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Lie Symmetries and Preliminary Classification of $u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b})$

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Abstract

Differential-difference equations (DDEs) of the form $u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b})$ with $k \geq 2$ are studied for Lie symmetries and preliminary classification. Explicit forms of equations are given for those admitting at least one intrinsic Lie symmetry. An algorithmic mechanism is also proposed to automate the symmetry calculation for fairly general DDEs via computer algebras.

1. Introduction

The Lie symmetry method [1, 2] for differential equations has by now been well established, though the same theory for differential-difference equations (DDEs) [3–7] or difference equations [8, 9] is much less studied or understood. Symmetry Lie algebras often go a long way to explain the behaviour of the corresponding system, just like the algebra on which its Lax pair (if any) lives would through the use of ISMs [10, 11]. To overcome the difficulties set by the infinite number of variables in DDEs, the concept of *intrinsic* symmetries [3] has been introduced to simplify the task of symmetry calculations. Our purpose here is to study the DDEs of the form

$$u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b}) , \qquad n \in \mathbf{Z} , \quad a \leq b , \quad k \geq 2 \tag{1.1}$$

where we use the superindex $^{(i)}$ to denote the i-th partial derivative with respect to (w.r.t.) t. We note that both a and b in (1.1) are integers and $\mathbf{Z}$ will always denote the set of all integers. We first look for the symmetries

$$\mathbf{X} = \xi(t, u_i : \ i \in \mathbf{Z})\partial_t + \sum_{n \in \mathbf{Z}} \phi_n(t, u_i : \ i \in \mathbf{Z})\partial_{u_n} \tag{1.2}$$

for system (1.1) and then consider the subsequent task of classification by means of the intrinsic Lie symmetries in the form

$$\mathbf{X} = \xi(t)\partial_t + \phi_n(t, u_n)\partial_{u_n} . \tag{1.3}$$

We note that systems (1.1) are so far only studied [7] for $k = 2$ with $b = a = -1$ w.r.t. the *intrinsic* Lie symmetries, and our objective here in this respect is to give an explicit list of DDEs (1.1) admitting at least one intrinsic Lie symmetry. This thus serves as a preliminary or semi classification. Complete classification of some more specific form of (1.1) is still under investigation. Another objective of ours is to devise an algorithmic mechanism to automate the calculation of Lie symmetries for DDEs so that the symmetries could be efficiently calculated for practical problems.

The paper is organized as follows. We first give in Section 2 the general Lie symmetry for (1.1). We will explain why the study of intrinsic Lie symmetries, particularly with regard to the classification, will not cause a significant loss of generality. Section 3 serves as a preliminary classification: the forms of system (1.1) are explicitly given for those bearing at least one intrinsic Lie symmetry. In Section 4, we will briefly propose an algorithmic mechanism for calculating intrinsic symmetries by means of computer algebras, along with several illustrative examples.

2. The general Lie symmetries for (1.1)

We call DDE (1.1) *nontrivial* if there exists at least one $n_0 \in \mathbf{Z}$ such that $F_{n_0}(t, u_{n_0+a}, \ldots, u_{n_0+b})$ is not a function of only variables t and u_{n_0}. Suppose system (1.1) is nontrivial for at least k=2 and $\mathbf{X}$ in (1.2) is a Lie symmetry of the system. Then ξ and ϕ_n in (1.2) must have the following form

$$\begin{aligned} &\xi = \xi(t), \quad \phi_n = \Big(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n\Big)u_n + \sum_{i\in\mathbf{Z}} c_{n,i}u_i + \beta_n(t)\ , \quad k \geq 2 \\ &\xi(t) = \alpha_2 t^2 + \alpha_1 t + \alpha_0\ , \qquad k \geq 3 \end{aligned} \tag{2.1}$$

where $c_{n,i}$, α_i and γ_n are all constants. Moreover, (1.2) with (2.1) is a Lie symmetry of (1.1) iff

$$\phi_n^{(k)} - k\dot{\xi}F_n - \xi\dot{F}_n + \sum_{i\in\mathbf{Z}} \phi_{n,u_i}F_i - \sum_{j=a}^{b} \phi_{n+j}F_{n,u_{n+j}} = 0 \tag{2.2}$$

is further satisfied.

We omit the derivation details of (2.1) and (2.2) as they are too lengthy to be put here. We note that nonintrinsic symmetries do exist for system (1). For example, the system $u_n^{(k)}(t) = \sum_{j=a}^{b} \mu_j e^{j\lambda t} u_{n+j}$ has the nonintrinsic Lie symmetry $\mathbf{X} = \partial_t + \sum_{n\in\mathbf{Z}} \Big\{ \sum_{i\in\mathbf{Z}} \alpha_i u_{n+i} - n\lambda u_n + \beta_n(t)\Big\}\partial_{u_n}$, and the system $u_n^{(k)}(t) = \sum_{j=a}^{b} \mu_j \lambda^{nj} u_{n+j}$ has the nonintrinsic Lie symmetry $\mathbf{X} = \partial_t + \sum_{n\in\mathbf{Z}} \Big\{ \sum_{i\in\mathbf{Z}} \alpha_i \lambda^{ni} u_{n+i} + \beta_n(t)\Big\}\partial_{u_n}$ if $\beta_n(t)$ satisfies the original DDEs respectively. In fact we can show that system (1.1) can only have intrinsic Lie symmetries unless it is linear or 'essentially' linear. We will not dwelt upon this here. Instead we quote without proof one of the results in [12] that *all the Lie symmetries of the system* $u_n^{(k)} = f(t, u_{n+c}) + g(t, u_{n+a}, \ldots, u_{n+c-1}, u_{n+c+1}, \ldots, u_{n+b})$ *for* $k \geq 2$ *and* $a \leq c \leq b$ *are intrinsic if* f *is nonlinear in* u_{n+c} *and the system is nontrivial.*

3. Systems bearing intrinsic Lie symmetries

The classification is in general a very laborious task. Due to the difficulties of handling nonintrinsic symmetries and their scarceness implied in Section 2 anyway, only intrinsic symmetries are known to be ever considered in this respect. For systems of type (1.1), the only known results in this connection have been for $k = 2$ along with $b = -a = 1$ [7]. We shall thus mostly consider the case of $k \geq 3$ in this section. Apart from two systems related to two special symmetry Lie algebras, we shall be mainly concerned with systems that bear at least one intrinsic Lie symmetry rather than the classification via the *complete* symmetry Lie algebras. For this purpose, we first show that, for any $k \geq 3$, a fiber-preserving transformation

$$u_n(t) = \Omega_n(y_n(\tilde{t}\,), t), \qquad \tilde{t} = \theta(t), \tag{3.1}$$

will transform system (1.1) into

$$\frac{d^k}{d\,\tilde{t}^k} y_n(\tilde{t}\,) = \tilde{F}_n(\tilde{t}, y_{n+a}, \ldots, y_{n+b})\ , \tag{3.2}$$

iff transformation (3.1) is given by

$$u_n(t) = A_n(\gamma t + \delta)^{k-1} y_n(\tilde{t}\,) + B_n(t), \quad \tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{3.3}$$

where $B_n(t)$ are arbitrary functions and A_n, α, β, γ and δ are arbitrary constants satisfying

$$\alpha\delta - \beta\gamma = \pm 1, \quad A_n \neq 0\ . \tag{3.4}$$

We now prove the above statement. First, since it is easy to show inductively

$$\begin{aligned} u_n^{(m)} &= \Omega_{n,y_n} \dot{\theta}^m y_n^{(m)} + \tbinom{m}{1} \Omega_{n,y_n y_n} \dot{\theta}^m y_n^{(m-1)} \dot{y}_n + \\ &\qquad +\text{terms of other powers of } y_n\text{'s derivatives}, \qquad m \geq 3\ , \end{aligned} \tag{3.5}$$

we must have $\Omega_{n,y_n y_n} = 0$ if (3.1) is to transform (1.1) into (3.2). Hence, (3.1) must take the form $u_n(t) = f_n(t) y_n(\tilde{t}) + B_n(t)$. Since it is not difficult to show inductively $u_n^{(m)}(t) = \lambda_m(t) y_n^{(m)} + \mu_m(t) y_n^{(m-1)} + \nu_m(t) y_n^{(m-2)} + \cdots$ for $m \geq 3$ in which λ_m, μ_m and ν_m are given correspondingly by the following more explicit formula

$$\begin{aligned} u_n^{(m)}(t) &= f_n(t) \dot{\theta}^m y_n^{(m)} + \left[\binom{m}{1} \dot{f}_n \dot{\theta}^{m-1} + \binom{m}{2} f_n \dot{\theta}^{m-2} \ddot{\theta} \right] y_n^{(m-1)} + \\ &\left[\tbinom{m}{2} \ddot{f}_n \dot{\theta}^{m-2} + 3\tbinom{m}{3} \dot{f}_n \dot{\theta}^{m-3} \ddot{\theta} + 3\tbinom{m}{4} f_n \dot{\theta}^{m-4} \ddot{\theta}^2 + \tbinom{m}{3} f_n \dot{\theta}^{m-3} \dddot{\theta} \right] y_n^{(m-2)} + \cdots\ , \end{aligned} \tag{3.6}$$

in order (3.1) transforms (1.1) into (3.2), it is clearly necessary that $\mu_k(t) = \nu_k(t) = 0$, read off from (3.6), be satisfied. Removing $\ddot{f}_n$ from these two equations, we have $\dot{f}_n \dot{\theta} \ddot{\theta} + \frac{k-2}{2} f_n \ddot{\theta}^2 + \frac{1}{3} f_n \dot{\theta} \dddot{\theta} = 0$, $\dot{f}_n \dot{\theta} + \frac{k-1}{2} f_n \ddot{\theta} = 0$ which are equivalent to $3\ddot{\theta}^2 = 2\dot{\theta}\dddot{\theta}$ and $\dot{f}_n \dot{\theta} + \frac{k-1}{2} f_n \ddot{\theta} = 0$. The solution of the above equations (with $\dot{\theta} \neq 0$) is elementary: it is given by

$$u_n(t) = A_n(t + \alpha_1)^{k-1} y_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \frac{\beta_1}{t + \alpha_1} + \gamma_1, \tag{3.7}$$

for $\ddot{\theta} \neq 0$, and is otherwise by

$$u_n(t) = A_n y_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \beta_1 t + \alpha_1,\ . \tag{3.8}$$

In (3.7) and (3.8), the function $B_n(t)$ and constants A_n, α_1, β_1 and γ_1 are arbitrary and satisfy $\beta_1 A_n \neq 0$. It is easy to verify that (3.7) and (3.8) together are equivalent to (3.3) with (3.4).

We now proceed to prove the sufficiency. For this purpose, we can show inductively for $m \geq 1$

$$x^{(m)} = A \sum_{i=0}^{m} \binom{m}{i} (-\beta_1)^i \mathsf{A}_{k-1-i}^{m-i} (t+\alpha_1)^{k-1-m-i} y^{(i)}, \tag{3.9}$$

where

$$x(t) = A(t+\alpha_1)^{k-1} y(\theta(t)), \quad \theta(t) = \frac{\beta_1}{t+\alpha_1} + \gamma_1, \quad k \geq 2$$
$$\mathsf{A}_n^i = n(n-1)\cdots(n-i+1), \quad \mathsf{A}_n^0 = 1 .$$

The proof is again meticulous but straightforward, and is thus skipped here for brevity. If we choose $m = k$, then each coefficient in front of $y^{(i)}$ in (3.9) for $0 \leq i < m$ has the factor $(m-k)$ and is thus zero. Hence, we obtain simply

$$x^{(k)} = A(-\beta_1)^k (t+\alpha_1)^{-k-1} y^{(k)} . \tag{3.10}$$

It is now clear from (3.10) that (3.7) will transform (1.1) into (3.2). Since the sufficiency regarding to (3.8) is almost trivial, we have completed the proof of the form-invariance of (1.1) under (3.1) and (3.3).

Notice that under $u_n(t) = y_n(\tilde{t})/\sigma_n(t) + B_n(t)$ with $\tilde{t} = \theta(t)$, the symmetry

$$\mathbf{X} = \xi(t)\partial_t + \Big[\big(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n\big)u_n + \beta_n(t)\Big]\partial_{u_n} \tag{3.11}$$

will read as

$$\mathbf{X} = \xi\dot{\theta}\partial_{\tilde{t}} + \Big\{\Big[\frac{k-1}{2}\dot{\xi} + \gamma_n + \frac{\dot{\sigma}_n}{\sigma_n}\xi\Big]y_n + \sigma_n\Big[\big(\frac{k-1}{2}\dot{\xi} + \gamma_n\big)B_n - \xi\dot{B}_n + \beta_n\Big]\Big\}\partial_{y_n} . \tag{3.12}$$

Of course, for (1.1) to become (3.2), we need $\sigma_n(t)$ to be given for $k \geq 3$ by (3.3) and for $k = 2$ [7] by $\sigma_n(t) = \dot{\theta}^{\frac{1}{2}}/A_n$ with $A_n \neq 0$. We can thus conclude from (2.1), (3.3) and (3.4) that there are exactly three canonical forms of $\xi(t) \neq 0$ given by

$$\text{(i)} \quad \xi(t) = 1, \quad \text{(ii)} \quad \xi(t) = t, \quad \text{(iii)} \quad \xi(t) = t^2 + 1 \tag{3.13}$$

because all other cases can be transformed via (3.3) into one of above three. In all these three cases, we may choose $B_n(t)$ such that

$$\xi(t)\dot{B}_n(t) = \Big(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n\Big)B_n + \beta_n(t) . \tag{3.14}$$

This means we may choose simply $\beta_n(t) = 0$ in all the three canonical cases in (3.13). If $\xi(t) = 0$, then (3.14) still has a solution $B_n = -\beta_n/\gamma_n$ if $\gamma_n \neq 0$. Hence the fourth and the last canonical cases are

$$\text{(iv)} \quad \xi(t) = 0,\ \gamma_n \neq 0,\ \beta_n(t) = 0, \qquad \text{(v)} \quad \xi(t) = 0,\ \gamma_n = 0,\ \beta_n(t) \neq 0 . \tag{3.15}$$

We note from (3.12) that systems in case (v) with β_n and $\tilde{\beta}_n$ are equivalent if $\tilde{\beta}_n(t) = \sigma_n(t)\beta_n(t)$. Also that in the case of $k = 2$, cases (ii) and (iii) are transformable to case

(i) due to allowed more general fiber transformations. Note furthermore that although in principle one should replace cases (iv) and (v) with the following more general form

$$\xi(t)=0,\ \gamma_m\neq 0,\ \beta_m(t)=0,\ \gamma_n=0,\ \beta_n(t)\neq 0,\quad m\in\mathbf{S},\ n\in\mathbf{Z}\backslash\mathbf{S}$$

for some subset $\mathbf{S}$ of $\mathbf{Z}$, such a mixture (which won't exist if the continuity in n is imposed on F_n as adopted in [7]) will lead to only $F_n=F_{d,n}$ for $n\in\mathbf{S}$ and $=F_{e,n}$ for $n\in\mathbf{Z}\backslash\mathbf{S}$, where $F_{d,n}$ and $F_{e,n}$ are just the F_n given in $(3.17)_d$ and $(3.17)_e$, respectively.

With the above preliminaries, we are ready to give the general forms of F_n such that (1.1) has at least one intrinsic Lie symmetry. The form of F_n is thus to be determined from (2.2), or more explicitly for $k\geq 2$,

$$\begin{aligned}&\frac{k-1}{2}\xi^{(k+1)}u_n+\beta_n^{(k)}+\Big[\gamma_n-\frac{k+1}{2}\dot{\xi}\Big]F_n-\xi\dot{F}_n-\\&\qquad\sum_{i=n+a}^{n+b}\Big[\Big(\frac{k-1}{2}\dot{\xi}+\gamma_i\Big)u_i+\beta_i\Big]F_{n,u_i}=0\ .\end{aligned}\tag{3.16}$$

In fact, for three canonical cases in (3.13) with $\beta_n=0$ and another two in (3.15), the solutions of (3.16) are

$$\begin{aligned}&\text{(i)}\quad F_n=e^{\gamma_n t}f_n(\zeta_{n+a},\ldots,\zeta_{n+b}),\quad \zeta_i=u_ie^{-\gamma_i t}\\&\text{(ii)}\quad F_n=t^{\gamma_n-(k+1)/2}f_n(\zeta_{n+a},\ldots,\zeta_{n+b}),\quad \zeta_i=u_it^{-\gamma_i-(k-1)/2}\\&\text{(iii)}\quad F_n=\frac{\exp(\gamma_n\tan^{-1}(t))}{(t^2+1)^{(k+1)/2}}f_n(\zeta_{n+a},\ldots,\zeta_{n+b}),\quad \zeta_i=u_i\frac{(t^2+1)^{(1-k)/2}}{\exp(\gamma_n\tan^{-1}(t))}\\&\text{(iv)}\quad F_n=u_{n+c}^{\gamma_n/\gamma_{n+c}}f_n(t,\zeta_{n+a},\ldots,\zeta_{n+c-1},\zeta_{n+c+1},\ldots,\zeta_{n+b}),\quad \zeta_i=\frac{u_i^{\gamma_{n+c}}}{u_{n+c}^{\gamma_i}}\\&\text{(v)}\quad F_n=\frac{\beta_n^{(k)}}{\beta_{n+c}}u_{n+c}+f_n(t,\zeta_{n+a},\ldots,\zeta_{n+c-1},\zeta_{n+c+1},\ldots,\zeta_{n+b}),\\&\qquad\ \zeta_i=u_i\beta_{n+c}-u_{n+c}\beta_i\end{aligned}\tag{3.17}$$

where c is any given integer in $[a,\,b]$, and the equivalent classes in case (v) are related by $\beta_n\sim\beta_n\sigma_n$ for the same σ_n as in (3.12). If $0\in[a,\,b]$, then case (iv) in $(3.17)_d$ may also be rewritten as

$$F_n=u_nf_n(t,\zeta_{n+a},\ldots,\zeta_{n-1},\zeta_{n+1},\ldots,\zeta_{n+b}),\quad \zeta_i=u_i^{\gamma_n}u_n^{-\gamma_i}\ .$$

Since (ii) and (iii) for $k=2$ in (3.13) are transformable to (i) there, formulas in (3.17), minus the cases of (ii)–(iii), also give the complete list for $k=2$. We also note that the form of F_n in (3.17) can be further refined by extending the symmetry algebras. Since the details of such undertakings would belong to the scope of complete classification which will be considered elsewhere in future, we shall limit our attention to providing two cases which are related to the Toda lattice [13] and the FPU system [14], respectively.

For this purpose, let us choose $\beta_n(t)=t^m$ in $(3.17)_e$ for $0\leq m<k$ and $k\geq 2$. Then F_n can be written as $F_n=f_n(t,\zeta_{n+a},\ldots,\zeta_{n+c-1},\zeta_{n+c+1},\ldots,\zeta_{n+b})$ with $\zeta_i=u_i-u_{n+c}$. It is obvious that this F_n also fits the form of $(3.17)_a$ with $\gamma_i=0$ if the function f_n is independent of t. Hence, for

$$F_n=f_n(\zeta_{n+a},\ldots,\zeta_{n+c-1},\zeta_{n+c+1},\ldots,\zeta_{n+b}),\quad \zeta_i=u_i-u_{n+c}\ ,\tag{3.18}$$

system (1.1) has the following $(k+1)$-dimensional nilpotent symmetry Lie algebra

$$\mathbf{X}_i = t^i \partial_{u_n},\ i = 0, 1, \ldots, k-1; \quad \mathbf{X}_k = \partial_t\ . \tag{3.19}$$

We now look for an additional symmetry $\mathbf{Y}$ of the form (3.11) such that the new Lie algebra formed by $\{\mathbf{X}_i, 0 \le i \le k,\ \mathbf{Y}\}$ contains $\{\mathbf{X}_i, 0 \le i \le k\}$ as its nilradical, i.e., $[\mathbf{X}_i, \mathbf{Y}] = \sum_{j=0}^{k} \alpha_{i,j} \mathbf{X}_j$. Since, for $0 \le i < k$,

$$[\mathbf{X}_k, \mathbf{Y}] = \dot{\xi}\partial_t + \Big(\frac{k-1}{2}\ddot{\xi}u_n + \dot{\beta}_n\Big)\partial_{u_n}, \quad [\mathbf{X}_i, \mathbf{Y}] = \Big(\gamma_n t^i + \frac{k-1}{2} t^i \dot{\xi} - i t^{i-1}\xi\Big)\partial_{u_n}\ , \tag{3.20}$$

we conclude that both ξ and $\gamma_n t + (k-1)t\dot{\xi}/2 - i\xi$ must be linear in t. Hence (3.20) implies $[\mathbf{X}_i, \mathbf{Y}]$ is linear in $\mathbf{X}_i$ for $0 \le i < k$ and $[\mathbf{X}_k, \mathbf{Y}] = \dot{\xi}\partial_t + \dot{\beta}_n \partial_{u_n}$ which in turn induces

$$\frac{d^{k+1}}{d\,t^{k+1}}\,\beta_n(t)\ =\ 0\ . \tag{3.21}$$

In order the structure constants in $[\mathbf{X}_i, \mathbf{Y}]$ be independent of n, we set $\gamma_n = \gamma$ and $\dot{\beta}_n(t) = \beta(t)$. In this way it is easy to see that $\mathbf{Y}$ can be any linear combination of

$$\begin{aligned}
&\mathbf{Y}_1 = t\partial_t + \Big(\frac{k-1}{2} + \gamma\Big)u_n \partial_{u_n},\ \gamma \ne -\frac{k-1}{2};\\
&\mathbf{Y}_2 = t\partial_t + \omega_n \partial_{u_n},\ \omega_i \ne \omega_j\ \forall i \ne j\ ;\\
&\mathbf{Y}_3 = t\partial_t + (k u_n + t^k)\partial_{u_n}; \quad \mathbf{Y}_4 = u_n \partial_{u_n}
\end{aligned} \tag{3.22}$$

modulus some linear combinations of $\mathbf{X}_i$. The FPU and Toda systems turn out to be related to $\mathbf{Y}_1$ and $\mathbf{Y}_2$, respectively. For symmetry operator $\mathbf{Y}_1$, (3.16) reduces to

$$\Big(\gamma - \frac{k+1}{2}\Big)F_n - t\dot{F}_n - \sum_i \Big(\frac{k-1}{2} + \gamma\Big)u_i F_{n,u_i} = 0$$

which is solved by (and is consistent with (3.18))

$$\begin{aligned}
&F_n = (u_{n+c} - u_{n+d})^{\frac{\gamma-(k+1)/2}{\gamma+(k-1)/2}} \times\\
&\qquad \times f_n(\zeta_{n+a}, \ldots, \zeta_{n+c-1}, \zeta_{n+c+1}, \ldots, \zeta_{n+d-1}, \zeta_{n+d+1}, \ldots, \zeta_{n+b}),\\
&\zeta_i = (u_i - u_{n+d})/(u_{n+c} - u_{n+d}), \qquad a \le c \ne d \le b\ .
\end{aligned} \tag{3.23}$$

When $b = -a = 1$, $\gamma = (1-3k)/2$ and $f_n(\zeta_{n+a}) = 1 - \zeta_{n+a}^2$, the FPU type system reads with $c = 1$ and $d = 0$ as

$$u_n^{(k)} = (u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2\ . \tag{3.24}$$

We note that for $k = 2$ eq.(3.24) can be rewritten as the original form of the FPU [14] system $\ddot{y}_n - (y_{n+1} + y_{n-1} - 2y_n)[1 + \omega(y_{n+1} - y_{n-1})] = 0$ under the transformation $y_n = u_n/\omega - n/(2\omega)$.

For $\mathbf{Y}_2$ in (3.22), eq.(3.16) is reduced to $kF_n + t\dot{F}_n + \sum_i \omega_i F_{n,u_i} = 0$ whose solution reads for any given $a \le c \ne d \le b$ as

$$F_n = \exp\Big[-k\frac{u_{n+c} - u_{n+d}}{\omega_{n+c} - \omega_{n+d}}\Big] \times \\ \times f_n(\zeta_{n+a}, \ldots, \zeta_{n+c-1}, \zeta_{n+c+1}, \ldots, \zeta_{n+d-1}, \zeta_{n+d+1}, \ldots, \zeta_{n+b}), \tag{3.25}$$

$$\zeta_i = (\omega_{n+c} - \omega_{n+d})u_i + (\omega_{n+d} - \omega_i)u_{n+c} + (\omega_i - \omega_{n+c})u_{n+d}.$$

When $b = -a = -c = 1$ and $d = 0$, then (3.25) reduces to

$$F_n = \exp\Big[-k\frac{u_{n-1} - u_n}{\omega_{n-1} - \omega_n}\Big] f_n(\zeta_{n+1}), \quad \zeta_{n+1} = \sum_{\substack{n-1,n,n+1 \\ \text{cyclic}}} (\omega_{n-1} - \omega_n)u_{n+1} \ . \tag{3.26}$$

Let furthermore $\omega_n = kn$ and $f_n(\zeta_{n+1}) = 1 - \exp(\zeta_{n+1}/k)$. Then (3.26) gives rise to the following Toda type system

$$u_n^{(k)}(t) = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}) \ .$$

4. Local overdeterminacy in explicit symmetry calculation

In this section, we shall consider a fairly large class of DDEs of the form

$$G_n(t, \partial^k u_i : \ |k| + |i - n| \le M) = 0, \qquad \forall n \in \mathbf{I} \ , \tag{4.1}$$

where $t = (t_1, \ldots, t_m)$, $\partial^k = \partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m}$, $|k| = k_1 + \cdots + k_m$, $\mathbf{I}$ is the index grid, and G_n is uniformly defined w.r.t. n in the sense that all partial differentiations of G_n commute with the index n. The purpose here is to propose a mechanism to find intrinsic Lie symmetries for uniformly defined DDEs *through* the use of such computer algebras that can deal with [15] systems of *finite* variables. Let the intrinsic symmetry be given by

$$\mathbf{X} = \xi(t)\partial_t + \eta(t, n, u_n)\partial_{n_n} \equiv \sum_{j=1}^{m} \xi_j \partial_{t_j} + \sum_{n \in \mathbf{I}} \eta_n \partial_{u_n} \ , \tag{4.2}$$

and let the Lie symmetry for (4.1) over $n \in \mathbf{J}$, a finite subset of $\mathbf{I}$, be denoted by

$$\mathbf{X}^{\mathbf{J}} = \xi^{\mathbf{J}}\partial_t + \sum_{i \in \mathbf{J}} \eta_i^{\mathbf{J}} \partial_{u_i} \ , \tag{4.3}$$

then our mechanism is based on the following observation:

If $\eta_n^{\mathbf{J}} = \eta^{\mathbf{J}}(t, n, u_n)$ is uniformly defined w.r.t. n with $\xi^{\mathbf{J}} = \xi^{\mathbf{J}}(t)$, then (4.2) with $\xi = \xi^{\mathbf{J}}$ and $\eta_i = \eta_i^{\mathbf{J}}$ ($\forall i \in \mathbf{I}$) is the intrinsic Lie symmetry of (4.1).

For convenience, we shall always denote by R_N the subsystem of N equations $G_j = 0$ for $j = n, \ldots, n + N - 1$. The algorithm proposed in this section for finding intrinsic Lie symmetries has in fact been applied to various DDEs, and *all* results have been consistent with the analytic ones whenever the later ones do exist. For instance, our consideration for the inhomogeneous Toda lattice [16]

$$\ddot{u}_n - \frac{1}{2}\dot{u}_n + (\frac{1}{4} - \frac{n}{2}) + \Big[\frac{1}{4}(n-1)^2 + 1\Big]e^{u_{n-1} - u_n} - \Big[\frac{1}{4}n^2 + 1\Big]e^{u_n - u_{n+1}} = 0$$

for R_4 will lead to the symmetry generators ∂_t, ∂_{u_n}, $e^{t/2}\partial_{u_n}$ and $e^{-t/2}\partial_t + (\frac{1}{2} - n)e^{-t/2}\partial_{u_n}$ which are exactly those obtained in [6] analytically. New but straightforward cases include applying the procedure to the discretized KZ equation

$$u_{n,xt} + \partial_x(u_{n,x}u_n) + u_{n+1} + u_{n-1} - 2u_n = 0$$

for R_3, which gives the symmetry $(\alpha t + \beta)\partial_t + (f(t) - \alpha x)\partial_x + (\dot{f}(t) - 2\alpha u_n)\partial_{u_n}$, and to the 2-dimensional system

$$\partial_x\partial_t u_n(x,t) = \exp\left(u_{n+1}(x,t) + u_{n-1}(x,t) - 2u_n(x,t)\right)$$

which gives the symmetry $(\alpha_1(t) + \beta_1(x) + (\alpha_2(t) + \beta_2(x))n - (1/2)n^2(\dot{f}(t) + g'(x)))\partial_{u_n} + f(t)\partial_t + g(x)\partial_x$ for arbitrary $f(t)$, $g(x)$, $\alpha_i(t)$ and $\beta_i(x)$. Likewise the 1-dimensional

$$\ddot{u}_n(t) = \exp\left(u_{n+1} + u_{n-1} - 2u_n\right), \qquad n \in \mathbf{Z}, \tag{4.4}$$

via R_5 will lead to the intrinsic Lie symmetry $(c_1 + c_2 t)\partial_t + (c_3 n + c_4 + (c_5 n + c_6)t - c_2 n^2)\partial_{u_n}$ for (4.4), which is again consistent with the analytic results in [7]. As for the similarity solutions, we only note that the reduction $u_n(t) = -(n^2 + a^2)\log t + w_n$ with $a \neq 0$, induced by the symmetry $-t\partial_t + (n^2 + a^2)\partial_{u_n}$, reduces (4.4) into $w_{n+1} + w_{n-1} - 2w_n = \log(n^2 + a^2)$ which has a solution $(n \neq 1, n \neq 0)$

$$w_n = \sum_{i=1}^{|n|-1} (|n| - i)\log(i^2 + a^2) + nw_1 - (n-1)w_0 + \chi(-n)|n|\log a^2,$$

where w_0 and w_1 are treated as arbitrary constants and χ is the step function defined by $\chi(t) = 0$ if $t \leq 0$ and =1 otherwise.

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