Symmetry in Nonlinear Mathematical Physics 1997, V.2, 483–487.

# One Geometric Model for Non-local Transformations

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**Abstract**

A geometric formulation of an important class of non-local transformations is presented.

The contemporary geometric theory of Bäcklund transformations (BT) [1–4] is based on certain integrability conditions for a system of differential equations which are determine it. These equations are written in terms of connections in the submanifold of a jet-bundle foliation. One of the first nontrivial examples of BT in three dimensions was described in paper [1]. In this paper, a geometric formulation of an important class of non-local transformations is presented. The generalization is obtained due to the use of non-local transformations of dependent variables and integrability conditions of a more general type.

**1.** In the fibered submanifold $(E^j, M^{'}, \rho^{'})$ with the $k$-jet bundle $J^1(M^{'}N^{'})$, we set an equation of order $t = 2 \leq l$:

$$\begin{aligned} &L_2^q\left(y, v, \underset{1}{v}, \underset{2}{v}\right) = 0, \\ &v = \left\{v^B\right\}, \quad M^{'} = R\left(1, \ldots, n-1\right), \quad \left(q = 1, \ldots, m^{'},\ B = 1, \ldots, m^{'}\right). \end{aligned} \tag{1}$$

We use here such notations:

$$\partial_\mu u = \frac{\partial u}{\partial x_\mu}, \qquad \{x_\mu\} = (x_0, x_1, \ldots, x_{n-1}).$$

Let now equation (1) be written in the form of n-th order exterior differential forms ($n$-forms)

$$\alpha^c = \frac{1}{n!}\alpha^c_{\mu_0\cdots\mu_{n-1}}(y, v)dy^{\mu_0} \wedge dy^{\mu_1} \wedge \ldots \wedge dy^{\mu_{n-1}}, \quad (c = 1, \ldots, r). \tag{2}$$

When this system of forms $\alpha^c$ is equal to zero, then (1) is fulfiled. So, the system of $\alpha^c = 0$ forms generates an ideal $I = \{\alpha^c\}$. The condition for system (2) to be closed is $d\alpha^c \subset I$. Let now consider 1-forms of connections which generate the standard basis of contact forms

$$\omega^A = du^A - H^A_\mu\left(x, v, \underset{1}{v}; u\right)dx^\mu, \qquad (\mu = 0, \ldots, n-1,\ A = 1, \ldots, m). \tag{3}$$

Let

$$x^\mu = y^\mu, \qquad (\mu = 0, \ldots, n-1).$$

The differential prolongation of (3) gives us such a system of contact forms:

$$\begin{aligned} \omega^A_{\mu_1} &= du^A_{\mu_1} - H^A_{\mu_1\mu}\left(x, v, \underset{1}{v}, \underset{2}{v}; u, \underset{1}{u}\right) dx^\mu, \dots, \\ \omega^A_{\mu_1\dots\mu_k} &= du^A_{\mu_1\dots\mu_k\mu} - H^A_{\mu_1\dots\mu_k\mu}\left(x, v, \underset{1}{v}, \underset{2}{v}; u, \dots, \underset{k}{u}\right) dx^\mu. \end{aligned} \tag{4}$$

Let us construct the set of $(n-1)$-forms

$$\Omega^B = \beta^B_A \wedge \omega^A, \quad (B = 1, \dots, m), \tag{5}$$

where $\beta^B_A$ are some $(n-2)$-forms of the type

$$\beta^B_A = \frac{1}{(n-2)!}\beta^B_{A\mu_1\dots\mu_{n-2}}(x, u) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_{n-2}}.$$

An extended ideal $I'$ is obtained by adding forms (5) to $I$:

$$I' = \{\alpha^A, \Omega^B\}.$$

Demand $\Omega^B$ be closed:

$$d\Omega^B \subset I'. \tag{6}$$

This condition is given by

$$d\Omega^B = \alpha^A f^B_A + \theta \wedge \Omega^B, \tag{7}$$

where $\theta$ is some 1-form such as, for example,

$$\theta = \theta_\mu dx^\mu, \qquad \theta_\mu = \text{const}.$$

Equation (7) is the basic for determining the functions $H^A_{\mu_1\dots\mu_k\mu}$ from connections (3) and for finding $\beta^B_A$ and $f^B_A$ coefficients . In [1], $\beta^B_{A\mu_1\dots\mu_{n-2}}$ satisfy such commuting conditions, from which it follows

$$\begin{aligned} I^{0A} &= -\beta^{0A}_c H^c_2 + \beta^{2A}_c H^c_1, \\ I^{1A} &= -\beta^{1A}_c H^c_1 + \beta^{0A}_c H^c_0, \\ I^{2A} &= -\beta^{2A}_c H^c_0 + \beta^{1A}_c H^c_2. \end{aligned} \tag{8}$$

It makes us possible to write the equation $L^q_2(x, v) = 0$ in the form of the conservation law

$$\partial_\mu I^{\mu A} = 0. \tag{9}$$

$H^c_k$ have a special structure, which allows us to write

$$I^{\mu A} = X_a(x, v)[v^A] \cdot f^{\mu a}\left(x, u, \underset{1}{u}\right), \qquad (\mu = 0, 1, 2;\ a = 1, \dots, r'), \tag{10}$$

and then obtain the incomplete Lie algebra

$$X_a(x, v) = X^c_a(x, v)\partial_{v^c}. \tag{11}$$

Additional conditions on $X_a$ allow it to become a complete Lie algebra.

**2.** In the space $E^1(y,v)$, let an invariance Lie algebra of infinitesimal operators for the equation $L_2^q(x,v)=0$ be given:

$$X_a = \xi_a^\mu(y,v)\partial_\mu + \eta_a^B(y,v)\partial_{v^B}, \quad [X_a,X_b]=\lambda_{ab}^c X_c, \ (a=1,\ldots,r;\ B=1,\ldots,m) \tag{12}$$

and the projection $\rho' : E^1 \to M'$ in a fibre bundle$(E', M', \rho')$ to each operator $X_a$ set the corresponding shorted operator

$$X_a^{\rho'} = \xi_a^\mu(y,v)\partial_\mu. \tag{13}$$

Operator (12) acting in $E^1$ is closely connected with an operator, which acts in $M'$ [5]

$$Q_a(y,v) = \xi_a^\mu(y,v)\partial_\mu - \eta_a^*(y,v), \quad \eta_a^* v^B = \eta_a^B. \tag{14}$$

Determine now $Q_a$-invariant solutions $v^{inv}$ of the equation $L_1^q(y,v)=0$:

$$Q_a(y,v)[v^B] = \xi_a^\mu(y,v)\partial_\mu v^B - \eta_a^B(y,v) = 0. \tag{15}$$

If $u^A(x) \neq v^B(y)$, then it follows from (15) that

$$Q_a(y,v)[u^A] = \Gamma_{ac}^A\left(y, v, \underset{1}{v}, \ldots\right) u^c \neq 0. \tag{15a}$$

In more general case, it is

$$Q_a(y,v)[u^A] = \Gamma_a^A\left(y, v, \underset{1}{v}, \ldots; u\right) \neq 0. \tag{15b}$$

Let now consider 1-forms of connections (3)

$$\omega^A = (\partial_{x^\mu} u^A)dx^\mu - H_\mu^A\left(x, v, \underset{1}{v}; u\right) dx^\mu. \tag{16}$$

Interior product of $\omega^A$ and the vector field

$$W_\nu = D_{y^\nu} h^\mu \partial_{x^\mu}, \tag{17}$$

where

$$\|W\| = \|Dh\| \cdot \left\|\underset{1}{\partial}\right\|, \qquad \underset{1}{\partial} \equiv \|\partial_0, \partial_1, \ldots, \partial_{n-1}\|^T,$$

is obtained [2–4] in the form

$$W_\nu \lrcorner\, \omega^A = D_{y^\nu} h^\mu \partial_{x^\mu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A. \tag{18}$$

Here, we use the notation

$$D_{y^\nu} h^\mu \partial_{x^\mu}|_M = \partial_{y^\nu}|_{M'}. \tag{19}$$

So

$$W_\nu \lrcorner\, \omega^A = \partial_{y^\nu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A. \tag{20}$$

For each fixed $\nu$, we multiply the scalar equation (20) by a 1-form $dy^\nu$ and then find the sum

$$\tilde{\omega}^A = \left[\partial_{y^\nu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A\right] dy^\nu = du^A - D_{y^\nu} h^\mu \cdot H_\mu^A \cdot dy^\mu. \tag{21}$$

An interior product of the vector field (13) and the form (21) is

$$X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A = \xi_a^\nu(y,v)\partial_{y^\nu}u^A - \eta_a^*(y,v)u^A - \left[\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A\right]. \tag{22}$$

Let us set

$$\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A \equiv \Gamma_{ac}^A\left(y,v,\underset{1}{v},\dots\right)u^c. \tag{23}$$

In a more general case, we have

$$\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A \equiv \Gamma_a^A\left(y,v,\underset{1}{v},\dots,u\right). \tag{24}$$

In new notations, (22) will have the form

$$X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A = Q_a(y,v)[u^A] - \Gamma_a^A\left(y,v,\underset{1}{v},\dots,u\right). \tag{25}$$

When (25) is equal to zero, it determines a linear connection along the vector field $X_a^{\rho'}$.

We calculate now the exterior differential of scalar (25) and mark a result as $\overset{\sim}{\omega}^A$. The interior product of this form and the vector field $X_a^{\rho'}$ is

$$\begin{aligned}X_b^{\rho'} \lrcorner \overset{\sim}{\omega}^A &= \xi_b^\mu(y,v)\partial_{y^\mu}\left(X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A\right) = \xi_b^\mu(y,v)\partial_{y^\mu}(Q_a[u^A]) - \xi_b^\mu(y,v)\partial_{y^\mu}\Gamma_a^A = \\ &= \left(\xi_b^\mu(y,v)\partial_{y^\mu} - \eta_b\right)Q_a[u^A] + \eta_b Q_a[u^A] - \xi_b^\mu(y,v)\partial_{y^\mu}\Gamma_a^A = Q_bQ_a[u^A] - Q_b\Gamma_a^A, \\ &\left(X_b^{\rho'} \lrcorner \overset{\sim}{\omega}^A = 0\right).\end{aligned} \tag{26}$$

With (26), let construct an equality

$$\{[Q_a,Q_b] - \lambda_{ab}^c Q_c\}\, u^A = \overset{\sim}{Q}_{[a}\,\Gamma_{b]}^A + \Gamma_{[a|u^c|}^A\Gamma_{b]}^c - \lambda_{ab}^c\Gamma_c^A\,|_{L_2(y,v)} = 0. \tag{27}$$

Here, $\overset{\sim}{Q}_a$ is the projection of the operator $Q_a$ on $E'$ (not differentiate with respect to $u$-variables)

$$\Gamma_{au^c}^A \equiv \partial_{u^c}\Gamma_a^A.$$

If the connection is linear, equation (27) is of the form

$$\{Q_{[a}\Gamma_{b]c}^A + \Gamma_{[a|K|}^A\Gamma_{b]c}^K - \lambda_{ab}^p\Gamma_{pc}^A\}u^c\,|_{L_2^q} = 0. \tag{27a}$$

So the equation, which determines the reducing of a non-local transformation system to the equation $L_2^q(y,v) = 0$, is ($X_a^{\rho'} \equiv X_a'$):

$$*\left\{\underset{X'_{[a}}{L}\,\underset{X'_{b]}}{L}\,u^A - \lambda_{ab}^c\,\underset{X'_c}{L}\,u^A\right\} \subset I'. \tag{28}$$

Here, we use the notation [2, 3]:

$$\underset{X}{L}\,u = X \lrcorner du, \qquad \Gamma = \left\{\alpha^c, X \lrcorner \overset{\sim}{\omega}^A\right\}.$$

The system of exterior differential equations $\alpha^c = 0$ gives us the representation of $L_2^q(y,v) = 0$. Condition (28) shows that

$$*\left\{X^{\rho'}_{[a}\lrcorner d(X^{\rho'}_{b]}\lrcorner\,\widetilde{\omega}^A) - \lambda^c_{ab}(X^{\rho'}_c\lrcorner\,\widetilde{\omega}^A)\right\} - f^A_c\cdot\alpha^c - *g^{As}_c\left(X^{\rho'}_s\lrcorner\,\widetilde{\omega}^A\right) = 0. \qquad (28a)$$

$*$ is the Hodge operator in $R(0,n)$.

Equation (28a) can be represented in terms of covariant derivatives $\nabla_{X_a}$ along the vector field $X_a$. If connections are linear, the covariant and Lie derivatives are identical. So, from

$$\{Q_{[a}\Gamma^A_{b]c} + \Gamma^A_{[a|K|}\Gamma^K_{b]c} - \lambda^p_{ab}\Gamma^A_{pc}\}u^c\,|_{L_2^q} = 0,$$

we obtain

$$\left\{\left[\nabla_{X^\rho_a}, \nabla_{X^\rho_b}\right] + \nabla_{[X_a,X_b]^\rho} - R(X_a,X_b)\right\}u^c = 0. \qquad (29)$$

Here,

$$R(X_a,X_b)u^c = \lambda^p_{ab}\Gamma^A_{pc}u^c \qquad (30)$$

is the tensor field of curvature of the corresponding connections [4].

**Theorem.** *The non-local transformation represented by system (25) via integrability conditions (27) for variables $u^A$ has, as a consequence, the equation $L_2^q(y,v) = 0$ when (28a) is fulfiled.*

# References


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