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Symmetries for a Class of Explicitly Space- and Time-Dependent (1+1)-Dimensional Wave Equations

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Abstract

The general d'Alembert equation $\Box u + f(t, x, u) = 0$ is considered, where $\Box$ is the two-dimensional d'Alembert operator. We classify the equation for functions f by which it admits several Lie symmetry algebras, which include the Lorentz symmetry generator. The conditional symmetry properties of the equation are discussed.

1 Introduction

In the present paper, we derive some results on the invariants of the nonlinear wave equation

$$\Box u + f(x_0, x_1, u) = 0, \tag{1.1}$$

where $\Box := \partial^2/\partial x_0^2 - \partial^2/\partial x_1^2$ and f is an arbitrary smooth function of its arguments, to be determined under some invariance conditions.

It is well known that Lie transformation groups play an important role in the investigation of nonlinear partial differential equations (PDEs) in modern mathematical physics. If a transformation leaves a PDE invariant, the PDE is said to possess a symmetry. A particular class of symmetries, known as the Lie point symmetries, has been studied by several authors (see, for example, the books of Ovsyannikov [10], Olver [9], Fushchych *et al.* [7], Ibragimov [8], Steeb [11]). Lie symmetries of nonlinear PDEs may be used to construct exact solutions and conservation laws for the equations (Fushchych *et al.* [7]). The classification of PDEs with respect to their Lie symmetry properties is an important direction in nonlinear mathematical physics. In particular the book of Fushchych, Shtelen and Serov [7] is devoted to the classification of several classes of nonlinear PDEs and systems of PDEs admitting several fundamental Lie symmetry algebras, such as the Poincaré algebra, the Euclidean and Galilean algebras, and the Schrödinger algebra. They mostly consider equations in (1+3)-dimensions as well as arbitrary-dimensional equations, usually excluding the $(1+1)$-dimensional cases. The classification of the $(1+1)$-dimensional wave equation (1.1) is the main theme in the present paper. The invariance of (1.1) with respect to the most general Lie point symmetry generator, Lie symmetry algebras of relativistic invariance, and conditional invariance is considered. We present the theorems without proofs. The proofs are given in Euler *et al.* [2].

2 The General Lie point symmetry generator

Before we classify (1.1) with respect to a particular set of Lie symmetry generators, we establish the general invariance properties of (1.1).

Theorem 1. *The most general Lie point symmetry generator for (1.1) is of the form*

$$Z = \{g_1(y_1) + g_2(y_2)\}\,\frac{\partial}{\partial x_0} + \{g_1(y_1) - g_2(y_2)\}\,\frac{\partial}{\partial x_1} + \{ku + h(x_0, x_1)\}\,\frac{\partial}{\partial u}, \tag{2.1}$$

where g_1, g_2, and h are arbitrary smooth functions of their arguments and $k \in \mathcal{R}$. One must distinguish between three cases:

a) For $g_1 \neq 0$ and $g_2 \neq 0$, the following form of (1.1) admits (2.1):

$$\Box u + \frac{\exp(k\varepsilon)}{g_1(y_1)g_2(y_2)}\left\{-4\int g_1(y_1)g_2(y_2)\frac{\partial^2 h}{\partial y_1 \partial y_2}\exp(-k\varepsilon)\,d\varepsilon + G(Y_1, Y_2)\right\} = 0. \tag{2.2}$$

Here, G is an arbitrary smooth function of its arguments, and

$$\frac{dy_1}{d\varepsilon} = 2g_1(y_1), \qquad \frac{dy_2}{d\varepsilon} = 2g_2(y_2),$$
$$Y_1 = \int \frac{dy_1}{g_1(y_1)} - \int \frac{dy_2}{g_2(y_2)},$$
$$Y_2 = u\exp(-k\varepsilon) - \int h\,(\varepsilon)\exp(-k\varepsilon)\,d\varepsilon,$$
$$y_1 = x_0 + x_1, \qquad y_2 = x_0 - x_1.$$

b) For $g_1 = 0$ and $g_2 \neq 0$, the following form of (1.1) admits (2.1):

$$\Box u + G(Y_1, Y_2)\, g_2(y_2)^{-1}\exp(k\varepsilon) = 0, \tag{2.3}$$

where G is an arbitrary smooth function of its arguments, and

$$Y_1 = x_0 + x_1, \qquad Y_2 = u\exp(-k\varepsilon) - \int h(\varepsilon)\exp(-k\varepsilon)d\varepsilon,$$
$$\frac{dy_2}{d\varepsilon} = 2g_2(y_2), \qquad y_2 = x_0 - x_1.$$

c) For $g_1 \neq 0$ and $g_2 = 0$, the following form of (1.1) admits (2.1):

$$\Box u + G(Y_1, Y_2)\, g_1(y_1)^{-1}\exp(k\varepsilon) = 0, \tag{2.4}$$

where G is an arbitrary smooth function of its arguments, and

$$Y_1 = x_0 - x_1, \qquad Y_2 = u\exp(-k\varepsilon) - \int h(\varepsilon)\exp(-k\varepsilon)d\varepsilon,$$
$$\frac{dy_1}{d\varepsilon} = 2g_1(y_1), \qquad y_1 = x_0 + x_1.$$

3 A particular Lie symmetry algebra

As a special case of the above general invariance properties, we now turn to the classification of (1.1) with respect to the invariance under the Lorentz, scaling, and conformal transformations, the Lie generators of which are given by

$$\begin{aligned} &L_{01} = x_1\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_1}, \qquad S = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \lambda u\frac{\partial}{\partial u}, \\ &K_0 = (x_0^2 + x_1^2)\frac{\partial}{\partial x_0} + 2x_0x_1\frac{\partial}{\partial x_1} + \alpha(x_0,x_1)\frac{\partial}{\partial u}, \\ &K_1 = -2x_0x_1\frac{\partial}{\partial x_0} - (x_0^2 + x_1^2)\frac{\partial}{\partial x_1} - \beta(x_0,x_1)\frac{\partial}{\partial u}. \end{aligned} \tag{3.1}$$

Here, α and β are arbitrary smooth functions and $\lambda \in \mathcal{R}$, to be determined for the particular Lie symmetry algebras. We are interested in the 4-dimensional Lie symmetry algebra spanned by $\{L_{01}, S, K_0, K_1\}$, the 3-dimensional Lie symmetry algebra spanned by $\{L_{01}, K_0, K_1\}$, the 2-dimensional case $\{L_{01}, S\}$ as well as the invariance of (1.1) under the Lorentz transformation generated by $\{L_{01}\}$. The following Lemma gives the conditions on α and β for the closure of the Lie algebras:

Lemma.
a) The generators $\{L_{01}, S, K_0, K_1\}$ span the 4-dimensional Lie algebra with commutation relations as given in the commutator table below if and only if

$$\alpha(x_0,x_1) = cx_0(x_0^2 - x_1^2)^{\lambda/2}, \qquad \beta(x_0,x_1) = cx_1(x_0^2 - x_1^2)^{\lambda/2}, \tag{3.2}$$

where c is an arbitrary real constant.

b) The generators $\{L_{01}, K_0, K_1\}$ span the 3-dimensional Lie algebra with commutations as given in the commutator table below if and only if

$$\begin{aligned} &\alpha(x_0,x_1) = (x_0 + x_1)\phi(y) + (x_0 + x_1)^{-1}\psi(y), \\ &\beta(x_0,x_1) = (x_0 + x_1)\phi(y) - (x_0 + x_1)^{-1}\psi(y), \end{aligned} \tag{3.3}$$

where ϕ and ψ are restricted by the condition

$$y^2\frac{d\phi}{dy} - y\frac{d\psi}{dy} + \psi = 0, \tag{3.4}$$

with $y = x_0^2 - x_1^2$.

Commutator Table

	L_{01}	S	K_0	K_1
L_{01}	0	0	$-K_1$	$-K_0$
S	0	0	K_0	K_1
K_0	K_1	$-K_0$	0	0
K_1	K_0	$-K_1$	0	0

Using the Lemma, we can prove the following four theorems:

Theorem 2. *Equation (1.1) admits the 4-dimensional Lie symmetry algebra spanned by the Lie generators* $\{L_{01}, S, K_0, K_1\}$ *given by (3.1) if and only if* α, β, *and equation (1.1) are of the following forms:*

a) For $\lambda \neq 0$,

$$\alpha(x_0, x_1) = c_1 x_0 (x_0^2 - x_1^2)^{\lambda/2}, \qquad \beta(x_0, x_1) = c_1 x_1 (x_0^2 - x_1^2)^{\lambda/2},$$

whereby (1.1) takes the form

$$\Box u - \lambda c_1 y^{(\lambda-2)/2} + y^{-2} c_2 \left(u - \frac{c_1}{\lambda} y^{\lambda/2} \right)^{(\lambda+2)/\lambda} = 0 \tag{3.5}$$

with $c_1, c_2 \in \mathcal{R}$ *and* $y = x_0^2 - x_1^2$.

b) For $\lambda = 0$,

$$\alpha(x_0, x_1) = c_1 x_0, \qquad \beta(x_0, x_1) = c_1 x_1,$$

whereby (1.1) takes the form

$$\Box u + y^{-1} \exp\left(-\frac{2}{c_1} u \right) = 0 \tag{3.6}$$

with $c_1 \in \mathcal{R} \backslash \{0\}$ *and* $y = x_0^2 - x_1^2$.

Theorem 3. *Equation (1.1) admits the 3-dimensional Lie symmetry algebra spanned by the Lie generators* $\{L_{01}, K_0, K_1\}$ *given by (3.1) if and only if* α, β, *and equation (1.1) are of the following forms:*

a) For f *linear in* u, *we yield*

$$\alpha(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_1 y^{-1} \ln y + k_4 \right\} + (x_0 + x_1)^{-1} \left\{ k_1 \ln y + k_2 y + k_3 \right\},$$

$$\beta(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_1 y^{-1} \ln y + k_2 \right\} - (x_0 + x_1)^{-1} \left\{ k_1 \ln y + k_2 y + k_3 \right\},$$

and (1.1) takes the form

$$\Box u - \frac{1}{y^2} \left(\frac{2k_1}{k_4 - k_2} u + \frac{2k_1 (k_3 + k_1)}{k_4 - k_2} y^{-1} + \frac{2k_1^2}{k_4 - k_2} y^{-1} \ln y - 4k_1 \ln y + k_5 \right) = 0,$$

where $y = x_0^2 - x_1^2$ *and* $k_1, \ldots, k_5$ *are arbitrary real constants with* $k_1 \neq 0$, $k_4 \neq k_2$.

b) For f *independent of* u, *we have*

$$\alpha(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_4 \right\} + (x_0 + x_1)^{-1} \left\{ k_2 y + k_3 \right\},$$

$$\beta(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_4 \right\} - (x_0 + x_1)^{-1} \left\{ k_2 y + k_3 \right\},$$

and (1.1) takes the form

$$\Box u + c y^{-2} = 0,$$

where k_2, k_3, k_4 *are arbitrary real constants and* $y = x_0^2 - x_1^2$.

c) For f nonlinear in u, it holds that

$$\alpha(x_0,x_1) = (x_0+x_1)\left\{k_3 y^{-1} + k_2\right\} + (x_0+x_1)^{-1}\left\{k_2 y + k_3\right\},$$

$$\beta(x_0,x_1) = (x_0+x_1)\left\{k_3 y^{-1} + k_2\right\} - (x_0+x_1)^{-1}\left\{k_2 y + k_3\right\},$$

whereby (1.1) takes the form

$$\Box u + y^{-2} g\left(u - k_2 \ln y + k_3 y^{-1}\right) = 0. \tag{3.7}$$

Here, k_2 and k_3 are arbitrary real constants, $y = x_0^2 - x_1^2$, and g is an arbitrary smooth function of its argument.

Theorem 4. *Equation (1.1) admits the 2-dimensional Lie symmetry algebra spanned by the Lie generators $\{L_{01}, S\}$ given by (3.1) if and only if (1.1) takes the following forms:*

a) For $\lambda = 0$, (1.1) takes the form

$$\Box u + y^{-1} g(u) = 0,$$

where g is an arbitrary function of its argument and $y = x_0^2 - x_1^2$.

b) For $\lambda \neq 0$, (1.1) takes the form

$$\Box u + u^{(\lambda-2)/\lambda} g\left(y^{-\lambda/2} u\right) = 0, \tag{3.8}$$

where g is an arbitrary function of its argument and $y = x_0^2 - x_1^2$.

Theorem 5. *Equation (1.1) admits the Lorentz transformation generated by $\{L_{01}\}$ if and only if (1.1) takes the form*

$$\Box u + g(y, u) = 0, \tag{3.9}$$

where g is an arbitrary function of its arguments and $y = x_0^2 - x_1^2$.

4 Lie symmetry reductions

In this section, we reduce the nonlinear equations stated in the above theorems to ordinary differential equations. This is accomplished by the symmetry Ansätze which are obtained from the first integrals of the Lie equations.

The invariants and Ansätze of interest are listed in Table 1 and the corresponding reductions in Table 2.

Remark. The properties of the reduced equations may, for example, be studied by the use of Lie point transformations and the Painlevé analysis. Some of the equations listed in Table 2 were considered by Euler [3]. In particular, the transformation properties of the equation

$$\ddot{\varphi} + f_1(\omega)\dot{\varphi} + f_2(\omega)\varphi + f_3(\omega)\varphi^n = 0,$$

where f_1, f_2, and f_3 are smooth functions and $n \in \mathcal{Q}$, were studied in detail by Euler [3].

Table 1

Generator	ω	$u(x_0,x_1)=f_1(x_0,x_1)\varphi(\omega)+f_2(x_0,x_1)$
L_{01}	$\omega=x_0^2-x_1^2$	$f_1=1, \qquad f_2=0$
S	$\omega=\dfrac{x_0}{x_1}$	$f_1=x_0^{\lambda}, \qquad f_2=0$
K_0	$\omega=\dfrac{x_0^2-x_1^2}{x_1}$	Theorem 2a: $f_1=1, \qquad f_2=\dfrac{c_1}{\lambda}\omega^{\lambda/2}x_1^{\lambda/2}$ Theorem 2b: $f_1=1, \qquad f_2=\dfrac{c_1}{2}\ln x_1$ Theorem 3c: $f_1=1, \qquad f_2=k_2\ln x_1-\dfrac{k_3}{\omega x_1}$
K_1	$\omega=\dfrac{x_0^2-x_1^2}{x_0}$	Theorem 2a: $f_1=1, \qquad f_2=\dfrac{c_1}{\lambda}\omega^{\lambda/2}x_0^{\lambda/2}$ Theorem 2b: $f_1=1, \qquad f_2=\dfrac{c_1}{2}\ln x_0$ Theorem 3c: $f_1=1, \qquad f_2=k_2\ln x_0-\dfrac{k_3}{\omega x_0}$

Table 2

We refer to ...	Reduced Equation
Theorem 2a	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}-c_1\lambda\omega^{(\lambda-2)/2}+c_2\omega^{-2}\left(\varphi-\dfrac{c_1}{\lambda}\omega^{\lambda/2}\right)^{(\lambda+2)/\lambda}=0$ S : $\omega^2(\omega^2+1)\ddot{\varphi}-2\omega(\omega^2-\lambda)\dot{\varphi}+\lambda(\lambda-1)\varphi-c_1\lambda(1-\omega^{-2})^{(\lambda-2)/2}$ $+c_2(1-\omega^{-2})^{-2}\left(\varphi-\dfrac{c_1}{\lambda}(1-\omega^{-2})^{\lambda/2}\right)^{(\lambda+2)/\lambda}=0$ K_0^- and K_1^+ : $\omega^2\ddot{\varphi}+2\omega\dot{\varphi}\mp c_2\omega^{-2}\varphi^{(\lambda+2)/\lambda}=0$
Theorem 2b	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+\omega^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$ S : $(\omega^2-1)\ddot{\varphi}+2\omega\dot{\varphi}+(1-\omega^2)^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$ K_0^+ and K_1^- : $\omega^2\ddot{\varphi}+2\omega\dot{\varphi}\pm\dfrac{c_1}{2}+\omega^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$

Table 2 (Continued)

We refer to ...	Reduced Equation
Theorem 3c	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+\omega^{-2}g\left(\varphi-k_2\ln\omega+k_3\omega^{-1}\right)=0$ K_0^- and K_1^+ : $\omega^2\ddot{\varphi}+2\omega\dot{\varphi}+4k_3\omega^{-2}-k_2\mp\omega^{-2}g\left(\varphi-k_2\ln\omega\right)=0$
Theorem 4a	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+\omega^{-1}g(\varphi)=0$ S : $(1-\omega^2)\ddot{\varphi}-2\omega\dot{\varphi}-(1-\omega^2)^{-1}g(\varphi)=0$
Theorem 4b	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+\varphi^{(\lambda-2)/\lambda}g\left(\omega^{-\lambda/2}\varphi\right)=0$ S : $2\omega^2(1-\omega^2)\ddot{\varphi}+2\omega(\lambda-\omega^2)\dot{\varphi}+\lambda(\lambda-1)\varphi$ $+\varphi^{(\lambda-2)/\lambda}g\left((\omega^2-1)^{-\lambda/2}\varphi\right)=0$
Theorem 5	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+g(\omega,\varphi)=0$

5 Conditional symmetries

An extension of the classical Lie symmetry reduction of PDEs may be realized as follows: Consider the compatibility problem posed by the following two equations

$$F\equiv\Box u+f(x_0,x_1,u)=0, \tag{5.1}$$

$$Q\equiv\xi_0(x_0,x_1,u)\frac{\partial u}{\partial x_0}+\xi_1(x_0,x_1,u)\frac{\partial u}{\partial x_1}-\eta(x_0,x_1,u)=0. \tag{5.2}$$

Here, (5.1) is the invariant surface condition for the symmetry generator

$$Z=\xi_0(x_0,x_1,u)\frac{\partial}{\partial x_0}+\xi_1(x_0,x_1,u)\frac{\partial}{\partial x_1}+\eta(x_0,x_1,u)\frac{\partial}{\partial u}.$$

A necessary and sufficient condition of compatibility on ξ_0, ξ_1, and η is given by the following invariance condition (Fushchych *et al.* [7], Euler *et al.* [4], Ibragimov [8])

$$Z^{(2)}\Big|_{F=0,\ Q=0}=0. \tag{5.3}$$

A generator Z satisfying (5.3) is known as a Q-conditional Lie symmetry generator (Fushchych *et al.* [7]). Note that conditional symmetries were first introduced by Bluman and Cole [1] in their study of the heat equation.

Let us now study the Q-symmetries of (1.1). It turns out that it is more convenient to transform (1.1) in light-cone coordinates, i.e., the transformation

$$x_1 \to \frac{1}{2}(x_0 + x_1), \qquad x_0 \to \frac{1}{2}(x_0 - x_1), \qquad u \to u.$$

Without changing the notation, we now consider the system (written in jet coordinates)

$$F \equiv u_{01} + f(x_0, x_1, u) = 0,$$
$$Q \equiv u_0 + \xi_1(x_0, x_1, u)u_1 - \eta(x_0, x_1, u) = 0,$$

where we have normalized ξ_0. After applying the invariance condition (5.3) and equating to zero the coefficients of the jet coordinates 1, u_1, u_1^2, u_1^3, u_{11}, and u_1u_{11}, we obtain the nonlinear determining equations:

$$\frac{\partial \xi_1}{\partial u} = 0, \qquad \frac{\partial \xi_1}{\partial x_0} = 0, \qquad \frac{\partial^2 \eta}{\partial u^2}\xi_1 = 0, \tag{5.4}$$

$$\frac{\partial^2 \eta}{\partial x_0 \partial u} - \frac{\partial^2 \xi_1}{\partial x_0 \partial x_1} + \frac{\partial^2 \eta}{\partial u^2}\eta - \frac{\partial^2 \eta}{\partial x_1 \partial u}\xi_1 = 0, \tag{5.5}$$

$$\frac{\partial f}{\partial x_0} + \xi_1 \frac{\partial f}{\partial x_1} + \eta \frac{\partial f}{\partial u} + f\left(\frac{\partial \xi_1}{\partial x_1} - \frac{\partial \eta}{\partial u}\right) + \frac{\partial^2 \eta}{\partial x_1 \partial u}\eta + \frac{\partial^2 \eta}{\partial x_0 \partial x_1} = 0. \tag{5.6}$$

According to (5.4), we need to consider two cases:

Case 1. $\dfrac{\partial^2 \eta}{\partial u^2} = 0$ and $\xi_1 = \xi_1(x_1)$.
By solving (5.5), η takes on the form

$$\eta(x_0, x_1, u) = \phi(z)u + h(x_0, x_1), \qquad z = x_0 + \int \frac{dx_1}{\xi_1(x_1)}, \tag{5.7}$$

where ϕ and h are arbitrary smooth functions of their arguments. The condition on f is given by (5.6), i.e., the following linear first order PDE

$$\begin{aligned} &\frac{\partial f}{\partial x_0} + \xi(x_1)\frac{\partial f}{\partial x_1} + (\phi(z)u + h(x_0, x_1))\frac{\partial f}{\partial u} + \left(\frac{d\xi_1}{dx_1} - \phi(z)\right) f \\ &\quad + \frac{u}{\xi_1(x_1)}\left(\phi'(z)\phi(z) + \phi''(z)\right) + \frac{h(x_0, x_1)}{\xi(x_1)} + \frac{\partial^2 h}{\partial x_0 \partial x_1} = 0. \end{aligned} \tag{5.8}$$

Since ϕ is not a constant, as in the case of a Lie symmetry generator (see Theorem 1), it is clear that there exist non-trivial Q-symmetry generators of the form

$$Z = \frac{\partial}{\partial x_0} + \xi_1(x_1)\frac{\partial}{\partial x_1} + \{\phi(z)u + h(x_0, x_1)\}\frac{\partial}{\partial u}.$$

For given functions ϕ, h, and ξ_1, the form of f may be determined by solving (5.8).

Case 2. $\dfrac{\partial^2 \eta}{\partial u^2} \neq 0$ and $\xi_1 = 0$.
The determining equations reduce to

$$\frac{\partial^2 \eta}{\partial x_0 \partial u} + \frac{\partial^2 \eta}{\partial u^2}\eta = 0, \qquad \frac{\partial f}{\partial x_0} + \eta\frac{\partial f}{\partial u} - \frac{\partial \eta}{\partial u}f + \frac{\partial^2 \eta}{\partial x_1 \partial u}\eta + \frac{\partial^2 \eta}{\partial x_0 \partial x_1} = 0 \tag{5.9}$$

Any solution of (5.9) determines f and η for which system (5.1)–(5.2) is compatible. In this case, the non-trivial Q-symmetry generators are of the form

$$Z = \frac{\partial}{\partial x_0} + \eta(x_0, x_1, u)\frac{\partial}{\partial u}.$$

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