K. McDonald
Princeton U.
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Positron Production in a Plasma Wakefield Accelerator

In a visit on Oct. 28, 1996 to the Center for Ultrafast Optical Studies I had interesting conversations with Gerard Mourou and Don Umstadter about possible mechanisms for positron production in a plasma that is illuminated with a strong laser. I argue that positron production via a trident process involving the plasma nuclei is possible only if the electrons have sufficient longitudinal momentum, as might arise from plasma-wakefield acceleration. Electrons with no longitudinal momentum cannot produce positrons no matter how large their 'quiver motion' induced by a laser passing through the plasma.

1 Introduction

A useful reference on the behavior of electrons in strong wave field is T.W.B. Kibble, Phys. Rev. **150**, 1060 (1966).

We characterize the strength of a wave field by the dimensionless parameter

$$\eta = \frac{e\mathcal{E}_{\rm rms}}{m\omega c},$$

where $\mathcal{E}$ is the electric field of the wave, m is the rest mass of an electron, and ω is the frequency of the wave.

An electron moving in such a wave is said to have an effective mass

$$\overline{m} = m\sqrt{1+\eta^2}.$$

A more precise meaning to this concept comes from the observation that if an electron had 4-momentum $p_\mu = (E, p_x, p_y, p_z)$ in the absence of the wave, then when it propagates in a wave of 4-momentum $k_\mu = \hbar(\omega, k_x, k_y, k_z)$ its effective 4-momentum (quasimomentum) is

$$q_\mu = p_\mu + \frac{m^2\eta^2}{2(p\cdot k)} k_\mu,$$

where $p{\cdot}k$ is the 4-vector product $p_\mu k^\mu$. The quasimomentum q_μ 'averages' over the transverse oscillations of the electron in the wave. It is readily verified that $q_\mu q^\mu = \overline{m}^2$, *i.e.*, the effective mass corresponding to the quasimomentum is $\overline{m}$.

A merit of the quasimomentum is that energy-momentum conservation in scattering process involving the electron can be correctly stated using q_μ rather than p_μ. In particular, all final-state electrons (and/or positrons) from an interaction within the wave should be described by their quasimomenta and not by their ordinary momenta.

2 Trident Production of Positrons

We consider the trident process involving an electron in a wave both of which are in a gas.

Here the gas serves to provide the electrons by ionization, and also to provide nuclei that can absorb momentum but not (or very little) energy in the trident process:

$$e + A \to e' + A' + e^+e^-.$$

For a very heavy nucleus A its final state A' has a different momentum but the same energy. Then the initial electron must provide the energy to create the e^+e^- pair as well as that for the final electron. The least energy required is when all three final-state electrons and positrons are at rest (*i.e.*, they have zero net longitudinal momentum; they must always have quiver motion when they are in the wave). Then the final energy is $3\overline{m}$. (I will set $\hbar = 1 = c$ in most of the following.)

I write $3\overline{m}$ and not $3m$ since the final-state particles are still in the wave and still have the associated 'quiver' energy. That is, when I speak of 'energy' as relevant to an 'interaction' within the wave, I use the quasienergy q_0, not the energy p_0 of the electron in the absence of the wave.

2.1 Electrons with Zero Net Momentum

If the initial electron had no net momentum when it is within the wave its quasienergy would be just $q_0 = \overline{m}$. Conservation of quasienergy does not permit this electron to create electron-positron pairs by the trident process.

2.2 Electrons with Longitudinal Momentum Due to the Ponderomotive Force of a Laser

To create a pair by the trident process the initial electron must have quasienergy of at least $3\overline{m}$. It must have net momentum for this to be possible.

In general an electron in a pulsed wave is not at rest even if it were at rest before the arrival of the wave because of ponderomotive forces that act on the electron as the wave overtakes it. These forces have both longitudinal and transverse components, and in highly focused pulses the transverse forces tend to the larger, and electrons are often ejected sideways from the core of the pulse. This latter situation is very unlikely to lead to positron production, so I consider only those electrons that experienced purely longitudinal ponderomotive forces. (The paper of Kibble is useful for the general case.)

If the ponderomotive force is purely longitudinal its effect is simply summarized by the quasimomentum concept. The idea is that the quasimomentum is the result of the transient ponderomotive forces experienced by the electron as the wavefront overtakes it. I consider an electron at rest before the wave arrives. Then its ordinary 4-momentum is

$$p_\mu = (m, 0, 0, 0).$$

The wave moves in the $+z$-direction in a gas of index very close to 1, so the wave 4-momentum is

$$k_\mu = (\omega, 0, 0, \omega), \qquad \text{and} \qquad p \cdot k = m\omega.$$

The quasimomentum is then

$$q_\mu = (m(1+\eta^2/2), 0, 0, m\eta^2/2) = (\overline{m}\gamma, 0, 0, \overline{m}\gamma\beta_z).$$

From this we learn that the net longitudinal velocity of the electron inside wave is

$$\beta_z = \frac{v_z}{c} = \frac{q_z}{q_0} = \frac{\eta^2/2}{1+\eta^2/2}.$$

That is, in a very strong wave the electron can take on relativistic longitudinal motion. Once the wave passes the electron by, however, the latter reverts to its original momentum and this process is not what is commonly meant by an 'accelerator'.

However, if the quasienergy $q_0 = m(1+\eta^2/2) \geq 3\overline{m}$ then we could have trident production while the electron is still in the wave. This is possible in principle if $\eta \geq \sqrt{16 + 12\sqrt{2}} = 5.74$, which requires an extremely intense wave!

2.3 Electrons with Longitudinal Momentum Due to Plasma-Wakefield Acceleration

For $\eta < 5.74$ the trident process is still possible within the wave provided the electron has quasienergy $q_0 \geq 3\overline{m}$. This might arise, for example, because of acceleration by the plasma wakefield effect.

Of possible amusement is the relation between the quasienergy of the accelerated electron while still in the wave and its energy as measured in the lab after the acceleration is over. If the electron is measured to have energy E in the lab its corresponding longitudinal momentum would be $p_z = \sqrt{E^2 - m^2}$. Then $p \cdot k = \omega(E - p_z)$, and the quasienergy when the electron was still in the wave was

$$q_0 = E + \frac{m\eta^2}{2(E - p_z)} = E + \frac{\eta^2}{2}(E + p_z),$$

recalling that $E^2 - p_z^2 = m^2$.

On applying the threshold condition, $E \geq 3\overline{m}$, we can eliminate E in favor of p_z to find the threshold momentum as measured out of the wave for trident production in the wave:

$$p_z \geq 2\sqrt{2}mc\frac{\sqrt{1+\eta^2}}{1+\eta^2}[1 - \eta^2(3\sqrt{2} - 4)/8].$$

We recognize that in the limit $\eta^2 \to 0$ the condition $p_z \geq 2\sqrt{2}mc$ is the equivalent of $E \geq 3mc^2$. For nonzero η the threshold momentum is lower! Indeed, for $\eta = \sqrt{16 + 12\sqrt{2}}$ an electron that emerges from the wave with zero longitudinal momentum (*i.e.*, one that was not accelerated) could have created positrons via the trident process in the wave as was noted in sec. 2b.

3 'Picturesque' Arguments

Here I review the 'picturesque' argument that the trident production is not possible via electrons inside the wave with zero net longitudinal momentum there. This time I do not invoke the quasimomentum directly. The key to the argument is that the final-state particles from an interaction in a wave must be created with quiver energy (and momentum) corresponding to the instantaneous phase of the wave at the interaction point.

If the particle has has zero net longitudinal momentum then its energy is the minimum possible for any particle in the wave, and there is no spare energy to be converted into the mass of the new particles.

The details of the motion for circular or linear motion don't matter for this argument.

3.1 A Loophole for Linear Polarization?

Maybe the electron creates the pair in a linearly polarized wave at a phase when its kinetic energy is high, but the final electron and the pair all appear with a lower kinetic energy corresponding to some other phase of the wave. This can't happen if the interaction takes place at a well-defined point, since the phase of the wave is a unique function space and time. Perhaps it could work if the final particles 'tunnel' to another space-time point before appearing and the instantaneous kinetic energy is lower at that point.

Without worrying about the probability of such tunneling, I examine whether there is any condition in which it might be allowed.

For this I need some details about the classical trajectory of electrons in a linearly polarized wave. The basic facts come from problem 2, sec. 47 of Landau and Lifshitz, 'The Classical Theory of Fields', but useful additional details are taken from E.S. Sarachik and G.T. Schappert, Phys. Rev. D **1**, 2738 (1970).

We consider a plane wave propagating in the $+z$-direction with field strength given by $\eta = e\mathcal{E}_{\rm rms}/m\omega c$, and we introduce the related parameter a given by

$$a^2 = \frac{\eta^2}{1+\eta^2}, \qquad 0 \le a^2 \le 1.$$

We consider only those trajectories with zero average momentum.

For circular polarization of the wave the electron trajectory is a circle in the plane perpendicular to the z axis, with radius ac/ω, velocity $\beta = v/c = a$ and Lorentz factor

$$\gamma_{\rm circ} = \frac{1}{\sqrt{1-a^2}} = \sqrt{1+\eta^2}.$$

For a wave linearly polarized in the x-direction the trajectory can be parametrized as

$$x = -\sqrt{2}\frac{ac}{\omega}\sin\alpha, \qquad z = \frac{a^2 c}{4\omega}\sin 2\alpha,$$

$$\text{where} \quad \alpha = \omega\tau\sqrt{1+\eta^2} = \frac{\omega\tau}{\sqrt{1-a^2}}, \qquad \text{and} \quad \tau = \text{ proper time}.$$

This is the famous figure-8 trajectory. Now

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma\frac{dx}{dt}, \qquad \text{so} \qquad \gamma\beta_x = \frac{1}{c}\frac{dx}{d\tau}, \qquad \text{and} \qquad \gamma^2\beta^2 = \left(\frac{1}{c}\frac{dx}{d\tau}\right)^2 + \left(\frac{1}{c}\frac{dz}{d\tau}\right)^2.$$

We shortly find that

$$\gamma^2\beta^2 = \frac{1}{1-a^2}\left[2a^2 - (kx)^2 + \frac{1}{4}[a^2 - (kx)^2]^2\right],$$

where $k = \omega/c$. A simpler form is obtained on noting that $\gamma^2 = 1 + \gamma^2\beta^2$:

$$\gamma_{\rm lin} = \frac{1 + \frac{1}{2}[a^2 - (kx)^2]}{\sqrt{1-a^2}}.$$

From the x-trajectory equation we note that $0 \leq (kx)^2 \leq 2a^2$, so

$$\gamma_{\rm min} = \frac{1 - a^2/2}{\sqrt{1-a^2}}, \qquad \text{and} \qquad \gamma_{\rm max} = \frac{1 + a^2/2}{\sqrt{1-a^2}}.$$

These values surround the result that $\gamma_{\rm circ} = 1/\sqrt{1-a^2}$ always for circular polarization. For small η, $\gamma_{\rm min} \approx 1 + \eta^4/8$, $\gamma_{\rm max} \approx 1 + \eta^2$, and $\gamma_{\rm circ} \approx 1 + \eta^2/2$.

Suppose an electron interacts with a nucleus with $\gamma_{\rm max}$ and reappears along with an electron-positron pair at a location where $\gamma_{\rm min}$ holds at that moment. The nucleus absorbs the excess momentum of the initial electron. Conservation of energy requires

$$\gamma_{\rm max} = 3\gamma_{\rm min} \qquad \Rightarrow \qquad 1 + a^2/2 = 3(1 - a^2/2) \qquad \Rightarrow \qquad a^2 = 1 \qquad \Rightarrow \qquad \eta \to \infty.$$

That is, the hypothetical tunneling process is not possible under any circumstances!

In sum, even in a wave an electron can produce positrons off nuclei only if the electron has sufficient longitudinal momentum that the corresponding (quasi)energy is three times the (effective) electron mass.

4 Brief Comparison with Positron Production in E-144

Thus we soon must confront the usual dilemma in positron sources. If there are enough nuclei to produce a lot of positrons there will be significant multiple scattering off the electrons bound to the nuclei, leading to poor emittance.

Now if the positrons were produced in a light-by-light scattering process, as demonstrated in SLAC experiment E-144, with no nuclei around there would be no multiple Coulomb scattering and the emittance would be more favorable. This, I believe, is the mechanism for future very high performance positron sources.