \sigma_y^2\right)\right\}.\end{aligned}$$

“Next-Order” Beam-Cavity Equation for Monopole & Dipole Modes

To summarize our results thus far, let us make explicit the beam-cavity equation, employing conventional notation where possible, for the usual monopole and dipole modes, but including the next-order corrections appropriate for a cavity with inversion symmetry in x & y. The monopole mode excitation satisfies

$$\left(\frac{d^2}{dt^2}+\frac{\omega_\lambda}{Q_L}\frac{d}{dt}+\omega_\lambda^2\right)V_c \approx 2\frac{\omega_\lambda}{Q_e}\frac{dV_F}{dt}+\frac{1}{2}\omega_\lambda\left[\frac{R}{Q}\right]_\lambda \frac{d}{dt}I_b(t)\langle X\rangle,$$

where V_c, the cavity voltage, and $[R/Q]$ are defined with respect to the axis distinguished by inversion symmetry. The form of the synchrous integral for this mode is constrained by symmetry,

$$w=\underset{monopole}{b_0}+\underset{normal\ quad}{b_2\left(x^2-y^2\right)}+\underset{normal\ octupole}{b_4\left(x^4-6x^2y^2+y^4\right)}+O\left(r_\perp^6\right)\mathrm{K}\quad ,$$

and thus

$$\langle X\rangle = 1+\frac{b_2}{b_0}\left\langle x^2-y^2\right\rangle+\frac{b_4}{b_0}\left\langle x^4-6x^2y^2+y^4\right\rangle+O\left(\left\langle r_\perp^6\right\rangle\right)\mathrm{K}\quad .$$

Including just the lowest order correction, this is

$$\langle X\rangle \approx 1+\frac{b_2}{b_0}\left(x_b^2-y_b^2+\sigma_x^2-\sigma_y^2\right).$$

Next, consider an x-dipole mode. The form of the synchronous integral is constrainted by symmetry

$$w=\underset{\substack{horizontal\\ dipole}}{b_1x}+\underset{normal\ sextupole}{b_3\left(x^3-3xy^2\right)}+\underset{normal\ decapole}{b_5\left(x^5-10x^3y^2+5xy^4\right)}+O\left(r_\perp^7\right)$$

We select an integration path at $x=x_0$, for our definitions of cavity voltage and [R/Q], and take the limit $x_0\to 0$, so that our description of the cavity is resolved in terms of the conventional quantities

$$V_c'=\lim_{x_0\to 0}\frac{V_c}{x_0}\equiv\frac{\partial V_c}{\partial x},\ \left[\frac{R_\perp}{Q}\right]=\lim_{x_0\to 0}\frac{1}{x_0^2}\left[\frac{R}{Q}\right],$$

i.e., $[R/Q]=2b_1^2x_0^2/\varepsilon_0\omega_\lambda$. Cavity excitation is then described by

$$\left(\frac{d^2}{dt^2}+\frac{\omega_\lambda}{Q_L}\frac{d}{dt}+\omega_\lambda^2\right)V_c' \approx 2\frac{\omega_\lambda}{Q_e}\frac{dV_F'}{dt}+\frac{1}{2}\omega_\lambda\left[\frac{R_\perp}{Q}\right]_\lambda \frac{d}{dt}I_b(t)X_b,$$

where the "inferred beam position" is

$$X_b=\lim_{x_0\to 0}x_0\langle X\rangle=\langle x\rangle+\frac{b_3}{b_1}\left\langle x^3-3xy^2\right\rangle+\frac{b_5}{b_1}\left\langle x^5-10x^3y^2+5xy^4\right\rangle+O\left(\left\langle r_\perp^7\right\rangle\right)\mathrm{K}\quad .$$

Explicitly, including only the lowest order correction,

$$X_b \approx x_b \left\{ 1 + \frac{b_3}{b_1} \left(x_b^2 - 3y_b^2 + 3\sigma_x^2 - 3\sigma_y^2 \right) \right\}.$$

The results presented here, for the monopole and dipole mode formulations, with only lowest-order corrections should be adequate for most work. However, as a further illustration of this formulation, let us consider a specific example, one of immediate relevance, through decapole order.

Example: Linac-Style BPM Cavity

As a practical example, we consider the *x*-dipole mode of the linac style *x*-cavity. This mode has symmetry odd in *x*, even in *y*, so that

$$\tilde{w}_{odd-even} = b_1 x \left\{ 1 + \underbrace{\frac{b_3}{b_1}\left(x^2 - 3y^2\right)}_{normal\ sextupole} + \underbrace{\frac{b_5}{b_1}\left(x^4 - 10x^2y^2 + 5y^4\right)}_{normal\ decapole} + O\left(r_\perp^6\right)\ldots \right\}.$$

As described in TN152, using the field-solving code, `GdfidL`, we evaluted the normal sextupole coefficient $\varepsilon_{NS} = b_3 / b_1$, and the normal decapole coefficient, $\varepsilon_{ND} = b_5 / b_1$ with the results

$$\varepsilon_{NS} = -4.2 \times 10^{-3}\,\text{cm}^{-2}, \qquad \varepsilon_D = -4.4 \times 10^{-3}\,\text{cm}^{-4}.$$

With these coefficients, we are in a position to determine the sensitivity of the cavity excitation to the beam first and second moments. We express the field integral as

$$\langle \tilde{w} \rangle \approx b_1 \left\{ \langle x \rangle + \varepsilon_{NS} \left\langle x^3 - 3y^2x \right\rangle + \varepsilon_{ND} \left\langle x^5 - 10x^3y^2 + 5y^4x \right\rangle \right\} \approx b_1 X_b,$$

where

$$\begin{aligned} X_b &= \langle x \rangle + \varepsilon_{NS} \left\langle x^3 - 3y^2x \right\rangle + \varepsilon_{ND} \left\langle x^5 - 10x^3y^2 + 5y^4x \right\rangle \\ &= x_b \left(1 + \delta_{NS} + \delta_{ND} \right), \end{aligned}$$

and the correction terms are

$$\begin{aligned} \delta_{NS} &= \varepsilon_{NS} \left\{ \left(x_b^2 - 3y_b^2 \right) + 3\left(\sigma_x^2 - \sigma_y^2 \right) \right\}, \\ \delta_{ND} &= \varepsilon_{ND} \left\{ \left(x_b^4 - 10x_b^2y_b^2 + 5y_b^4 \right) + 15\left(\sigma_x^2 - \sigma_y^2 \right)^2 + 10\left(x_b^2 - 3y_b^2 \right)\left(\sigma_x^2 - \sigma_y^2 \right) \right\}. \end{aligned}$$

Let us consider first the case of beam-size small compared to beam-offset. In this case, we have $\delta_{NS} \approx \varepsilon_{NS}\left(x_b^2 - 3y_b^2\right)$, and $\delta_{ND} \approx \varepsilon_{ND}\left(x_b^4 - 10x_b^2y_b^2 + 5y_b^4\right)$. The sextupole correction is a maximum at fixed-radius, when *x*=0, giving $\delta_{NS} \approx -3y_b^2\varepsilon_{NS}$ and in the case of a beam misteered as far as the beam pipe radius (1 cm), this amounts to only a 1% correction. Actually, in this case, since *x*=0, the effect is negligible. If one asks when the *absolute* sextupole correction is a maximum, the result is $x_b\delta_{NS} \approx \varepsilon_{NS} x_b^3$, and in this extreme case, the correction is still only 0.5% in amplitude. The conclusion is qualitatively the same for the decapole correction. The maximum fractional correction occurs for x=0, giving $\delta_{ND} \approx 5y_b^4\varepsilon_{ND}$, or a 2% correction, but to a negligible signal. A useful rule of thumb for this cavity is that for gross vertical misteering, the correction amounts to at most 1%. In fact, however, our interest is in operating the cavity at beam-sizes and offsets a small fraction of the pipe radius, $\sigma_x \approx \sigma_y \approx 100\ \mu\mathrm{m}$, and $x_{\mathrm{b}} \approx y_{\mathrm{b}} \approx 100\ \mu\mathrm{m}$. We expect $x_{\mathrm{b}} \approx y_{\mathrm{b}} \approx 1\ \mathrm{mm}$ in the worst operationally functional case. At the level of 100 μm offsets, the decapole terms are negligible, and the sextupole terms contribute a scale factor correction $\delta_{NS} \approx 1\% \times 10^{-4}$. Since corrections at this level are negligible, the main conclusion from such an analysis is a caution concerning operation with large vertical misteering.

3. Form Factor

Next we would like to integrate our cavity equation,

$$\left(\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_{L\lambda}}\frac{d}{dt} + \omega_\lambda^2\right)V_c \approx 2\frac{\omega_\lambda}{Q_{e\lambda}}\frac{dV_F}{dt} + \frac{1}{2}\omega_\lambda\left[\frac{R}{Q}\right]_\lambda \frac{d}{dt} I_b(t)\langle X\rangle,$$

for the particular case of a Gaussian current profile. We set $V_F = 0$ corresponding to no drive; it is straightforward to add it back later, using superposition, as V_F enters in the same form as $I_b(t)\langle X\rangle$. One may check, by differentiation, that

$$V_c(t) = \frac{1}{2}\frac{\omega_\lambda}{\Omega_\lambda}\left[\frac{R}{Q}\right]_\lambda \int_{-\infty}^{t} dt' \sin\left[\Omega_\lambda(t-t')\right] e^{-\nu(t-t')/2} \frac{d}{dt'}\left[I_b\langle X\rangle\right],$$

is a solution, where we abbreviate

$$\nu_\lambda = \frac{\omega_\lambda}{Q_{L\lambda}}, \qquad \Omega_\lambda = \sqrt{\omega_\lambda^2 - \nu_\lambda^2}\ .$$

Integrating by parts, we obtain

$$V_c(t) = 2k_{l\lambda}\int_{-\infty}^{t} dt' g(t-t')\left[I_b\langle X\rangle\right]_{t'},$$

where the loss factor is

$$k_{l\lambda} = \frac{1}{4}\omega_\lambda\left[\frac{R}{Q}\right]_\lambda,$$

and the Green’s function is

$$g(t) = \left\{\cos\Omega_\lambda t - \frac{\nu_\lambda}{2\Omega_\lambda}\sin\Omega_\lambda t\right\}e^{-\nu_\lambda t/2} = \Re\left\{\left(1 + j\frac{\nu_\lambda}{2\Omega_\lambda}\right)e^{-\Gamma_\lambda t}\right\},$$

with

$\Gamma_\lambda = \frac{1}{2}\nu_\lambda - j\Omega_\lambda .$

Typically, our interest is not in the variation of the cavity voltage during the time the beam is passing through the cavity. Rather our interest is to know, after a bunch has passed, at what amplitude and phase a cavity mode is ringing. To determine this, we rewrite our expression for cavity voltage as

$$V_c(t) = 2k_{l\lambda}\Re\left(1 + j\frac{\nu_\lambda}{2\Omega_\lambda}\right)\int_{-\infty}^{t} dt' e^{-\Gamma_\lambda(t-t')}\left[I_b\langle X\rangle\right]_{t'} = 2k_{l\lambda}\Re\left(1 + j\frac{\nu_\lambda}{2\Omega_\lambda}\right)e^{-\Gamma_\lambda t}\langle X\rangle\int_{-\infty}^{t} dt' e^{\Gamma_\lambda t'} I_b(t')$$

After the beam has passed, the integral

$$\int_{-\infty}^{\infty} dt' e^{\Gamma_\lambda t'} I_b(t') = \int_{-\infty}^{\infty} dt' e^{\Gamma_\lambda t'} \frac{Q_b}{(2\pi)^{1/2}\sigma_t}\exp\left(-\frac{(t'-t_b)^2}{2\sigma_t^2}\right) = e^{\Gamma_\lambda t_b} Q_b \exp\left(\tfrac{1}{2}\Gamma_\lambda^2\sigma_t^2\right) ,$$

determines the excitation amplitude. Using $\Gamma_\lambda^2 = \left(\frac{1}{2}\nu_\lambda\right)^2 - \Omega_\lambda^2 - j\nu_\lambda\Omega_\lambda = -\omega_\lambda^2 - j\nu_\lambda\Omega_\lambda$, we have

$$V_c(t) = 2k_{l\lambda}\Re\left(1 + j\frac{\nu_\lambda}{2\Omega_\lambda}\right)e^{-\Gamma_\lambda(t-t_b)} Q_b\langle X\rangle \exp\left(-\tfrac{1}{2}\omega_\lambda^2\sigma_t^2\right)\exp\left(-\tfrac{1}{2}j\nu_\lambda\Omega_\lambda\sigma_t^2\right).$$

To simplify the notation, let us denote

$$\eta = \left(1 + j\frac{\nu_\lambda}{2\Omega_\lambda}\right)\exp\left(-\tfrac{1}{2}j\nu_\lambda\Omega_\lambda\sigma_t^2\right),$$

so that

$$V_c(t) = 2k_{l\lambda}Q_b\langle X\rangle\exp\left(-\tfrac{1}{2}\omega_\lambda^2\sigma_t^2\right)\Re\,\eta e^{-\Gamma_\lambda(t-t_b)}.$$

(For work we will be doing $\eta \approx 1$ is an excellent approximation, as it is whenever $Q_{L\lambda} >> 1$.) Thus one sees that after bunch passage, the cavity rings at the resonance frequency Ω_λ (with damping correction). It is helpful to represent this result in terms of a cavity phasor “in the rotating frame”, *i.e.*, to express,

$$V_c(t) = \Re e^{j\Omega_\lambda(t-t_b)}\tilde{V}_c(t),$$

where the phasor $\tilde{V}_c(t)$ has a step-rise and exponential decay,

$$\tilde{V}_c(t) = \hat{V}_c H(t-t_b)\exp^{-\frac{1}{2}\nu_\lambda(t-t_b)},$$

with amplitude

$$\hat{V}_c = 2k_{l\lambda}Q_b\langle X\rangle\eta\exp\left(-\tfrac{1}{2}\omega_\lambda^2\sigma_t^2\right).$$

Actually, we already know that for a point bunch, without damping the induced voltage amplitude is $2k_{l\lambda}Q_b$, and the character of step-rise and exponential fall is not surprising. The quantity $\eta \approx 1$. Thus all we have really calculated here is the "form factor" $\exp\left(-\frac{1}{2}\omega_\lambda^2\sigma_t^2\right)$. It is often convenient to express this as $\exp\left(-\frac{1}{2}k_\lambda^2\sigma_z^2\right)$, with $\sigma_z = c\sigma_t$ the rms bunch length, and $k_\lambda = \omega_\lambda / c$.

Generalized Form Factor

We may go one step farther with the form factor concept, and employ it to account for higher multipole corrections as well. We introduce a quantity *Σ,* such that

$$\exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right)\langle\tilde{w}\rangle \approx \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma^2\right)\langle\tilde{w}\rangle_0,$$

where $\langle\tilde{w}\rangle_0$ is the leading order part of the field integral, and multipole corrections are included in $\Sigma^2 - \sigma_z^2$.

For illustration let us consider the case of the cavity with inversion symmetry in *x* & *y*. For a predominantly monopole mode, and including only the next-order, right-quadrupole correction,

$$\exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma^2\right) \approx \exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right)\left\{1 + \frac{b_2}{b_0}\left(x_b^2 - y_b^2 + \sigma_x^2 - \sigma_y^2\right)\right\},$$

or

$$\Sigma^2 \approx \sigma_z^2 - \frac{2}{k_\lambda^2}\frac{b_2}{b_0}\left(x_b^2 - y_b^2 + \sigma_x^2 - \sigma_y^2\right). \qquad \text{(monopole mode with right-quad component)}$$

For the case of a predominantly *x*-dipole mode, and including only the next-order, right-sextupole correction,

$$\exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma^2\right) \approx \exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right)\left\{1 + \frac{b_3}{b_1}\left(x_b^2 - 3y_b^2 + 3\sigma_x^2 - 3\sigma_y^2\right)\right\},$$

or

$$\Sigma^2 \approx \sigma_z^2 - \frac{2}{k_\lambda^2}\frac{b_3}{b_1}\left(x_b^2 - 3y_b^2 + 3\sigma_x^2 - 3\sigma_y^2\right). \qquad \text{(dipole mode with right-sextupole component)}$$

Expression of multipole contributions in this fashion is helpful, in that it permits direct quantitative comparison (once the multipole coefficients b_i are known) with a more familiar dependence, that of the usual form factor dependence on bunch length.

For the example of interest here, the linac *x*-cavity,

$$\langle\tilde{w}\rangle_0 = b_1 x_b,$$
$$\tfrac{1}{2}k_\lambda^2\left(\Sigma^2 - \sigma_z^2\right) \approx \delta_{NS} + \delta_{ND} \approx \varepsilon_{NS}\left\{\left(x_b^2 - 3y_b^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)\right\}.$$

Employing the values computed with the help of `GdfidL`, $\varepsilon_{NS} = b_3 / b_1 \approx -4.2\times10^{-3}\,\mathrm{cm}^{-2}$, and

$k_\lambda \approx 0.59858\,\mathrm{cm}^{-1}$ as for 2856 MHz, we have,

$$\Sigma^2 = \sigma_z^2 + 0.023\left\{\left(x_b^2 - 3y_b^2\right) + 3\left(\sigma_x^2 - \sigma_y^2\right)\right\}.$$

In this form, the relative importance of nonlinearities is made quite explicit. The form factor correction amounts to 1%, for $\Sigma \approx 0.24\,\mathrm{cm}$, and in order to see such an effect from bunch length, one would have unacceptable energy-spread in the linac.[7] For the sextupole correcion to be most pronounced, at fixed beam-size, the beam would be misteered in the vertical on the order of 0.9 cm. The beam pipe has a 1 cm radius, and thus for a 100 µm spot-size, the beam would still have a 10σ clearance, even at this level of offset. Thus in principle it is strictly possible to obtain a correction from the sextupole term at the level of 1%, without scraping. However, these conditions do not correspond to normal operation, and such a level of vertical misteering in an x-cavity would be diagnosed with the help of the adjacent *y*-cavity, and could be corrected.

4. Single-Bunch Scalings for Monopole & Dipole Modes

It is helpful to reduce our single bunch results from the last section to some simple rules of thumb. Energy deposited in the cavity by a single bunch is

$$U = \frac{\left|\hat{V}_c\right|^2}{\omega_\lambda [R/Q]} = k_{l\lambda} Q_b^2 \langle X \rangle^2 \exp\left(-k_\lambda^2 \sigma_z^2\right),$$

and the power waveform radiating from the cavity rises as a step function, and decays exponentially with e-folding time $T_f/2 = 1/\nu_\lambda = Q_L/\omega_\lambda$. In the absence of drive $V_c = V_F + V_R = V_R$,, and the peak power radiated out of the cavity is

$$-P_w = \frac{|V_c|^2}{Q_{e\lambda}[R/Q]_\lambda} = \frac{\omega_\lambda}{Q_{e\lambda}} U.$$

(Peak power dissipated in the cavity walls is $P_d = \omega_\lambda U / Q_{w\lambda}$.) Using the relations

$$\frac{1}{Q_{e\lambda}} = \frac{\beta}{Q_{w\lambda}}, \qquad \frac{1}{Q_{L\lambda}} = \frac{1+\beta}{Q_{w\lambda}}, \quad T_f = \frac{2Q_{L\lambda}}{\omega_\lambda} = \frac{2Q_{w\lambda}}{\omega_\lambda}\frac{1}{1+\beta} = \frac{Q_{e\lambda}}{\omega_\lambda}\frac{2\beta}{1+\beta},$$

we may write this as

$$-P_w = \frac{2\beta}{1+\beta}\frac{U}{T_f}.$$

To make this more explicit, consider the usual monopole and dipole modes, but including the next-order corrections appropriate for a cavity with inversion symmetry in *x* & *y*.

Example: Linac φ-Cavity

[7] For further discussion of the form factor see "Introduction to the Phase-Cavity for the Linac Resonant Cavity Beam Position Monitor System", ARDB Technical Note 154.

For the monopole mode $k_{l\lambda} = \omega_\lambda [R/Q]/4$ is just the usual loss factor evaluated on the centered orbit,

$$\langle X \rangle \approx 1 + \frac{b_2}{b_0}\left(x_b^2 - y_b^2 + \sigma_x^2 - \sigma_y^2\right),$$

and abbreviating this as $\exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right)\langle X \rangle = \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma^2\right)$, we have

$$-P_w = \frac{2\beta}{1+\beta}\frac{1}{T_f}k_{l\lambda}Q_b^2\exp\left(-k_\lambda^2\Sigma^2\right).$$

For the case of the linac φ-cavity, we have cylindrical symmetry to an excellent approximation, so that the quadrupole term $b_2 \approx 0$. Using cylindrical pillbox scalings, as described in TN154, one finds $Q_w \approx 1.73 \times 10^4$, $[R/Q] \approx 220\ \Omega$ and unloaded shunt impedance $R_s \approx 3.77\ \text{M}\Omega$. Altenmueller and Brunet quote $Q_L \approx 1200$ and $[R/Q] \approx 248\ \Omega$,[8] corresponding to $Q_e \approx 1371$ and $\beta = Q_w / Q_e \approx 7.0$. Field decrement time due to copper losses is then $T_0 = 2Q_w/\omega_0 \approx 1.07\ \mu\text{s}$, and the loaded fill time is $T_f = 2Q_L/\omega \approx 133\ \text{ns}$. The loss-factor is $k_l = \omega_\lambda [R/Q]/4 \approx 1.11\ \text{V/pC}$. With these parameters we have

$$U \approx 1.11\ \mu\text{J}\ Q_b^2(\text{nC}), \qquad P_w \approx 1.75\frac{U_C}{T_f} \approx 1.75\frac{1.11\ \mu\text{J}\ Q_b^2(\text{nC})}{133\ \text{ns}} \approx 14.6\ W\ Q_b^2(\text{nC}),$$

with a power e-folding time of $T_f/2 \approx 66.5\ \text{ns}$. It is conventional to refer to power in units of decibels relative to 1 mW, dBm, so that

$$10\log_{10} P_w(\text{mW}) \approx 41.6\ \text{dBm} + 20\log_{10} Q_b(\text{nC}) \ .$$

For an SLD high-charge bunch of 5 nC (3.1×10^{10} electrons) this provides 56 dBm. For a single bunch of 17.5 pC (1.1×10^8 electrons) this provides 6.5 dBm. (As a point of reference, a train of such bunches arriving at 2856MHz corresponds to 50 mA).

Example: Linac x-Cavity

For the case of a predominantly x-dipole mode we have, again defining quantities on a reference orbit at $x = x_0$, and taking the limit $x_0 \to 0$, we have

$$X_b = \lim_{x_0 \to 0} x_0 \langle X \rangle \approx x_b \left\{1 + \frac{b_3}{b_1}\left(x_b^2 - 3y_b^2 + 3\sigma_x^2 - 3\sigma_y^2\right)\right\},$$

which we will express as

$$X_b \exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right) = x_b \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma^2\right).$$

Making use of

[8] O. Altenmueller and P. Brunet, "Some RF Characteristics of the Beam Phase Reference Cavity" SLAC TN-64-51, Sept. 1964 (available from the SLAC Library).

$$k_{\perp\lambda} = \lim_{x_0 \to 0} \frac{1}{x_0^2} k_{l\lambda} = \frac{1}{4}\omega_\lambda \left[\frac{R_\perp}{Q}\right],$$

we find

$$U = k_{\perp\lambda} Q_b^2 x_b^2 \exp\left(-k_\lambda^2 \Sigma^2\right),$$

and

$$-P_w = \frac{2\beta}{1+\beta}\frac{1}{T_f} k_{\perp\lambda} Q_b^2 x_b^2 \exp\left(-k_\lambda^2 \Sigma^2\right).$$

For a numerical example, we consider the linac *x*-cavity, putting aside the generalized form factor (Σ-term) for now. TN152 provides discussion of the cavity parameters. A closed pillbox estimate gives $Q_w \approx 2.26\times 10^4$ $\left[R_\perp / Q\right] \approx 17.8\ \Omega\mathrm{cm}^{-2}$, $R_\perp \approx 4.0\times 10^5\ \Omega/\mathrm{cm}^2$, $k_\perp \approx 8.07\times 10^{-2}\ V/\mathrm{pC\,cm}^2$. As for external coupling we make use of the value $Q_L \approx 325$ based on the technical note of Brunet, *et al.,*[9] recognizing that there is some question as to the actual value.[10] This corresponds to $Q_e = 332$, and $\beta \approx 45.2$. The natural field decrement time is $T_0 = 2Q_w/\omega \approx 1.67\,\mu\mathrm{s}$, and the loaded fill-time is $T_f = 2Q_L/\omega \approx 36.2\,\mathrm{ns}$.

Energy deposited by a single bunch is then

$$U \approx 8.07\times 10^{-16} J\left[Q_b(\mathrm{nC}) x_b(\mu\mathrm{m})\right]^2,$$

and this energy damps after bunch passage, with e-folding time of $T_f/2 \approx 18.1$ ns. Peak power radiated is

$$P_w \approx 1.96\frac{U}{T_f} \approx 4.37\times 10^{-8}\,\mathrm{W}\left[Q_b(\mathrm{nC}) x_b(\mu\mathrm{m})\right]^2,$$

using $\beta \approx 45.2$ and $T_f \approx 36.2$ ns. This is in fair agreement with the Blue Book value for sensitivity 100 $\mu\mathrm{W/mA^2mm^2}$. (For Blue Book loaded *Q*, $T_f/2 \approx 34\,\mathrm{ns}$ and the coefficient is $2.9\times 10^{-8}\,\mathrm{W}$, or 1.9 dB lower---lower since single-bunch peak power prefers lower loaded *Q*, not being helped by resonance.). In dB relative to milliwatt we have,

$$10\log_{10} P_w(\mathrm{mW}) \approx -43.6\,\mathrm{dBm} + 20\log_{10}\left[Q_b(\mathrm{nC}) x_b(\mu\mathrm{m})\right].$$

For an SLD high-charge bunch of 5 nC (3.1×10^{10} electrons) offset by 1 μm, this provides -30 dBm. For a single bunch of 17.5 pC (1.1×10^8 electrons) this provides -79 dBm. (As a point of

[9] P. Brunet, J. Dobson, M. J. Lee, and C. B. Williams, "Microwave Beam Position Monitors", SLAC TN 64-45, July, 1964 (available from the SLAC Library).

[10] *The Two-Mile Accelerator*, R. B. Neal, ed., (W. A. Benjamin, New York, 1968) pp. 500-516. We will refer to this reference as the "Blue Book". For the Blue Book value of $Q_L \approx 600$, $\beta \approx 24$, and $T_f \approx 67\,\mathrm{ns}$. The report by Brunet, *et al.*, quotes $Q_L \approx 300$ (p. 26) and lists $Q_L \approx 325$ in their Fig. 12 (p. 23), there labelled as a cold test result.

reference, a train of such bunches arriving at 2856 MHz corresponds to 50 mA).

5. Excitation by a Bunch Train

Next, we would like to proceed on to the case of a multi-bunch train. First though, there is one integral that is worth noting.

Baseband Integral

In thinking about the processing of cavity emitted waveforms, there is one particularly simple and noteworthy result that follows immediately from our analysis. In processing a waveform such as

$$V_c(t) = \Re e^{j\Omega_\lambda t}\tilde{V}_c(t),$$

we prefer to abstract a linear analog of the signal at a frequency lower than the carrier frequency Ω_λ. In the simplest ideal case, we would mix the signal with a local oscillator output at the same frequency, phased with the beam. In this case we access an output waveform that is just $\tilde{V}_c(t)$. The integral of this waveform is then

$$\int_{-\infty}^{+\infty} dt\,\tilde{V}_c(t) = \hat{V}_c T_f = 2k_{l\lambda}T_f Q_b\langle X\rangle \exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right) = R_{L\lambda}Q_b\langle X\rangle \exp\left(-\tfrac{1}{2}k_\lambda^2\sigma_z^2\right),$$

with $R_{L\lambda} = Q_{L\lambda}[R/Q]_\lambda$. For a monopole cavity, this is

$$\int_{-\infty}^{+\infty} dt\,\tilde{V}_{cm}(t) = R_L Q_b \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma_m^2\right),$$

and for a dipole cavity this is

$$\int_{-\infty}^{+\infty} dt\,\tilde{V}_{cd}(t) = R_{L\perp} Q_b x_b \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma_d^2\right).$$

It is straightforward to extend these single bunch results to the case of a bunch train, when the beam-phase and (micro) bunch-length are not chirped along the macropulse (and where our LO phase is steady). Using superposition, we can see that the downmixed waveform for a bunch train satisfies the same relations, with Q_b replaced with the total macropulse charge, and $Q_b x_b$ replaced with the charge-weighted beam-centroid. Thus the charge-weighted position can be obtained in a straightforward fashion. Appreciating that

$$\frac{Q_b x_b \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma_d^2\right)}{Q_b \exp\left(-\tfrac{1}{2}k_\lambda^2\Sigma_m^2\right)} = x_b\left\{1 + \frac{b_3}{b_1}\left(x_b^2 - 3y_b^2 + 3\sigma_x^2 - 3\sigma_y^2\right)\right\} = X_b,$$

we arrive at the simplest algorithm for monitoring of beam-position,

$$X_b = k_x \frac{\int_{-\infty}^{+\infty} dt\, \tilde{V}_{cd}(t)}{\int_{-\infty}^{+\infty} dt\, \tilde{V}_{cm}(t)},$$

where the calibration constant is, theoretically, $k_x = R_L / R_{L\perp}$.

Practically speaking, however, one must carry the analysis of bunch trains farther, as cavity tuning and beam chirp are real issues to be dealt with. In practice, too, some resolution within the macropulse may be desirable, *i.e.*, it may not be the case that the downstream application (the BPM analyzing software+target+detector in E158) requires only charge-weighted position. Beam yaw may influence detector statistics and thus may require monitoring. Thus we wish to analyze the actual transient waveforms produced, including the possibility of cavity mistuning.

Waveform Calculation

To determine the transient signal produced by a bunch train, we simply superimpose the transient voltages induced by the individual bunches. Let us number our bunches n=0,1,..., N-1, and let us refer to time t_{bn} the arrival time of the n-th bunch (just like the variable t_b used previously), so that t_{b0} is the arrival time of the lead bunch. Let us express the timing of the n-th bunch with respect to bunch #0 as $\tau_n = t_{bn} - t_{b0}$. The bunch spacing is determined from the linac front-end and cannot deviate too much from $f_{RF}\tau_n = 1$, without degrading the beam energy spectrum, with $f_{RF} \approx 2856\,\text{MHz}$ the linac operating frequency.[11] At the same time, we should take note that f_{RF} may differ from $\Omega_\lambda / 2\pi$, the cavity resonance frequency, in the case of a cavity temperature excursion. We also have the condition $f_{RF} \neq \Omega_\lambda / 2\pi$ when we consider parasitic, non-resonant modes.

The voltage waveform induced by the n-th bunch is

$$V_{cn}(t) = \Re e^{-\Gamma_\lambda (t - t_{bn})} H(t - t_{bn}) \hat{V}_{cn},$$

where

$$\hat{V}_{cn} = 2k_{l\lambda} Q_{bn} \langle X_n \rangle \exp\left(-\tfrac{1}{2} k_\lambda^2 \sigma_{zn}^2\right).$$

We may then express the total voltage at any time as

$$V_c(t) = \sum_{n=0}^{N-1} V_{cn}(t) = \Re \sum_{n=0}^{t_{bn}<t} e^{-\Gamma_\lambda (t - t_{bn})} \hat{V}_{cn} = \Re e^{-\Gamma_\lambda (t - t_{b0})} \sum_{n=0}^{t_{bn}<t} e^{\Gamma_\lambda \tau_n} \hat{V}_{cn}$$

The sum in the second line is over those bunches that have already passed as of time t.

This sum may be employed to study analytically and numerically the effects of systematics due to τ_n, Σ_n, and Q_{bn} variations with n. In the meantime, however, let us consider an idealized

[11] As a point of reference, in a 200 ns pulse, there are 571 bunches. If by some means the beam is bunched 100 kHz off in frequency (uniform 12 fs bunch spacing error) then the phase-error across the train is 7°. This is a small phase drift, particularly given that such a frequency error on the part of the beam would result in a 3% loss of no-load voltage in the linac proper.

case corresponding to a "top-hat" current profile, a uniformly bunched beam, steady or small beam-size along the bunch train, and steady or small centroid offset along the train. We permit for the time-being a frequency deviation between the beam and cavity resonance. Summing a geometric series, we have

$$V_c(t) = \Re \exp\left[\Gamma(t - t_{b0})\right] \hat{V}_{c0} \frac{\exp(\Gamma\tau n) - 1}{\exp(\Gamma\tau) - 1},$$

where n indexes the last bunch that passed, *i.e.*, $t_{bn} < t < t_{bn+1}$. At early times this gives a voltage waveform rising linearly in time and output power rising quadratically. As time goes on, the system reaches a steady-state between power deposited by the beam, and power absorbed by the walls, and radiated down the waveguide. After the bunch train has gone by, the voltage decays exponentially.

In the case of a cavity perfectly tuned to resonance with the perfectly bunched beam, we have

$$V_c(t) \approx \Re \exp^{j\Omega(t - t_{b0})} \frac{\hat{V}_{c0}}{\Gamma\tau} = \Re \exp^{j\Omega(t - t_{b0})} \hat{V}_{eff},$$

where, using $\Gamma\tau = \nu\tau/2 = \pi/Q_L$, we may write

$$\hat{V}_{eff} = \frac{Q_L}{\pi} \hat{V}_{c0} = 2k_{l\lambda} Q_{beff} \langle X \rangle \exp\left(-\tfrac{1}{2} k_\lambda^2 \sigma_z^2\right),$$

and the effective charge is just the amount of charge passing through the cavity in one fill time,

$$Q_{beff} = \frac{Q_L}{\pi} Q_b.$$

Appreciating that the effect of resonant enhancement can be large, let us consider the effect of detuning from resonance.

Small Detuning

We consider a small detuning $\delta_\Omega = \Delta\Omega/\Omega << 1$, so that

$$\Gamma\tau = \tfrac{1}{2}\nu\tau - j\Omega\tau \approx \frac{\pi}{Q_L} - 2\pi j\delta_\Omega - 2\pi j N_h,$$

where the cavity harmonic is $N_h = 1$. Our voltage in steady-state is

$$V_c(t) \approx \Re \exp\left[j\Omega(t - t_{b0})\right] \frac{\hat{V}_{c0}}{\left(\pi/Q_L - 2\pi j\delta_\Omega\right)},$$

and this may be expressed in terms of the cavity tuning angle, ψ,

$$\tan\psi = 2Q_L\delta_\Omega = 2Q_L \frac{\Delta\Omega}{\Omega}$$

according to

$$V_c(t) \approx \Re \exp\left[j\Omega\left(t - t_{b0}\right)\right] \frac{Q_L}{\pi} \hat{V}_{c0} \cos\psi e^{j\psi},$$

Evidently tuning error introduces both a phase and an amplitude error. The amplitude error is second order in the tuning angle, and the phase-error is just the tuning angle.

As an example, for the linac φ-cavity, Altenmueller and Brunet report a tuning angle variation with temperature is 1.2° / °F (cavity detuning of 25 kHz, corresponding to $\delta_\Omega \approx 9 \times 10^{-6}$ for a 1° F cavity temperature error.) Reduction of amplitude by 0.1% corresponds to 2.6° or 2.1°F. At first sight, one is inclined to conclude that if 1 μm resolution is desired with offsets as large as 100 μm, and if the phase-reference signal is employed for amplitude normalization, then this reference amplitude should be accurate to better than 1%. Good temperature regulation (1-2°F) would be required to accomplish this. On second thought, while good temperature regulation is indeed in order, actual errors would involve the convolution of two caviy signals, and if the cavities are similarly mistuned the effect of mistuning may cancel. We will return to this subject in a later note, under the heading of signal analysis with errors. For now we continue on to consider just the basic features that are the ingredients for such an analysis.

Large Detuning

Let us consider the case of large mistuning, $\sin(\Omega\tau/2) > \nu\tau/2$, corresponding to a parasitic mode, and no accidental harmonic. In this case,

$$\exp(\Gamma\tau) - 1 \approx \exp(-j\Omega\tau) - 1 \approx -2j\sin(\Omega\tau/2)\exp(-j\Omega\tau/2),$$

and the steady-state voltage takes the form

$$V_c(t) \approx \Re \exp\left[j\Omega\left(t - t_{bn}\right)\right] \hat{V}_{c0} \frac{j\exp^{(j\Omega\tau/2)}}{2\sin(\Omega\tau/2)}.$$

Thus roughly speaking, one loses the factor of Q_L/π enhancement and this is replaced with a factor of order unity. If one wishes to be more precise, one can compute

$$\eta = \left|\frac{1}{\exp(\Gamma\tau) - 1}\right| = \frac{1}{\sqrt{e^{\nu\tau} - 2e^{\nu\tau/2}\cos(\Omega\tau) + 1}} = \frac{e^{-\nu\tau/4}}{2^{1/2}} \frac{1}{\sqrt{\cosh(\nu\tau/2) - \cos(\Omega\tau)}}$$
$$\approx \frac{1}{2} \frac{1}{\sqrt{\sin^2(\Omega\tau/2) + (\nu\tau/4)^2}},$$

as a figure of merit. This is to be compared to Q_L/π with the design mode (both multiplied by their respective [*R/Q*] values). In fact, this would not be a bad thing to compute from a field-solver mode-listing. In that case, with Q_e not available, one would have to make a judicious choice---whether based on symmetry the mode of interest would be out-coupled or not, that is to say, whether $\nu \approx \Omega/Q_w$ or is much lower. So, in principle, one could have a poorly coupled (high Q_e) falling squarely on a beam-harmonic; however, in such a case, only a slight mistuning would take the mode off resonance.

Example Waveforms

The formulas provided make it quite simple to calculate waveforms, with almost no computational cost. Let us note a simple recursive algorithm, as it is the fastest. We evaluate our sum just after the passage of the *m*-th bunch, rearranging it a bit,

$$\begin{aligned}
\tilde{V}_c\left(t_{bm+}\right) &= e^{-\Gamma_\lambda\left(t_{bm+}-t_{b0}\right)}\sum_{n=0}^{m} e^{\Gamma_\lambda \tau_n}\hat{V}_{cn} = e^{-\Gamma_\lambda\left(t_{bm+}-t_{b0}\right)}\left\{e^{\Gamma_\lambda \tau_m}\hat{V}_{cm} + \sum_{n=0}^{m-1} e^{\Gamma_\lambda \tau_n}\hat{V}_{cn}\right\} \\
&= \hat{V}_{cm} + e^{-\Gamma_\lambda\left(t_{bm-1+}+\tau_m-t_{b0}\right)}\sum_{n=0}^{m-1} e^{\Gamma_\lambda \tau_n}\hat{V}_{cn} \\
&= \hat{V}_{cm} + e^{-\Gamma_\lambda \tau_m}\tilde{V}_c\left(t_{bm-1+}\right)
\end{aligned}$$

This series starts after the 0-th bunch, with $\tilde{V}_c\left(t_{b0+}\right) = \hat{V}_{c0}$. We can illustrate this with a few examples as in Figs. 2 (a)-(d), Fig. 3 (a)-(b), and Fig. 4. These figures depict the modulus of the complex amplitude. [in revision, replace with linac x-cavity plots, for illustration, as these particular plots are already incorporated in TN154]

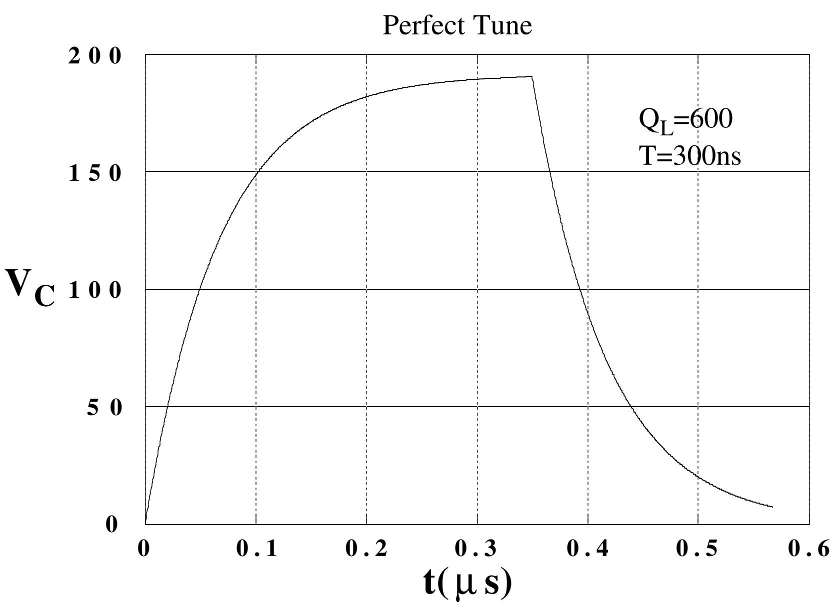

FIGURE 2. (a) Ideal waveform (in units where $2k_lQ_n$=1) for a 300 ns bunch train and a perfectly tuned cavity with Q_L=600. The vertical axis corresponds to $\left|\tilde{V}_c\right|$ in the notation of the text.

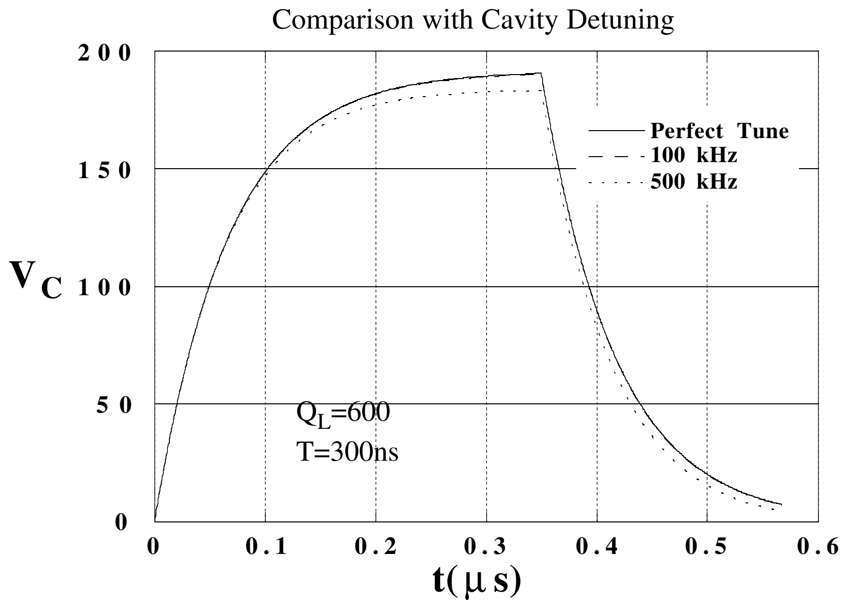

FIGURE 2. (b) Result of cavity detuning; as in (a): in units where $2k_lQ_n$=1, for a 300 ns bunch train and a perfectly tuned cavity with Q_L=600.

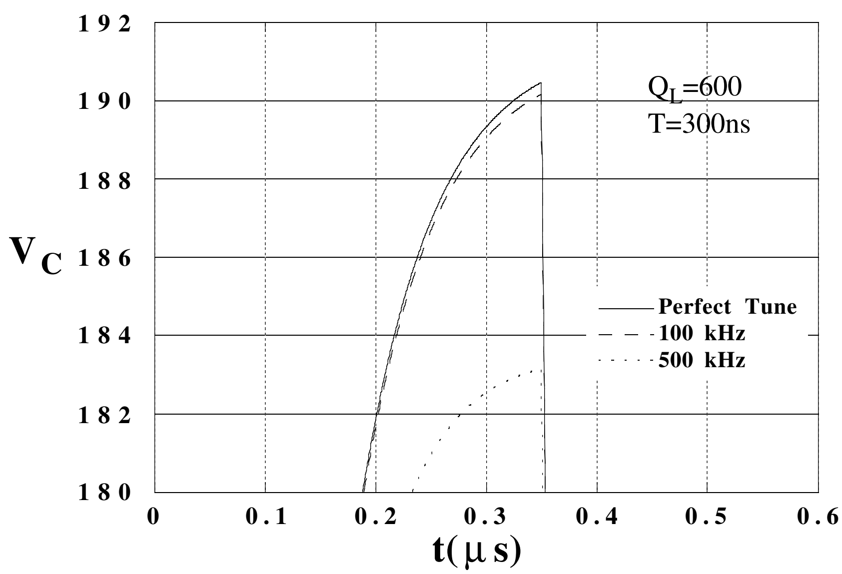

FIGURE 2. (c) A closeup of (b).

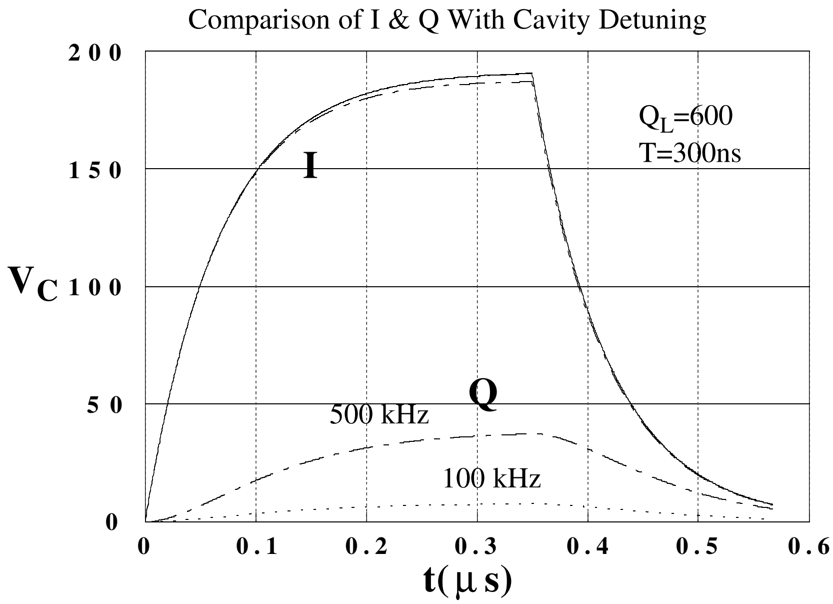

FIGURE 2. (d) A look at the I & Q waveforms corresponding to (b) and (c). The notation is $I = \Re\tilde{V}_c$, $Q = \Im\tilde{V}_c$.

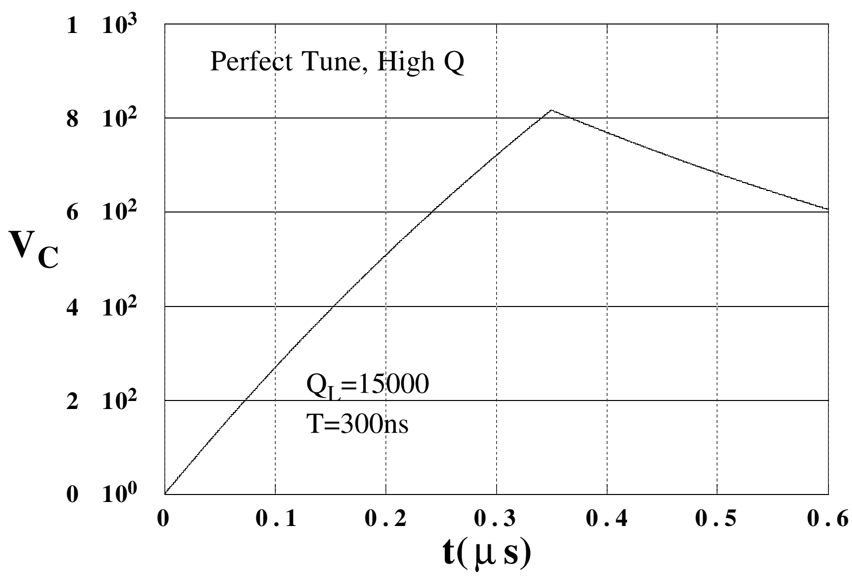

FIGURE 3. (a) Response of a high *Q*, perfectly tuned cavity.

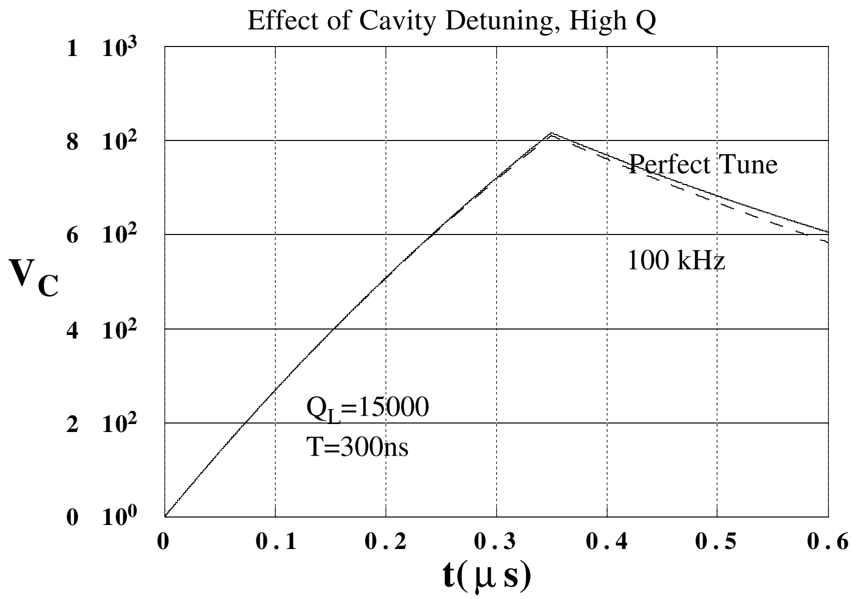

FIGURE 3. (b) Response of a high *Q*, cavity, for perfect tuning, and 100 kHz detuning.

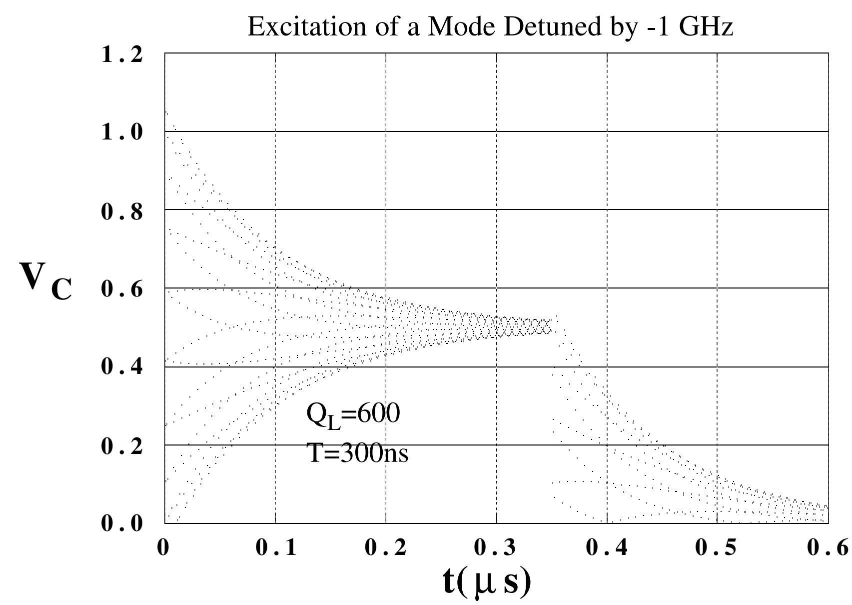

FIGURE 4. Response of a mode detuned by -1 GHz.

In the foregoing, we referred to effective charge, Q_{beff}; let us give the expression in terms of the pulse current $I_b = Q_b f_{RF}$, and let us consider some examples. The effective charge for a single bunch is the charge that has passed in one fill time

$$Q_{beff} = \frac{Q_L}{\pi} Q_b = I_b T_f,$$

in terms of which the steady-state amplitude is

$$\hat{V}_{eff} = \frac{Q_L}{\pi} \hat{V}_{c0} = \frac{Q_L}{\pi} 2k_{l\lambda} Q_b \langle X \rangle \exp\left(-\tfrac{1}{2} k_\lambda^2 \sigma_z^2\right) = 2k_{l\lambda} Q_{beff} \langle X \rangle \exp\left(-\tfrac{1}{2} k_\lambda^2 \sigma_z^2\right).$$

Monopole Mode

For the case of a monopole mode, we have

$$\frac{Q_L}{\pi} 2k_{l\lambda} Q_b = Q_L \left[\frac{R}{Q}\right] I_b = R_L I_b,$$

and

$$\hat{V}_{eff} = R_L I_b \exp\left(-\tfrac{1}{2} k_\lambda^2 \Sigma^2\right),$$

with $R_L = Q_L[R/Q]$ the loaded shunt impedance.

So for example, for the φ-cavity, we have $Q_{eff} \approx 382\, Q_b$ and thus may take over directly the expression given in the previous section,

$$10 \log_{10} P_w(\text{mW}) \approx 41.6 \text{ dBm} + 20 \log_{10} Q_{eff}(\text{nC}).$$

It is also helpful to have the expression in terms of current. Since the fill time is 133 ns, 1 nC of equivalent charge corresponds to 7.52 mA of real beam current. Thus

$$10 \log_{10} P_w(\text{mW}) \approx 24.1 \text{dBm} + 20 \log_{10} I_b(\text{mA}).$$

This provides a nominal 257 mW/mA^2 of sensitivity. The Blue Book quotes a value a factor of 250 mW/mA^2, as does the original technical note on the subject.[12] In passing we could note that the required sensitivity for the original application of the cavity BPM's was 200 mW/mA^2 . Nominal conditions were intended to be 300 mA corresponding to 17 kW; 50 mA would have correspond to about 500 W.

Dipole Mode

For the case of a predominantly dipole mode we have

$$\hat{V}'_{eff} = \frac{Q_L}{\pi} \hat{V}'_{c0} = 2k_{\perp\lambda} Q_{beff} x_b \exp\left(-\tfrac{1}{2} k_\lambda^2 \Sigma^2\right) = R_{\perp L} I_b x_b \exp\left(-\tfrac{1}{2} k_\lambda^2 \Sigma^2\right),$$

with $R_{\perp L} = Q_L[R_\perp / Q]$.

For the linac x-cavity, $Q_{eff} = 103.5 Q_b$ for $Q_L \approx 325$. ($Q_{eff} = 191 Q_b$, for $Q_L \approx 600$). This gives us

[12] O. Altenmueller and P. Brunet, "Some RF Characteristic of the Beam Phase Reference Cavity" SLAC TN-64-51, Sept. 1964 (available from the SLAC Library).

$$P_w \approx 5.73\times10^{-11}\,\mathrm{W}\left[I_b(\mathrm{mA})x_b(\mu\mathrm{m})\right]^2, \qquad \Leftrightarrow \qquad 57.3\mu\mathrm{W}/\mathrm{mA}^2\mathrm{mm}^2$$

or

$$10\log_{10} P_w(\mathrm{mW}) \approx -72.4\,\mathrm{dBm} + 20\log_{10}\left[I_b(\mathrm{mA})x_b(\mu\mathrm{m})\right].$$

This corresponds to 57% of the sensitivity quoted in the Blue Book, 100 μW/mA^{2}mm^2. Had we employed $Q_L \approx 600$, the result would agree to 5%. Brunet, *et al.*, list the "experimental sensitivity" 85 μW/mA^{2}mm^2 (their Fig. 12, p. 23).

WKB Approach

Before leaving the matter of bunch train excitation, let us note an alternate method of computing the transient cavity voltage in this case. Starting from our cavity excitation (we write it in generic form),

$$\left\{\frac{d^2}{dt^2} + \frac{\omega_0}{Q_L}\frac{d}{dt} + \omega_0^2\right\}V_c = \frac{1}{2}\omega_0\left[\frac{R}{Q}\right]\frac{dI_b}{dt},$$

we approximate our expression for beam current using the first harmonic component, $I_{b0} \approx \Re\tilde{I}_{b0}e^{j\omega t}$, and similarly represent the cavity voltage as $V_c \approx \Re\tilde{V}_c e^{j\omega t}$. We make the slowly varing envelope (WKB) approximation,

$$\left|\frac{d\tilde{I}_{b0}}{dt}\right| << \omega\left|\tilde{I}_{b0}\right|,$$

with the result

$$\frac{dV_c}{dt} = \Re\left(\frac{d\tilde{V}_c}{dt} + j\omega\tilde{V}_c\right)e^{j\omega t}, \qquad \frac{d^2V_c}{dt^2} \approx \Re\left(2j\omega\frac{d\tilde{V}_c}{dt} - \omega^2\tilde{V}_c\right)e^{j\omega t},$$

so that

$$2j\omega\frac{d\tilde{V}_c}{dt} + \left(\omega_0^2 - \omega^2 + j\frac{\omega\omega_0}{Q_L}\right)\tilde{V}_c \approx \frac{1}{2}j\omega\omega_0\left[\frac{R}{Q}\right]\tilde{I}_{b0}.$$

In terms of tuning angle, filling time, and loss factor,

$$\tan\psi = Q_L\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) \approx 2Q_L\left(\frac{\omega_0-\omega}{\omega_0}\right), \qquad T_f = \frac{2Q_L}{\omega_0}, \qquad k_l = \frac{1}{4}\omega_0\left[\frac{R}{Q}\right],$$

this is

$$\frac{d\tilde{V}_c}{dt}+\frac{1}{T_f}\left(1-j\tan\psi\right)\tilde{V}_c \approx k_l \tilde{I}_{b0}.$$

Abbreviating

$$\Delta=\frac{1}{T_f}\left(1-j\tan\psi\right),$$

we integrate this equation to find

$$\tilde{V}_c(t)e^{\Delta t}=\tilde{V}_c\left(t_0\right)e^{\Delta t_0}+k_l\int_{t_0}^{t} dt'\,\tilde{I}_{b0}\left(t'\right)e^{\Delta t'}.$$

For an initially unexcited cavity, and a beam current waveform arriving at t=0, we have

$$\tilde{V}_c(t)=k_l e^{-\Delta t}\int_0^t dt'\,\tilde{I}_{b0}\left(t'\right)e^{\Delta t'}.$$

In making connection with the formulation in terms of a bunch train, the identity

$$\sum_{n=-\infty}^{+\infty}Q_b\delta(t-nT)=\frac{Q_b}{T}\left\{1+2\sum_{m=0}^{\infty}\cos\left(2\pi m\frac{t}{T}\right)\right\},$$

is helpful. This implies that a train of bunches each of charge Q_b, spaced at an interval T, and therefore of average current $I_b=Q_b/T$, corresponds to a Fourier component, or rf current, $\tilde{I}_b=2I_b$.

With this result it is straightforward to make comparison with the results depicted in the foregoing figures. In fact let us consider a slightly more general current waveform,

$$\tilde{I}_{b0}(t)=2I_b\begin{cases}0\,; & t<0\\ e^{-\alpha t}\,; & 0<t<T_p\\ 0\,; & T_p<t\end{cases}$$

one with a bit of current variation over the pulse. Integrating, we find

$$\tilde{V}_c(t)=\frac{2k_l I_b}{(\Delta-\alpha)}\begin{cases}\left(e^{-\alpha t}-e^{-\Delta t}\right) & ; \quad 0<t<T_p\\ \left(e^{-\alpha T_p}-e^{-\Delta T_p}\right)e^{-\Delta\left(t-T_p\right)} & ; \quad T_p<t\end{cases}.$$

For the sake of comparison, total pulse charge at time t is

$$Q_{tot}(t) = \frac{1}{2}\int_0^t \tilde{I}_{b0}(t) = I_b \begin{cases} \left(\dfrac{1-e^{-\alpha t}}{\alpha}\right); & 0 < t < T_p \\ \left(\dfrac{1-e^{-\alpha T_p}}{\alpha}\right); & T_p < t \end{cases}$$

In this connection, let us note that in general one has the relation

$$\frac{1}{2}\int_{-\infty}^{+\infty} dt\, \tilde{I}_{b0} \approx \frac{1}{2k_l}\int_{-\infty}^{+\infty} dt \left(\frac{d\tilde{V}_c}{dt} + \frac{1}{T_f}(1 - j\tan\psi)\tilde{V}_c \right) \approx \frac{(1 - j\tan\psi)}{2k_l T_f}\int_{-\infty}^{+\infty} dt\, \tilde{V}_c \approx \frac{(1 - j\tan\psi)}{R_L}\int_{-\infty}^{+\infty} dt\, \tilde{V}_c,$$

so that the cavity phasor, integrated over the full pulse provides a measure of pulse charge. For a tuned cavity this reduces to the result obtained by superimposing the single bunch waveform integrals,

6. Common Mode

For application of a cavity as a pickup, the concern is resolution. One's first thought might be that thermal noise is a limit. This is far from the case, as a simple calculation will show. For comparison, at 112°F (44°C, 318°K) $k_B T \approx 27\,\mathrm{meV} \approx 4.4 \times 10^{-21}$ J. In a 1 GHz bandwidth one expects 4.4×10^{-12} W or -114 dBm. The natural full bandwidth of the x-cavity is $\Delta f = f_0 / Q_L \approx 10\,\mathrm{MHz}$ and in this bandwidth the noise power is 4.0×10^{-14} W or -134 dBm. With the latter number, a beam-induced signal will drop into the noise at $Q_{eff} x_b \approx 30\,\mathrm{pC-nm}$ for the linac x-cavity.[13] Practically speaking, signals induced in other cavity modes become an issue long before this theoretical noise level is reached.[14]

To reduce pickup from other modes, one can in principle employ symmetry, filtering, and gating. One concern with the asymmetric output coupling employed with the linac-type x-cavities is the inability of the pickup to distinguish, by symmetry, excitation of modes not varying linearly with position. The function of symmetric coupling is illustrated with the cartoon of Fig. 5.

For the position cavity, the concern is the TM110 mode. Unlike the position mode (TM210) this mode has non-zero reading with beam centered. We may refer to it as a "common mode". The matter of otuput-coupling for this mode in the actual cavity is illustrated in Fig. 6. One can see that a coupler to the dipole mode will also couple to the monopole mode.

[13] The figure is 2 pC for the φ–cavity and its 2 MHz bandwidth.

[14] Environmental noise may be an issue depending on the environment, and the shielding and cabling employed for the pickups and electronics.

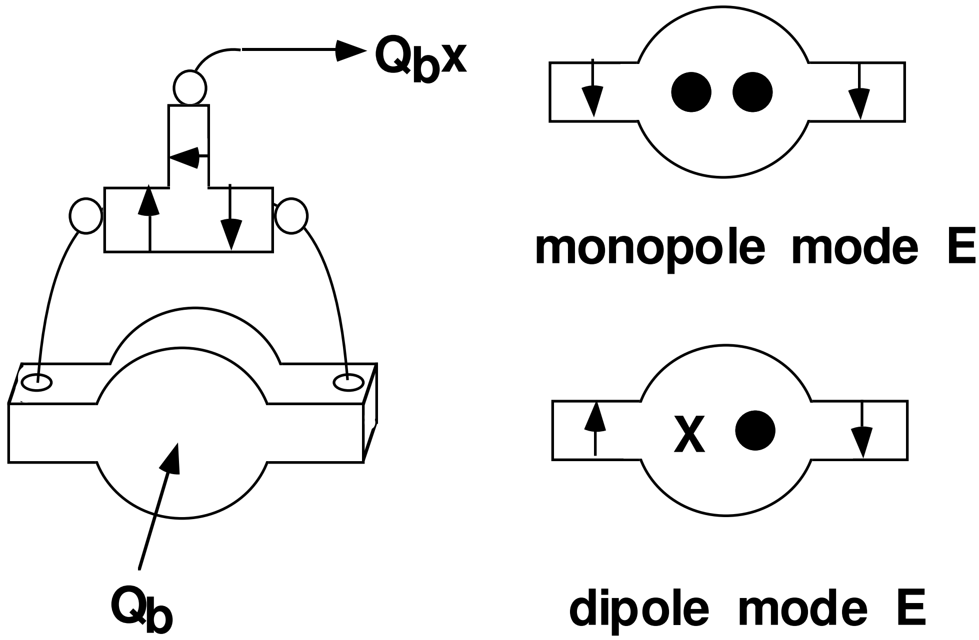

FIGURE 5. To illustrate that with symmetric couplers, a hybrid ("magic") tee, and carefully chosen guide lengths one can cancel modes of like symmetry. Isolation of 25 dB is not demanding.

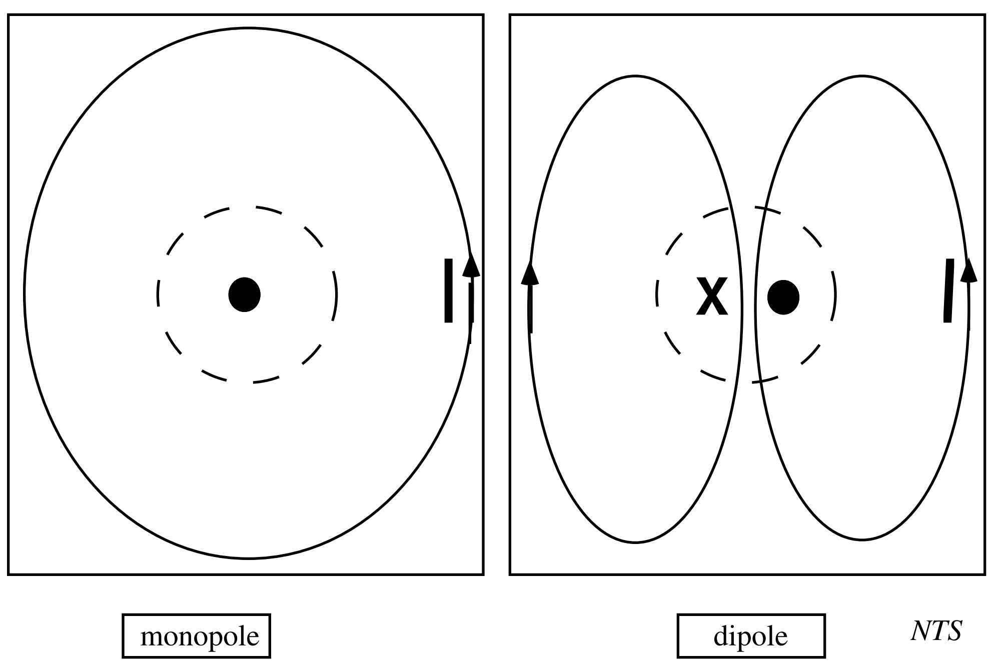

FIGURE 6. This cartoon illustrates loop coupling to the two modes. Electric field is indicated by the solid dots and the cross; magnetic field lines correspond to the circles; the beam aperture is indicated by the dotted circle. The coupling loop depicted by the solid bar, enclosed some flux from each mode. As a result, the voltage waveform induced on the output waveguide includes contributions from both modes.

The TM110 mode has a resonant frequency of $f_M = 1.8875\,\text{GHz}$, computed from the pillbox model. Skin-depth is $\delta \approx 1.6\ \mu\text{m}$, and theoretical wall Q is $Q_w \approx 1.66\times10^4$. Since the output coupling loop does not distinguish this mode from the dipole mode particularly, one may presume that the loaded Q for this mode is also rather low. For the meantime, we suppose $Q_L = 600$. We have $[R/Q] \approx 215\,\Omega$, and loss factor is $k_l \approx 0.64\,\text{V/pC}$. Shunt impedance (unloaded) is $R_s = 3.6\,\text{M}\Omega$. Because of the favorable transit angle, this mode would make a nice accelerating mode, strongly coupling to the beam, except that, happily, it is way off resonance with the beam. Even with the low $Q_L = 600$, the full bandwidth of the mode is 3.2 MHz, and it lies far from 2856 MHz. For single-bunch operation or tests of the cavity, the presence of this mode could greatly interfere with position detection, unless a notch filter or a high-pass filter were employed.

Let us indicate the limits on position resolution, due to this mode, in several different operating schemes.

Single-Bunch Mode

Single-bunch mode operation is interesting because this mode is available during SLC running for pre-E158 tests. One would like to have a reasonable expectation, then, of what tests are useful to perform.

Suppose in the simplest (and worst) case, that one performs power detection on the unfiltered output of the *x*-cavity signal. Assuming comparable guide-couplings for the modes, one may estimate the position resolution by equating the energy deposited in the dipole mode with that in the monopole mode $U_d = k_\perp (Q_b x_b)^2 = U_m = k_l Q_b^2$. Using $k_l \approx 0.64\,\text{V/pC}$ and $k_\perp \approx 8.07\times10^{-2}\ V/\text{pC}-\text{cm}^2$, one can see that these are equal at the (unphysical) value $x_b \approx 2.8\,\text{cm}$!

Next, let's consider filtering followed by power detection. We suppose a very sharply cut-off high-pass filter (a piece of cut-off waveguide, for example). In this case, we are samping the high-frequency tail of a Lorentzian response function that we may compute by putting our beam-cavity equation back in the frequency domain,

$$\tilde{V}_c(\omega) = \frac{2 j \omega_\lambda \omega k_l}{\omega_\lambda^2 - \omega^2 + j\omega_\lambda \omega / Q_{L\lambda}} \tilde{I}_b(\omega).$$

The power spectrum then takes the form

$$P_\lambda(\omega) = \frac{\omega_\lambda}{Q_{e\lambda}} \frac{\left|\tilde{V}_c(\omega)\right|^2}{\omega_\lambda [R/Q]} = \frac{(\omega_\lambda \omega)^2}{\left(\omega_\lambda^2 - \omega^2\right)^2 + \left(\omega_\lambda \omega / Q_{L\lambda}\right)^2} \frac{\left(k_l Q_b^2\right)}{Q_{e\lambda}} e^{-\omega^2 \sigma_t^2}.$$

Thus the response of a $1888\,\text{MHz}$ with $Q_L \approx 600$, evaluated at 2856 MHz is on the order of

$$P_\lambda(\omega) \approx \frac{\left(k_l Q_b^2\right)/Q_{e\lambda}}{\left(\omega_\lambda/\omega - \omega/\omega_\lambda\right)^2} e^{-\omega^2\sigma_t^2} \approx \left(k_l Q_b^2\right)/Q_{e\lambda},$$

whereas the dipole mode spectrum, on-resonance is

$$P_\lambda(\omega) \approx \frac{Q_{L\lambda}^2}{Q_{e\lambda}} \left(k_\perp Q_b^2 x_b^2\right).$$

In other words, the "tail" of the Lorentzian line profile, evaluated off-resonance but at a not-much different frequency, is down by a factor of Q_L^2. So, for our example, and assuming a sharp enough rolloff on our filter (so that we suffer only the tail of the adjacent mode) we could contemplate resolving offsets on the order of $x_b \approx 2.8\,\text{cm}/600 \approx O(50\,\mu\text{m})$.

If in addition, by means of symmetric output coupling we were able to subtract out the symmetric mode, so that the voltage picked up took the form $V_d + \varepsilon V_m$, then our position resolution would be $x_b \approx O(50\,\mu\text{m})\varepsilon$. With a hybrid ("magic") tee, 25 dB of isolation would be considered not hard, and this corresponds to $\varepsilon \approx 6\times10^{-2}$, and $x_b \approx O(3\,\mu\text{m})$. None of these considerations consulted the charge of the bunch; they do assume however that it is a single bunch, and not a bunch train.[15]

Multi-Bunch Mode

Let us consider next the situation for a bunched beam driving such a mode off resonance. This is the case of operational interest. Excitation of a mode off-resonance has been calculated in a previous section, and the relevant figure of merit is

$$\eta \approx \frac{1}{2}\frac{1}{\sqrt{\sin^2(\pi\Omega/\Omega_0) + (\pi\Omega/2\Omega_0 Q_L)^2}},$$

for a mode with (damping-corrected) resonant angular frequency Ω_0, and loaded Q, Q_L, driven by a beam bunched at angular frequency Ω_0. This is to be compared to Q_{L0}/π for the design mode.The argument of the sine function is 119°, so that $\eta \approx 0.6$. This is to say that destructive interference is such that the steady-state amplitude in the common mode is equivalent to the amplitude produced by one single bunch with $0.6\times$ the nominal single bunch charge. Meanwhile, for the design mode $Q_{L0}/\pi \approx 100$. In this case, to compare common-mode and dipole-mode signals, we should compare $k_\perp(Q_{beff}x_b)^2$ and $k_l Q_b^2 \eta^2$, and we obtain $x_b \approx O(2\times10^2\,\mu\text{m})$.

Now, however, filtering doesn't help, since the signal we are considering is a driven oscillation at the beam frequency. With symmetrized output coupling, and a tee with isolation of 25 dB, we can bring this estimate to $x_b \approx O(10\,\mu\text{m})$. If critical coupling had been employed ($Q_e = Q_w = 2Q_L$) we would have $Q_L \approx 7.5\times10^3$, corresponding to $Q_{beff} = 2.4\times10^3 Q_b$, and resolution approaching $x_b \approx O(10\,\mu\text{m})$, without symmetric coupling; in this case, with symmetric coupling and 25 dB of isolation we would have $x_b \approx O(0.5\,\mu\text{m})$. Note that reducing the coupling, and raising the fill time, will reduce temporal resolution.

Phase & Amplitude

The foregoing discussion was based on comparison of signal power levels; this generally leads to somewhat too pessimistic an answer. The mere presence of an offset in a signal does not imply that one cannot detect small changes in the total signal. In addition, however, more is available to us in the way of information than just the amplitude of the emitted signal. We can employ phase-information. In fact, it is possible in principle to perform vector subtraction to remove the common mode. To illustrate this in the simplest way, let us compute the "baseband integral" of the down-mixed waveform corresponding to the common mode. Returning to the case

[15] We could add, in passing, that an alternative to filtering employed by Shintake [ref], consists of designing a cavity a heavily de-Q'd common mode, and gating the output waveform well after the evanescence of the monopole mode.

of a single bunch for which the induced voltage may be expressed as

$$V_c(t) = \Re \exp\left[j\Omega_c(t-T)\right]\tilde{V}_c(t),$$

where Ω_c is the angular frequency of the common mode. The phasor

$$\tilde{V}_c(t) = H(t-T)\exp\left\{-\tfrac{1}{2}\nu(t-T)\right\}\hat{V}_c,$$

and $\hat{V}_c \approx 2k_l Q_b$. We neglect the form factor correction for this term, as the term itself represents already a small effect. We suppose that this signal is mixed with an LO, phased with the beam, at frequency Ω. In this case the integral of the downmixed waveform gives

$$\int_{-\infty}^{+\infty} dt\,\tilde{V}_c(t)e^{j\Delta\Omega(t-T)} = 2k_l Q_b \int_{-\infty}^{+\infty} dt'\,e^{(j\Delta\Omega-\nu/2)t'}H(t') = \frac{2k_l Q_b}{\nu/2 - j\Delta\Omega} = \frac{R_L Q_b}{1 - jT_f\Delta\Omega},$$

with $\Delta\Omega = \Omega_c - \Omega$. Taking $\Omega_c \approx 2\pi \times 1.89\ \text{GHz}$ and $\Omega \approx 2\pi \times 2.856\ \text{GHz}$, with $T_f \approx 100\ \text{ns}$, one has $T_f\Delta\Omega \approx 6.1\times 10^2$, in agreement with the foregoing estimates.

The feature to note about this result is that the phase of this signal is fixed with respect to the beam, and independent of offset. Thus if a reliable phase-reference is available upstairs (for example the phase-cavity signal itself), the signal can be balanced out in a tee. The idea fails if for any reason the common mode phase or amplitude drift appreciably with respect to the reference signal. A sketch of the idea is seen in Fig. 7. The idea relies on the assumption that the driven oscillation in the common mode as seen upstairs has an amplitude and phase fixed in relation to the monopole mode of the phase cavity. Assuming the same cable runs, and equal cable lengths, this may be achievable in a diurnally-reliable fashion. In the case that consistent monitoring is required only over the scale only of minutes it seems even more promising. The phase-shifter and attenuator would require initial adjustment (frequent adjustment if diurnal issues appear). The shortest procedure for adjustment would be to adjust the phase-shifter and attenuator to zero the output of the tee. Maximum sensitivity to position would then be available at whatever the current location of the beam. One could devise an iterative procedure to center the beam accurately (as opposed to monitoring changes accurately) and we leave this to a later note. This idea requires further analysis; for example, of the character of cancellation achievable with transient waveforms from modes with different time constants. It appears that a concept such as this could be tested using the LI30 BPM.

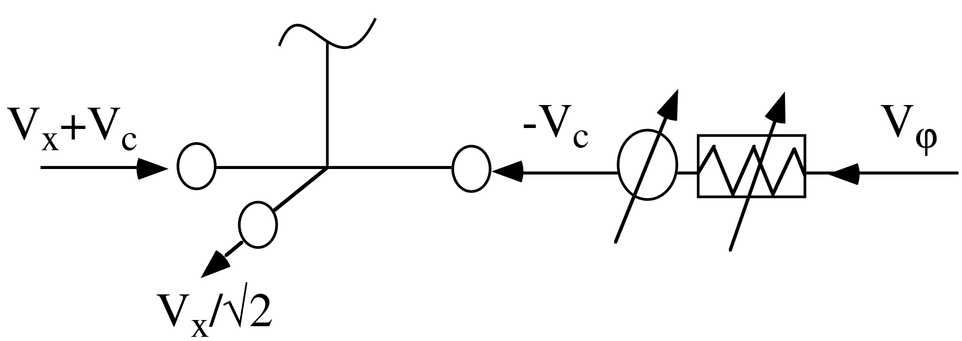

FIGURE 7. In principle one may subtract out a steady common mode signal, by making use of the phase-cavity signal and a magic tee.

Other Modes

In the BPM cavity there is also a y-sensitive mode, the TM120 mode, with fields given by wall Q is $Q_w \approx 1.82\times10^4$ Loss factor is $k_\perp \approx 1.17\times10^{-1}\ V/\mathrm{pC-cm}^2$, about 40% larger than for the TM210 mode, due to the fact that the y-dimension of the cavity is smaller. Given the expected poor external coupling of this mode, we may suppose that the loaded Q of the mode is just the wall Q. The natural field decrement time of the mode is then $T_0 = 2Q_w/\omega \approx 1.88\,\mu\mathrm{s}$, and one may expect a voltage below that for resonant response by a factor of $\Delta\Omega T_0 \approx 2\pi\times232\,\mathrm{MHz}\times1.88\,\mu\mathrm{s} \approx 2.7\times10^3$. This together with the absence of good coupling suggests that this mode is of no particular concern.

7. Summary

In this note we have formulated the problem of cavity excitation and output coupling for an arbitrary isolated mode (non-degenerate) of an asymmetric cavity. Corrections due to higher multipole content have been quantified. Single-bunch and multibunch (resonant & nonresonant) operation have been treated. Examples have been given relevant to the linac-cavity BPM system. A broader view of this subject can be found in TN 134

Appendix: Review of Beam Interaction with a Cavity

This Appendix is excerpted from Appendix C of "Introduction to Electrodynamics for Microwave Linear Accelerators", D. Whittum, submitted to *Proceedings of the Joint US-CERN-Japan School, RF Engineering for Particle Accelerators* (World Scientific, to be published). It is intended as background. The complete article can be found at the ARDB Tech. Note web site.

The development of the theory of the accelerator cavity in Sec. 3 proceeded from an isolated resonance of a closed, lossless cavity, and considered perturbations: a beam, Eq. (3.5), wall losses, Eq. (3.6), a waveguide, Eq. (3.12), and iris-coupling to another cavity, Eq. (3.26). Since these results are central to the development of linac theory it is useful to set down a more formal treatment of these perturbations.

We consider fields and sources of the form

$$\vec{E}(\vec{r},t) = \Re\left(\tilde{E}(\vec{r},\omega)e^{j\omega t}\right),\ \vec{H}(\vec{r},t) = \Re\left(\tilde{H}(\vec{r},\omega)e^{j\omega t}\right),$$
$$\vec{J}(\vec{r},t) = \Re\left(\tilde{J}(\vec{r},\omega)e^{j\omega t}\right),\ \rho(\vec{r},t) = \Re\left(\tilde{\rho}(\vec{r},\omega)e^{j\omega t}\right),$$

and the charge density is subject to continuity,

$$\tilde{\rho}(\vec{r},\omega) = -\frac{1}{j\omega}\vec{\nabla}\bullet\tilde{J}(\vec{r},\omega).$$

Maxwell's equations reduce to the form

$$\vec{\nabla}\times\tilde{E} = -j\omega\mu\tilde{H}, \qquad \vec{\nabla}\times\tilde{H} = j\omega\varepsilon\tilde{E} + \tilde{J}.$$

Gauss's law is satisfied identically, as one can check by taking the divergence. We consider a closed cavity with perfectly conducting walls in the absence of source terms. Boundary conditions are $\hat{n}\times\tilde{E} = 0$, $\hat{n}\bullet\tilde{H} = 0$, and the fields each separately satisfy the Helmholtz equation,

$$\left(\nabla^2 + k^2\right)\tilde{E} = 0,\ \left(\nabla^2 + k^2\right)\tilde{H} = 0,$$

where $k^2 = \omega^2\mu\varepsilon$. These amount to eigenvalue equations for the mode frequencies. Let us enumerate the modes and label them with index λ. The eigenvalues we denote k_λ, and the mode frequencies, ω_λ. We select basis functions $\tilde{E}_\lambda$ to be real, and adopt the normalization

$$\int d^3\vec{r}\tilde{E}_\lambda\bullet\tilde{E}_{\lambda'} = \delta_{\lambda,\lambda'}.$$

To demonstrate this for $\lambda\neq\lambda'$, one employs, in the case of non-degenerate modes, the Helmholtz equation, and conducting boundary conditions:

$$\begin{aligned}\left(k_\lambda^2 - k_{\lambda'}^2\right)\int d^3\vec{r}\,\tilde{E}_\lambda \bullet \tilde{E}_{\lambda'} &= \int d^3\vec{r}\left(\tilde{E}_\lambda \nabla^2 \tilde{E}_{\lambda'} - \tilde{E}_{\lambda'} \nabla^2 \tilde{E}_\lambda\right) \\ &= \int d^3\vec{r}\,\vec{\nabla} \bullet \left\{\tilde{E}_\lambda \times \left(\vec{\nabla} \times \tilde{E}_{\lambda'}\right) - \tilde{E}_{\lambda'} \times \left(\vec{\nabla} \times \tilde{E}_\lambda\right)\right\} \\ &= \int d\vec{S} \bullet \left\{\tilde{E}_\lambda \times \left(\vec{\nabla} \times \tilde{E}_{\lambda'}\right) - \tilde{E}_{\lambda'} \times \left(\vec{\nabla} \times \tilde{E}_\lambda\right)\right\} \\ &= 0.\end{aligned}$$

In the case of degenerate modes, we assume Gram-Schmidt orthogonalization has been employed. We define the basis functions for the magnetic field according to

$$\vec{\nabla} \times \tilde{H}_\lambda = k_\lambda \tilde{E}_\lambda, \qquad \vec{\nabla} \times \tilde{E}_\lambda = k_\lambda \tilde{H}_\lambda,$$

and one can show that these too satisfy orthonormality. For this idealized system, a general *vacuum* oscillation may be expressed as

$$\vec{E}(\vec{r},t) = \sum_\lambda \tilde{E}_\lambda(\vec{r})\tilde{e}_\lambda(t), \qquad \vec{H}(\vec{r},t) = \sum_\lambda \tilde{H}_\lambda(\vec{r})\tilde{h}_\lambda(t),$$

where the time evolution is given by Faraday's law in the time-domain,

$$Z_0 \frac{\partial h_\lambda}{\partial t} = -\omega_\lambda e_\lambda,$$

and Ampere's law,

$$\frac{\partial e_\lambda}{\partial t} = \omega_\lambda Z_0 h_\lambda,$$

corresponding to an undamped simple harmonic oscillation in each mode,

$$\left(\frac{\partial^2}{\partial t^2} + \omega_\lambda^2\right) e_\lambda = 0.$$

On average, equal amounts of energy are stored in the electric and magnetic fields.

Next let us consider a weakly perturbed cavity, one for which the original basis functions are no longer exact solutions. Let the fields in steady-state at angular frequency ω take the form $\tilde{E}(\vec{r},\omega), \tilde{H}(\vec{r},\omega)$. We define mode amplitudes for the perturbed problem according to

$$\tilde{e}_\lambda(\omega) = \int d^3\vec{r}\,\tilde{E}(\vec{r},\omega) \bullet \tilde{E}_\lambda(\vec{r}), \qquad \tilde{h}_\lambda(\omega) = \int d^3\vec{r}\,\tilde{H}(\vec{r},\omega) \bullet \tilde{H}_\lambda(\vec{r}),$$

with integrals over the unperturbed cavity volume. To obtain equations for these mode amplitudes, we take the dot product of Faraday's law with $\tilde{H}_\lambda$, and we take the dot product of Ampere's law with $\tilde{E}_\lambda$, and we integrate each over the volume. Faraday's law takes the form

$$
\begin{aligned}
-j\mu\omega\tilde{h}_\lambda &= \int d^3\vec{r}\left(\vec{\nabla}\times\tilde{E}\right)\bullet\tilde{H}_\lambda = \int d^3\vec{r}\left\{\vec{\nabla}\bullet\left(\tilde{E}\times\tilde{H}_\lambda\right)+\tilde{E}\bullet\vec{\nabla}\times\tilde{H}_\lambda\right\} \\
&= \int d\vec{S}\bullet\tilde{E}\times\tilde{H}_\lambda + \int d^3\vec{r}\,\tilde{E}\bullet\left(k_\lambda\tilde{E}_\lambda\right) = k_\lambda\tilde{e}_\lambda + \int d\vec{S}\bullet\vec{\tilde{E}}\times\tilde{H}_\lambda .
\end{aligned}
$$

and Ampere's law takes the form

$$
\begin{aligned}
j\omega\varepsilon\tilde{e}_\lambda &= \int d^3\vec{r}\left(\vec{\nabla}\times\tilde{H}\right)\bullet\tilde{E}_\lambda - \int d^3\vec{r}\,\tilde{J}\bullet\tilde{E}_\lambda \\
&= \int d^3\vec{r}\left\{\tilde{H}\bullet\left(\vec{\nabla}\times\tilde{E}_\lambda\right)-\vec{\nabla}\bullet\left(\tilde{H}\times\tilde{E}_\lambda\right)\right\} - \int d^3\vec{r}\,\tilde{J}\bullet\tilde{E}_\lambda \\
&= k_\lambda\tilde{h}_\lambda + \int d\vec{S}\bullet\left(\tilde{E}_\lambda\times\vec{\tilde{H}}\right) - \int d^3\vec{r}\,\vec{\tilde{J}}\bullet\tilde{E}_\lambda .
\end{aligned}
$$

Abbreviating,

$$
\tilde{J}_\lambda = \int d^3\vec{r}\,\tilde{J}\bullet\tilde{E}_\lambda ,
$$

these are just

$$
-j\mu\omega\tilde{h}_\lambda = k_\lambda\tilde{e}_\lambda + \int d\vec{S}\bullet\tilde{E}\times\tilde{H}_\lambda , \tag{A.1}
$$

$$
j\varepsilon\omega\tilde{e}_\lambda = k_\lambda\tilde{h}_\lambda + \int d\vec{S}\bullet\tilde{E}_\lambda\times\tilde{H} - \tilde{J}_\lambda . \tag{A.2}
$$

The integrals are over the unperturbed cavity boundary. We consider next different kinds of perturbations.

A.1 Perturbation Due to Lossy Walls

With the addition of finite conductivity to the cavity surface, the boundary conditions on the physical fields (as opposed to our modes λ that were derived for different boundary conditions) are amended to read $\tilde{E} = -Z_s\hat{n}\times\tilde{H}$, where recall that $Z_s = R_s\left(1 + j\,\mathrm{sgn}\,\omega\right)$, with R_s the surface resistance. This permits us to compute the surface integrals. We have

$$
\begin{aligned}
\int_{S_w} d\vec{S}\bullet\tilde{E}\times\tilde{H}_\lambda &= \int_{S_w} dS\,\hat{n}\bullet\left(-Z_s\hat{n}\times\tilde{H}\right)\times\tilde{H}_\lambda = \int_{S_w} dS\,Z_s\hat{n}\bullet\left\{\left(\tilde{H}\bullet\tilde{H}_\lambda\right)\hat{n}-\left(\hat{n}\bullet\tilde{H}_\lambda\right)\tilde{H}\right\} \\
&= Z_s\int_{S_w} dS\,\tilde{H}\bullet\tilde{H}_\lambda ,
\end{aligned}
$$

where use is made of conducting boundary conditions on the unperturbed modal pattern, $\hat{n}\bullet\tilde{H}_\lambda = 0$; likewise, $\hat{n}\times\tilde{E}_\lambda = 0$, so that

$$
\int_{S_w} d\vec{S}\bullet\left(\tilde{E}_\lambda\times\tilde{H}\right) = \int_{S_w} dS\,\hat{n}\bullet\left(\tilde{E}_\lambda\times\tilde{H}\right) = \int_{S_w} dS\,\tilde{H}\bullet\left(\hat{n}\times\tilde{E}_\lambda\right) = 0 .
$$

To simplify the wall loss term, we make the approximation

$$\int_{S_w} dS\, \tilde{H} \bullet \tilde{H}_\lambda \approx h_\lambda \int_{S_w} dS\, \tilde{H}_\lambda^2,$$

thereby discarding any *mode coupling* through the lossy wall currents. This is appropriate, for example, for an isolated resonance, one that does not overlap another mode in the frequency domain. In terms of the *wall quality factor*, $Q_{w\lambda}$,

$$\frac{1}{Q_{w\lambda}} = \frac{\delta}{2} \int_{S_w} dS\, \tilde{H}_\lambda^2 = \frac{\delta}{2} \frac{\int_{wall} dS\, \tilde{H}_\lambda^2}{\int_{volume} dV\, \tilde{H}_\lambda^2}. \tag{A.3}$$

we may write

$$\int_{S_w} d\vec{S} \bullet \tilde{E} \times \tilde{H}_\lambda = Z_s \int_{S_w} dS\, \tilde{H} \bullet \tilde{H}_\lambda = \frac{\mu}{Q_w}(1 + j\,\mathrm{sgn}\,\omega)|\omega| h_\lambda .$$

Putting the results together, we have

$$j\varepsilon\omega\tilde{e}_\lambda = k_\lambda \tilde{h}_\lambda - \tilde{J}_\lambda, \qquad -j\mu\omega\tilde{h}_\lambda = k_\lambda \tilde{e}_\lambda + \frac{\mu}{Q_w}(1 + j\,\mathrm{sgn}\,\omega)|\omega| h_\lambda . \tag{A.4}$$

We consider a weak perturbation, $Q_w >> 1$, neglecting terms which are second-order in perturbations. In addition, we approximate $|\omega| \approx \omega_\lambda$ in the term of order $1/Q_w$. The effect of this approximation must be slight, since $Q_w >> 1$. Note that wall-losses have produced a shift in the real part of the cavity frequency, we subsume that into a *redefinition* of ω_λ,

$$\frac{\omega_\lambda^2}{1 + 1/Q_w} \to \omega_\lambda^2. \tag{A.5}$$

We arrive at

$$\left\{\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right\}\tilde{e}_\lambda = -j\omega\frac{1}{\varepsilon}\tilde{J}_\lambda,$$

or, in the time-domain,

$$\left(\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_w}\frac{d}{dt} + \omega_\lambda^2\right)e_\lambda = \frac{1}{\varepsilon}\frac{d}{dt}J_\lambda. \tag{A.6}$$

A.2 Perturbation Due to Beam

Next we evalute the perturbation due to the current source in Eq. (3.6). For a ballistic "pencil" beam, the current density is

$$\vec{J} = \hat{z}\delta^2(\vec{r}_\perp)I_b\left(t - \frac{z}{V}\right), \tag{A.7}$$

where I_b is the beam current waveform. In the frequency domain,

$$\tilde{J}(\vec{r},\omega) = \hat{z}\delta^2(\vec{r}_\perp)\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}e^{-j\omega t}I_b\left(t - \frac{z}{V}\right) = \hat{z}\delta^2(\vec{r}_\perp)\tilde{I}_b(\omega)e^{-j\omega z/V}.$$

With this we may compute the source term $\tilde{J}_\lambda$,

$$\begin{aligned}\tilde{J}_\lambda &= \int d^3\vec{r}\,\tilde{J}\bullet\tilde{E}_\lambda = \int d^3\vec{r}\left(\hat{z}\delta^2(\vec{r}_\perp)\tilde{I}_b(\omega)e^{-j\omega z/V}\right)\bullet\tilde{E}_\lambda \\ &= \tilde{I}_b(\omega)\int_{cavity} dz\,E_{z\lambda}(\vec{r}_\perp = 0, z)e^{-j\omega z/V} = \tilde{I}_b(\omega)w^*,\end{aligned}$$

and we abbreviate,

$$\tilde{w} = \int_{cavity} dz\,E_{z\lambda}(\vec{r}_\perp = 0, z)e^{j\omega z/v}.$$

This coefficient we may relate to the more conventional quantity, [*R/Q*], as follows. The voltage drop (or gain) experienced by a particle travelling at speed V in the z-direction, and passing z=0 at time t=t_0 is

$$V_c(t_0) = \int_{cavity} dz\,E_z\left(\vec{r}_\perp = 0, z, t_0 + \frac{z}{V}\right). \tag{A.8}$$

We may express this in terms of the modal decomposition,

$$E_z\left(\vec{r}_\perp = 0, z, t_0 + \frac{z}{V}\right) = \sum_\lambda E_{z\lambda}(\vec{r}_\perp = 0, z)e_\lambda\left(t_0 + \frac{z}{V}\right),$$

and in the frequency domain, as

$$\begin{aligned}\tilde{V}_c(\omega) &= \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}e^{-j\omega t_0}\tilde{V}_c(t_0) = \int_{cavity} dz\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}e^{-j\omega t_0}\,e_\lambda\left(t_0 + \frac{z}{V}\right)E_{z\lambda}(\vec{r}_\perp = 0, z) \\ &= \tilde{e}_\lambda(\omega)\int_{cavity} dz\,E_{z\lambda}(\vec{r}_\perp = 0, z)e^{j\omega z/V},\end{aligned}$$

and we assume for simplicity that only a single mode, λ, is excited. Evidently then $\tilde{V}_c = \tilde{e}_\lambda\tilde{w}$, and our cavity equation, Eq. (A.5), may be re-expressed in terms of the more meaningful normalizations, $\tilde{V}_c$, $\tilde{I}_b$, according to

$$\left\{\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right\}\underbrace{\tilde{V}_c}_{w\tilde{e}_\lambda} = -\frac{j\omega}{\varepsilon}\underbrace{\tilde{I}_b|w|^2}_{w\tilde{J}_\lambda},$$

or

$$\left\{\omega^2 - j\frac{\omega\omega_\lambda}{Q_w} - \omega_\lambda^2\right\}\tilde{V}_c = -j\omega\omega_\lambda\left[\frac{r}{Q}\right]\tilde{I}_b. \tag{A.9}$$

Here we introduce

$$\left[\frac{r}{Q}\right] = \frac{|w|^2}{\varepsilon_0\omega_\lambda}.$$

The $[r/Q]$ for the coupling of the beam to this mode λ may be expressed in a normalization-independent manner as

$$\left[\frac{r}{Q}\right] = \frac{\left|\int_{cavity} dz\, E_{z\lambda}(\vec{r}_\perp = 0, z)e^{j\omega_\lambda z/V}\right|^2}{\varepsilon_0\omega_\lambda \int_{cavity} d^3\vec{r}\,\vec{E}_\lambda \bullet \vec{E}_\lambda} = \frac{\left|\tilde{V}_c\right|^2}{2\omega_\lambda U}, \tag{A.10}$$

where U is the stored energy in the cavity corresponding to cavity voltage $\tilde{V}_c$. To summarize, in the time-domain,

$$\left(\frac{d^2}{dt^2} + \frac{\omega_\lambda}{Q_w}\frac{d}{dt} + \omega_\lambda^2\right)V_c = \omega_\lambda\left[\frac{r}{Q}\right]\frac{dI_b}{dt}. \tag{A.11}$$

Let us note explicitly the relation

$$\left[\frac{R}{Q}\right] = 2\left[\frac{r}{Q}\right],$$

between circuit and accelerator notations. In these notes we have consistently distinguished between these two using upper-case and lower-case notations. In reading the literature it pays to check the definition being employed.

A.3 Cell-to-Cell Coupling

Since we are working to first order in perturbations, there is no impediment to considering each perturbation in isolation, later to return and add them all together to describe the complete system. Accordingly, let us return to the closed lossless cavity, and consider perturbation by iris-coupling to another cavity. The evolution of the mode amplitudes is governed by Faraday's law,

$$-\mu_0 \frac{\partial h_\lambda}{\partial t} = k_\lambda e_\lambda + \int_{iris} \left(\hat{n} \times \vec{E}\right) \bullet \vec{H}_\lambda \, dA,$$

and Ampere's law,

$$\varepsilon_0 \frac{\partial e_\lambda}{\partial t} = k_\lambda h_\lambda - J_\lambda + \int_{iris} \left(\hat{n} \times \vec{E}_\lambda\right) \bullet \vec{H} \, dA = k_\lambda h_\lambda.$$

In the last line we have set the current term to zero and taken account of the conducting boundary conditions satisifed by the unperturbed mode electric field. Combining these two results we obtain a single equation for the electric field amplitude,

$$\frac{\partial^2 e_\lambda}{\partial t^2} + \omega_\lambda^2 e_\lambda = -\omega_\lambda c \int_{iris} \left(\hat{n} \times \vec{E}\right) \bullet \vec{H}_\lambda \, dA. \tag{A.12}$$

Evidently the effect of the iris is known once the variation of the tangential electric field in the vicinity of the iris is known, and the integral computed. Note that if we take the variation in electric field near the iris to be just the unperturbed variation, then the tangential field is zero, and so is the integral. Evidently then the important feature of the iris is the redistribution of electric field lines, some passing into the next cavity, but some terminating on the iris edge.

To compute this variation we should in principle solve Maxwell's equations for the new geometry. Happily however, for a *small* iris, the Helmholtz equation for the potentials reduces simply to Laplace's equation $\left(\nabla^2 + k^2\right)\varphi = 0 \approx \nabla^2 \varphi$. This approximation is adequate because, near the iris, spatial variations occur on the length-scale of the iris. The rf, meanwhile, is incapable of resolving details much smaller than a wavelength. Thus our problem is considerably simplified. One has a circular aperture in a plane, and far from the aperture one has normal fields above and below, call them E_1 and E_2. One intuits that if $E_1 \neq E_2$, some field lines will terminate on the plane, and some tangential field will result. Following Jackon's derivation,[17] we simply write down the solution for this tangential field in the iris,

$$\vec{E}_\perp(\vec{r}, z=0) = \frac{\left(E_1 - E_2\right)}{\pi} \frac{\vec{r}}{\sqrt{a^2 - r^2}},$$

where z=0 locates the plane of the iris, $\vec{r}$ is the coordinate in the plane of the iris, a is the iris radius, E_1 is the asymptotic field to the left of the iris (in cavity #1), and E_2 is the asymptote to the right of the iris (in cavity #2). The basis functions and normalization for the unperturbed TM_{010} mode are

$$\vec{E}_\lambda = J_0\left(\beta_c r\right) E_{0\lambda} \hat{z}, \qquad \vec{H}_\lambda = -J_0'\left(\beta_c r\right) E_{0\lambda} \hat{\phi}, \quad E_{0\lambda} = \left(\pi L R^2 J_1^2\left(j_{01}\right)\right)^{-1/2}.$$

The asymptotic field components are then $E_1 \approx e_1 E_{0\lambda}$, $E_2 \approx e_2 E_{0\lambda}$, and we amend our notation, denoting by e_1 the excitation of the fundamental mode in cavity #1, and e_2 that for cavity #2. (There is no ambiguity here, since we are not considering other modes λ, to which previously the subscripts on the e's referred.). With this it remains only to compute the port integral appearing in Eq. (A.12),

$$\int_{port} \left(\hat{n} \times \vec{E}\right) \bullet \vec{H}_\lambda \, dA \approx \int_0^{2\pi} d\phi \int_0^a r dr \frac{(e_1 - e_0)}{\pi} \frac{J_0'(\beta_c r)}{\sqrt{a^2 - r^2}} r E_{0\lambda}^2$$

$$\approx \frac{2}{3}(e_2 - e_1) E_{0\lambda}^2 a^3 \beta_c .$$

Our equation for the excitation of cavity #1 then takes the form,

$$\frac{\partial^2 e_1}{\partial t^2} + \omega_0^2 e_1 = \frac{1}{2} \kappa \omega_0^2 (e_2 - e_1), \tag{A.13}$$

$$\frac{\partial^2 e_2}{\partial t^2} + \omega_0^2 e_2 = -\frac{1}{2} \kappa \omega_0^2 (e_2 - e_1), \tag{A.14}$$

and the second line follows from the first, by symmetry. The coupling constant is

$$\kappa = \frac{4}{3\pi} \frac{a^3}{LR^2} \frac{1}{J_1^2(j_{01})} \approx 1.57 \frac{a^3}{LR^2}. \tag{A.15}$$

In retrospect it should be clear that the simpler, and intuitive derivation in Sec. 3, did no great injustice to this calculation, merely failing to produce an explicit result for κ.

A.4 Perturbation by Connecting Guide

To compute the coupling of a cavity to a waveguide, one again has a port-integral to compute. However, in this case it is convenient to let our unperturbed modal basis refer to the solution of the Helmholtz equation, with *open-circuit* boundary conditions at the port, and to designate the port surface, S_p, as a plane located in the smooth connecting guide. Since the waveguide modes form a complete set we may expand the electric field on S_p,

$$\tilde{E}_\perp(r_\perp) = \sum_a \tilde{E}_{\perp a}(r_\perp) V_a , \quad \tilde{H}_\perp(r_\perp) = \sum_a \tilde{H}_{\perp a}(r_\perp) Z_{ca} I_a . \qquad \text{(at port } S_p\text{)}$$

where the subscript $\perp$ denotes components lying in the transverse plane in the waveguide, *i.e.*, S_p. In a similar way we may expand the cavity modes,

$$\tilde{E}_{\perp\lambda}(r_\perp) = \sum_a \tilde{E}_{\perp a}(r_\perp) V_{a\lambda}, \qquad \tilde{H}_{\perp\lambda}(r_\perp) = \sum_a \tilde{H}_{\perp a}(r_\perp) Z_{ca} I_{a\lambda},$$

where the mode coefficients are just the overlap integral of the unperturbed cavity fields with the waveguide mode,

$$I_{a\lambda} = Z_{ca} \int_{S_p} d^2\vec{r}_\perp \tilde{H}_\lambda(r_\perp) \bullet \tilde{H}_{\perp a}(r_\perp), \ V_{a\lambda} = \int_{S_p} d^2\vec{r}_\perp \tilde{E}_\lambda(r_\perp) \bullet \tilde{E}_{\perp a}(r_\perp).$$

Making use of the orthogonality of the waveguide modes we have then

$$\int_{S_p} d\vec{S} \bullet \tilde{E} \times \tilde{H}_\lambda = \int d^2 \vec{r}_\perp \left\{ \sum_a \tilde{E}_{\perp a} V_a \right\} \times \left\{ \sum_b \tilde{H}_{\perp b} Z_{cb} I_{b\lambda} \right\} = \sum_a V_a I_{a\lambda},$$

and similarly,

$$\int_{S_p} d\vec{S} \bullet \tilde{E}_\lambda \times \tilde{H} = \int d^2 \vec{r}_\perp \left\{ \sum_a \tilde{E}_{\perp a} V_{a\lambda} \right\} \times \left\{ \sum_b \tilde{H}_{\perp b} Z_{cb} I_b \right\} = \sum_a V_{a\lambda} I_a.$$

Thus the mode equations take the form

$$j\varepsilon\omega\tilde{e}_\lambda = k_\lambda \tilde{h}_\lambda + \underbrace{\sum V_{a\lambda} I_a}_{\int_{S_p} d\vec{S} \bullet \tilde{E}_\lambda \times \tilde{H}}, \qquad -j\mu\omega\tilde{h}_\lambda = k_\lambda \tilde{e}_\lambda + \underbrace{\sum V_a I_{a\lambda}}_{\int_{S_p} d\vec{S} \bullet \tilde{E} \times \tilde{H}_\lambda} = k_\lambda \tilde{e}_\lambda,$$

and in the last equality we implemented the open-circuit boundary condition on the modal basis, $I_{a\lambda}$=0. Combining these two results, we obtain a single equation for $\tilde{e}_\lambda$,

$$\left(\omega^2 - \omega_\lambda^2\right)\tilde{e}_\lambda = -j\omega \frac{1}{\varepsilon} \sum_a V_{a\lambda} I_a. \tag{A.16}$$

This should be augmented by one additional condition, continuity of transverse electric field at the port. The electric field at the port may be expressed in either basis,

$$\tilde{E}_\perp(r_\perp) = \sum_a \tilde{E}_{\perp a}(r_\perp) V_a \approx \sum_\lambda \tilde{E}_\lambda(r_\perp)\, \tilde{e}_\lambda,$$

and, taking the dot-product with a guide-mode, we have

$$V_a = \sum_\lambda V_{a\lambda}\, \tilde{e}_\lambda. \tag{A.17}$$

This relation closes the system of equations and permits one to solve self-consistently for the cavity fields and the forward and reverse signals in the waveguide. For example, we may compute the *impedance looking into the cavity*, from S_p, (*i.e.* in the direction in which we have taken positive current to flow)

$$Z_{11} = \frac{V_1}{I_1}.$$

Insofar as other modes are below cut-off we may solve approximately for the mode excitation in terms of the driving current in the fundamental mode,

$$\tilde{e}_\lambda = I_1 \frac{1}{\varepsilon\omega_\lambda} \frac{V_{1\lambda}}{j\left(\omega/\omega_\lambda - \omega_\lambda/\omega\right)}, \tag{A.18}$$

so that

$$V_1 = \sum_\lambda V_{1\lambda}\tilde{e}_\lambda = I_1 \sum_\lambda \frac{1}{\varepsilon\omega_\lambda} \frac{V_{1\lambda}^2}{j(\omega/\omega_\lambda - \omega_\lambda/\omega)}.$$

Next we define the *external Q*, $Q_{e\lambda}$, for mode λ coupling to the fundamental mode of the connecting guide,

$$\frac{1}{Q_{e\lambda}} = \frac{V_{1\lambda}^2}{\varepsilon\omega_\lambda Z_{c1}} = \frac{1}{\varepsilon\omega_\lambda Z_{c1}} \left(\int_{S_p} d^2 \vec{r}_\perp \tilde{E}_\lambda \bullet \tilde{E}_{\perp 1} \right)^2, \tag{A.19}$$

in terms of which

$$Z_{11} = \frac{V_1}{I_1} = Z_{c1} \sum_\lambda \frac{1/Q_{e\lambda}}{j(\omega/\omega_\lambda - \omega_\lambda/\omega)}. \tag{A.20}$$

Notice that Z_{11} is large on-resonance, corresponding to an open circuit, just as we expect. Meanwhile, off-resonance, Z_{11} is small, corresponding to a short-circuit. This motivates the standard nomenclature for our choice of reference plane: the *plane of the detuned short*.

To make contact with Eq. (3.12), let us define $V_F = nV_+$, $V_R = nV_-$, where n may be thought of as a *turns ratio*, transforming to an impedance $Z_c' = n^2 Z_{c1}$. Continuity of transverse electric field takes the form, $V_1^+ + V_1^- = V_{1\lambda}\tilde{e}_\lambda$, and this suggests defining a *cavity voltage*, $V_c = nV_{1\lambda}\tilde{e}_\lambda$, in terms of which continuity reads,

$$V_c = V_F + V_R.$$

The port current driving term may be expressed as, $Z_{c1}I_1 = V_1^+ - V_1^-$, and in the time domain our system takes the form

$$\left\{ \frac{d^2}{dt^2} + \omega_\lambda^2 \right\} e_\lambda = -\frac{1}{\varepsilon} \frac{d}{dt} \sum_a V_{a\lambda} I_a,$$

and in the transformed units,

$$\left\{ \frac{d^2}{dt^2} + \omega_\lambda^2 \right\} V_c = \frac{\omega_\lambda}{Q_e} \frac{d}{dt} (V_F - V_R). \tag{A.21}$$

These calculations and the results Eqs. (A.1)-(A.21) complete the formal analysis underpinning the coupled cavity model introduced in the text, having developed with some additional rigor the geometric origin of the coupled cavity circuit parameters: Q_W, κ, Q_e, n, $[R/Q]$. Historically, the formal treatment of a cavity in this way was first set down by Slater; however, there were a number of approximations in his approach that, for the pure mathematician are somewhat unsatisfying. The first complete treatment of the problem is credited, by Collin, to

Kurokawa.[16] We have side-stepped the full mathematical complexity of the problem by electing to consider a single isolated, narrow-band cavity resonance. This is in fact not the totality of the problem as one can see by inspecting a measured cavity impedance. Such an impedance will indeed include narrow spikes corresponding to cavity resonances, but broadband portions as well. Thus there is a complementary problem of interest, calculation of *broadband impedance*. Such terms are important for a complete treatment of beam dynamics. They are not essential, however, for understanding the coupling of the accelerating mode, and its observational features, seen by means of couplers looking into the cavity.

There is one additional perturbation calculation that is sufficiently fundamental that it should be included, and that is Slater's theorem.

A.5 Perturbation to a Conducting Boundary (Slater's Theorem)

We consider the effect on a mode resonant frequency of indenting a cavity wall. Given the development of Sec. 1, it is most straightforward to proceed by employing two results: momentum conservation and adiabatic invariance.

We consider a lossless cavity that is ringing in one mode. Momentum conservation stipulates that if we slowly indent a cavity wall we must do work on the mode such that

$$\delta U = F^a \delta\xi_a = \int_S \langle T^{ab} \rangle dS_b \delta\xi_a,$$

where S is the region of wall being indented, F^a is the average force on the wall, $\langle T^{ab} \rangle$ is the time-averaged stress-tensor and $\delta\vec{\xi}$ is the wall displacement. Tangential wall displacements have no effect since they carry conductor into itself. Thus we may consider a displacement along the outward normal $\hat{n}$, $\delta\vec{\xi} = \hat{n}\,\delta\xi$, *i.e.*, $\delta\xi_a = n_a\,\delta\xi$. Noting that $dS_b = n_b\,dS$, we have simply

$$\delta U = \int_S \langle T^{nn} \rangle dS\,\delta\xi. \tag{A.22}$$

The diagonal element of the stress-tensor for the normal component at a conducting surface takes the simple form

$$T^{nn} = \frac{1}{2}\varepsilon_0 \vec{E}^2 - \frac{1}{2}\mu_0 \vec{H}^2,$$

and is just the electromagnetic pressure on the wall. Next we employ a mode expansion in the exact modal basis for the conducting boundaries as they are at time t,

$$\vec{E} = e_\lambda(t)\vec{E}_\lambda(\vec{r}),\ \vec{H} = h_\lambda(t)\vec{H}_\lambda(\vec{r}).$$

Recognizing that the mode resonant frequency may be changing in time, we choose to express the mode amplitude in terms of the cavity phase,

---

16 K. Kurokawa, "The expansions of electromagnetic fields in cavities", *IRE Trans. Microwave Theory Tech.*, vol. **MTT-6**, pp.178-187 (1958). See also, Kaneyuki Kurokawa, "My Personal Recollections of Microwaves", *IEEE MTT-S Newsletter* **145** (IEEE, Piscataway, 1997) pp. 57-69.

$$\theta(t) = \int^{t} dt' \, \omega_\lambda(t'),$$

according to

$$e_\lambda(t) = \Re\left\{\tilde{e}_\lambda(t) e^{j\theta}\right\}.$$

Maxwell's equations then take the form

$$\left\{\frac{d^2}{dt^2} + \omega_\lambda^2(t)\right\} e_\lambda(t) = 0,$$

and after some algebra this may be reduced to

$$\frac{d}{dt}\left\{\tilde{e}_\lambda(t)\sqrt{\omega_\lambda(t)}\right\} = 0,$$

in the limit of an adiabatically varying resonant frequency. Taking account of similar results for the magnetic field amplitude, and noting the expression for stored energy,

$$U = \frac{1}{2}\int_V \left\{\mu_0 \vec{H}^2 + \varepsilon_0 \vec{E}^2\right\} dV = \int_V \left\langle \varepsilon_0 \vec{E}^2 \right\rangle dV = \frac{1}{2}\varepsilon_0 \left|\tilde{e}_\lambda(t)\right|^2$$

we find

$$\frac{\delta U}{U} = -\frac{\delta\omega_\lambda}{\omega_\lambda} \Leftrightarrow \delta\left(\frac{U}{\omega_\lambda}\right) = 0. \tag{A.23}$$

The adiabatic invariance of U/ω_λ ("photon number") together with momentum conservation permit us to solve for the increment in resonant frequency due to an indentation $\delta\xi$. We express the time-averaged stress on the wall as

$$\left\langle T^{nn}\right\rangle = \frac{1}{4}\varepsilon_0 \left|\tilde{e}_\lambda(t)\right|^2 \left(\vec{E}_\lambda^2 - \vec{H}_\lambda^2\right) = \frac{1}{2} U \left(\vec{E}_\lambda^2 - \vec{H}_\lambda^2\right),$$

to find

$$\frac{\delta\omega_\lambda}{\omega_\lambda} = -\frac{\delta U}{U} = -\frac{\int_S \left\langle T^{nn}\right\rangle dS}{U}\delta\xi = \frac{1}{2}\int_S \left(\vec{H}_\lambda^2 - \vec{E}_\lambda^2\right) dS \, \delta\xi. \tag{A.24}$$

We may integrate this formally to obtain Slater's theorem,

$$\omega_\lambda^2 = \omega_{\lambda 0}^2 \left(1 + \int_V \left(\tilde{H}_\lambda^2 - \tilde{E}_\lambda^2\right) \delta V\right), \tag{A.25}$$

for the perturbation to the resonant frequency of a mode due to an excluded volume V. In applying this result, it is good to keep in mind that the integration represented here employs mode profiles that vary with the surface displacement, "δV"$\equiv dS\,\delta\xi$.