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Discrete Symmetries and Supersymmetries – Powerful Tools for Studying Quantum Mechanical Systems

J. NIEDERLE

Institute of Physics, Academy Sci. Czech Republic, Prague
E-mail: niederle@fzu.cz

Abstract

Dicrete symmetries (DS) of the Schrödinger-Pauli equation are applied to reduction of this equation and search for its hidden supersymmetries. General problems of using of DS in quantum mechanics are discussed.

1 Introduction

It is well-known that quantum mechanical systems are usually described in terms of differential equations and that symmetries of these equations form powerful tools for their studies. They are used to separate variables, to find out solutions of linear and nonlinear differential equations as well as to solve associated labelling problems, to derive spectra and related complete sets of functions of linear differential operators, to derive the corresponding conservation laws, to guide constructions of new theories, i.e., to figure out differential equations invariant with respect to a given symmetry, and so on.

Let us recall that in quantum mechanics the statement: "The physical system S has a symmetry group G" means that there is a group of transformations leaving the equation of motion of system S as well as the rules of quantum mechanics invariant. In particular no transformation from symmetry group G is allowed to produce an observable effect. Thus if system S is described by an observable A in states $|\psi >$, $|\phi >$, ..., then the system S' obtained by a symmetry transformation $g \in G$, $g : S \to S'$, is desribed by the corresponding observable A' in the states $|\psi' >$, $|\phi' >$, ... and the equality

$$\left|< \psi'|A'|\phi' >\right|^2 = |< \psi|A|\phi >|^2 \tag{1.1}$$

holds. Thus, as shown by E. Wigner [1], to any symmetry g there exists a unitary or antiunitary operator U_g (representing g in the Hilbert space H of the system S) such that

$$\begin{array}{l} |\psi' >= U_g|\psi > \quad \text{and} \\ A' = U_g A U_g^+ \end{array} \tag{1.2}$$

describe the effect of g, i.e., the change $S \to S'$.

There are two types of symmetries: *continuous* (e.g., rotations) and *discrete* (e.g., parity transformation). For continuous symmetries any $g \in G$ is a function of one or more continuous parameters α^i, $i = 1, 2, \ldots, n$, $g(\alpha^1, \alpha^2, \ldots, \alpha^n)$ and any U_g can be expressed in terms of Hermitian operators B_1, B_2, … via $e^{i\alpha^j B_j}$, where each of B_j is an observable, i.e., a constant of motion, due to continuity of parameters α^j, since for a given quantum-mechanical system described by Hamiltonian H

$$\left[e^{i\alpha^j B_j}, H\right] = 0 \;\Leftrightarrow\; \sum_{n=0} \frac{(i\alpha^j)^n}{n!} \left[B_j^n, H\right] = 0 \;\Leftrightarrow\; [B_j, H] = 0. \tag{1.3}$$

Now, if $g \in G$ is a discrete symmetry, it does not depend on continuous parameters. The corresponding operator U_g can still be written as e^{iB} or Ke^{iB}, where K is an anti-unitary operator, but in fact $[B, H] = 0$ is only a sufficient condition for $\sum_{n=0} \frac{(i)^n}{n!} [B^n, H] = 0$ but not necessary. However, all discrete symmetries in physics fulfil the condition $U_g^2 = 1$. Thus if U_g is unitary ($U_g U_g^+ = U_g^+ U_g = 1$) it is also Hermitian $U_g^+ = U_g$ and therefore an observable. This is not true for $U_g^2 \neq 1$.

Now we are ready to review some results derived by A. G. Nikitin and myself [2, 3, 4].

2 Involutive symmetries and reduction of the physical systems

Consider the free Dirac equation

$$L_0\psi = (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.1}$$

with

$$\gamma_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3, \qquad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}.$$

It is invariant w.r.t the complete Lorentz group. Involutive symmetries form a finite subgroup of the Lorentz group consisting of 4 reflections of x_μ, 6 reflections of pairs of x_μ, 4 reflections of triplets of x_μ, reflection of all x_μ and the identity transformation.

If the coordinates x_μ in (2.1) are transformed by these involutive symmetries, function $\psi(x)$ cotransforms according to a projective reprezentation of the symmetry group, i.e., either via $\psi(x) \to R_{kl}\psi(x)$ or via $\psi(x) \to B_{kl}\psi(x)$. Here R_{kl} and $B_{kl} = CR_{kl}$ are linear and antilinear operators respectively which commutes with L_0 and consequently transform solutions of (2.1) into themselves. The operators $R_{kl} = -R_{kl}$ form a reprezentation of the algebra $so(6)$ and C is the operator of charge conjugation $C\psi(x) = i\gamma_2\psi^*(x)$. Among the operators B_{kl} there are six which satisfy the condition that $(B_{kl})^2 = -1$ and nine for which $(B_{kl})^2 = 1$. We shall consider further only B_{kl} fulfilling the last condition (for the reason mentioned in the Introduction and since otherwise B_{kl} cannot be diagonalized to real γ_5 and consequently used for reduction). As shown in [2] the operators R_{kl}, B_{kl} and C form a 25–dimensional Lie algebra. It can be extended to a 64-dimensional real Lie algebra or via non-Lie symmetries (for details see [3]).

Let us discuss now only one example how to use discrete symmetries to reduce a physical system into uncoupled subsystems (for the other examples see [2]). Let the system be a spin $\frac{1}{2}$ particle interacting with a magnetic field described by the Dirac equation

$$L\psi(x) = (\gamma^\mu(i\partial_\mu - eA_\mu) - m)\,\psi(x) = 0. \tag{2.2}$$

Eq. (2.2) is invariant w.r.t. discrete symmetries provided $A_\mu(x)$ cotransforms appropriately. For instance,

$$A_\mu(-x) = -A_\mu(x) \tag{2.3}$$

for $x \to -x$ and $\psi(x) \to \widehat{R}\psi(x) = \gamma_5\widehat{\theta}\psi(x) = \gamma_5\psi(-x)$. Then, diagonalizing symmetry operator $\widehat{R}$ by means of the operator

$$W = \frac{1}{\sqrt{2}}(1+\gamma_5\gamma_0)\frac{1}{\sqrt{2}}(1+\gamma_5\gamma_0\widehat{\theta}), \tag{2.4}$$

the equation (2.2) is reduced to the block diagonal form:

$$(-\mu(i\partial_0 - eA_0) - \vec{\sigma}(i\vec{\partial} - e\vec{A})\widehat{\theta} - m)\psi_\mu(x) = 0, \tag{2.5}$$

where $\mu = \pm 1$ and ψ_μ are two-component spinor satisfying $\gamma_5\psi_\mu = \mu\psi_\mu$.

If equations (2.5) admit again a discrete symmetry then they can further be reduced to one-component uncoupled subsystems.

3 Discrete symmetries and supersymmetries

It was shown in [4] that extended, generalized and reduced supersymmetries appear rather frequently in many quantum-mechanical systems. Here I illustrate only one thing – appearance of extended supersymmetry in the Schrödinger-Pauli equation describing a spin $\frac{1}{2}$ particle interacting with a constant and homogeneous magnetic field $\vec{H}$:

$$\widehat{H}\psi(x) = \left[(-i\vec{\partial} - e\vec{A})^2 - \frac{1}{2}eg\vec{\sigma}.\vec{H}\right]\psi(x) = 0 \tag{3.1}$$

This system is exactly solvable (for details see [4]). One standard supercharge of this equation is

$$\begin{aligned} &Q_1 = \vec{\sigma}(-i\vec{\partial} - e\vec{A}),\\ &Q_1^2 = \widehat{H}. \end{aligned} \tag{3.2}$$

Three other supercharges can be constructed due to the fact that (3.1) is invariant w.r.t. space reflections R_a of x^a, $a = 1, 2, 3$. It was found in [4] that they are of the form:

$$\begin{aligned} &Q_2 = iR_3\vec{\sigma}.(-i\vec{\partial} - e\vec{A}),\\ &Q_3 = iCR_4\vec{\sigma}.(-i\vec{\partial} - e\vec{A}),\\ &Q_4 = iCR_2\vec{\sigma}.(-i\vec{\partial} - e\vec{A}). \end{aligned}$$

They are integrals of motion for (3.1) (notice that without the usual "fermionic" operators) and responsible for degeneracy of the energy spectrum of the system. For many other examples see [4].

References

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[4] Niederle J. and Nikitin A.G., Extended supersymmetries for the Schrödinger-Pauli equation (submitted for publication).