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Fermionic Symmetries of the Maxwell Equations with Gradient-like Sources

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Abstract

The Maxwell equations with gradient-like sources are proved to be invariant with respect to both bosonic and fermionic representations of the Poincar'e group and to be the kind of Maxwell equations with maximally symmetric properties. Nonlocal vector and tensor-scalar representations of the conformal group are found, which generate the transformations leaving the Maxwell equations with gradient-like sources being invariant.

1. Introduction

The relationship between the massless Dirac equation and the Maxwell equations attracts the interest of investigators [1–21] since the creation of quantum mechanics. In [8, 11], one can find the origin of the studies of the most interesting case where mass is nonzero and the interaction in the Dirac equation is nonzero too. As a consequence, the hydrogen atom can be described [11, 19–21] on the basis of the Maxwell equations. Starting from [10], the Maxwell equations with gradient-like sources have appeared in consideration. From our point of view, it is the most interesting kind of the Maxwell equations especially in studying the relationship with the massless Dirac equation. Below we investigate the symmetry properties of this kind of the Maxwell equations.

2. The Maxwell equations with gradient-like sources

Let us choose the γ^μ matrices in the massless Dirac equation

$$i\gamma^\mu\partial_\mu\Psi(x)=0; \quad x\equiv(x^\mu)\in R^4, \quad \Psi\equiv(\Psi^\mu), \quad \partial_\mu\equiv\frac{\partial}{\partial x^\mu}, \quad \mu=0,1,2,3, \tag{1}$$

obeying the Clifford-Dirac algebra commutation relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}, \quad \gamma^{\mu\dagger}=g^{\mu\nu}\gamma_\nu, \quad \mathrm{diag}\, g=(1,-1,-1,-1), \tag{2}$$

in the Pauli-Dirac representation (shortly: PD-representation):

$$\gamma^0=\begin{vmatrix}1&0\\0&-1\end{vmatrix}, \quad \gamma^k=\begin{vmatrix}0&\sigma^k\\-\sigma^k&0\end{vmatrix};$$
$$\sigma^1=\begin{vmatrix}0&1\\1&0\end{vmatrix}, \quad \sigma^2=\begin{vmatrix}0&-i\\i&0\end{vmatrix}, \quad \sigma^3=\begin{vmatrix}1&0\\0&-1\end{vmatrix}. \tag{3}$$

Note first of all that after the substitution of ψ by the following column

$$\psi = \text{column}\left|E^3 + iH^0, \quad E^1 + iE^2, \quad iH^3 + E^0, \quad -H^2 + iH^1\right|, \tag{4}$$

the Dirac equation (1) is transformed into equations for the system of electromagnetic $\left(\vec{E}, \vec{H}\right)$ and scalar $\left(E^0, H^0\right)$ fields:

$$\begin{aligned} &\partial_0 \vec{E} = \text{curl } \vec{H} - \text{grad } E^0, \qquad \partial_0 \vec{H} = -\text{curl } \vec{E} - \text{grad } H^0, \\ &\text{div } \vec{E} = -\partial_0 E^0, \qquad \text{div } \vec{H} = -\partial_0 H^0 \end{aligned} \tag{5}$$

(three other versions of the treatment of these equations are given in [16], the complete set of linear homogeneous connections between the Maxwell and the Dirac equations is given in [14, 17, 18]). In the notations $(E^\mu) \equiv \left(E^0, \vec{E}\right), (H^\mu) \equiv \left(H^0, \vec{H}\right)$, Eqs.(5) have a manifestly covariant form

$$\partial_\mu E_\nu - \partial_\nu E_\mu + \varepsilon_{\mu\nu\rho\sigma} \partial^\rho H^\sigma = 0, \qquad \partial_\mu E^\mu = 0, \qquad \partial_\mu H^\mu = 0. \tag{6}$$

In terms of the complex 4-component object

$$\mathcal{E} = \text{column}\left|E^1 - iH^1, \quad E^2 - iH^2, \quad E^3 - iH^3, \quad E^0 - iH^0\right|, \tag{7}$$

Eqs.(5) have the form

$$\partial_\mu \mathcal{E}_\nu - \partial_\nu \mathcal{E}_\mu + i\varepsilon_{\mu\nu\rho\sigma} \partial^\rho \mathcal{E}^\sigma = 0, \qquad \partial_\mu \mathcal{E}^\mu = 0. \tag{8}$$

The free Maxwell equations are obtained from Eqs. (5), (6), (8) in the case of $E^0 = H^0 = 0$.

The unitary operator V

$$\begin{aligned} &V \equiv \begin{vmatrix} 0 & C_+ & 0 & C_- \\ 0 & iC_- & 0 & iC_+ \\ C_+ & 0 & C_- & 0 \\ C_- & 0 & C_+ & 0 \end{vmatrix}, \quad V^\dagger \equiv \begin{vmatrix} 0 & 0 & C_+ & C_- \\ C_+ & iC_+ & 0 & 0 \\ 0 & 0 & C_- & C_+ \\ C_- & iC_- & 0 & 0 \end{vmatrix}; \\ &C_\pm \equiv \frac{C \pm 1}{2}; \quad VV^\dagger = V^\dagger V = 1. \end{aligned} \tag{9}$$

(in the space where the Clifford-Dirac algebra is defined as a real one, this operator is unitary) transforms the ψ from (4) into the object $\mathcal{E}$ (7):

$$\psi \equiv \begin{vmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} \mathcal{E}^3 + \mathcal{E}^{*3} - \mathcal{E}^0 + \mathcal{E}^{*0} \\ \mathcal{E}^1 + \mathcal{E}^{*1} + i\mathcal{E}^2 + i\mathcal{E}^{*2} \\ \mathcal{E}^0 + \mathcal{E}^{*0} - \mathcal{E}^3 + \mathcal{E}^{*3} \\ -i\mathcal{E}^2 + i\mathcal{E}^{*2} - \mathcal{E}^1 + \mathcal{E}^{*1} \end{vmatrix} \equiv V^{-1}\mathcal{E}, \qquad \mathcal{E} \equiv \begin{vmatrix} \mathcal{E}^1 \\ \mathcal{E}^2 \\ \mathcal{E}^3 \\ \mathcal{E}^0 \end{vmatrix} = V\psi, \tag{10}$$

and the γ^μ matrices (3) in the PD-representation into the bosonic representation (shortly: B-representation)

$$\gamma^\mu \longrightarrow \widetilde{\gamma}^\mu \equiv V\gamma^\mu V^\dagger. \tag{11}$$

Here C is the operator of complex conjugation: $C\Psi = \Psi^*$. The unitarity of the operator V can be proved by means of relations

$$Ca = (aC)^* = a^* C, \quad (AC)^\dagger = CA^\dagger = A^T C \tag{12}$$

for an arbitrary complex number a and a matrix A.

Let us write down the explicit form of the Clifford-Dirac algebra generators in the B-representation

$$\widetilde{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} C, \quad \widetilde{\gamma}^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} C,$$
$$\widetilde{\gamma}^2 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} C, \quad \widetilde{\gamma}^3 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} C. \tag{13}$$

In the B-representation, the complex number i is represented by the following matrix operator

$$\widetilde{i} = ViV^{\dagger} = i\Gamma, \quad \Gamma \equiv \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} = \Gamma^{\dagger} = \Gamma^{-1}, \quad \Gamma^2 = 1. \tag{14}$$

The above-mentioned facts mean that the Maxwell equations (5), (6), (8) can be rewritten in the Dirac-like form

$$\widetilde{i}\widetilde{\gamma}^{\mu}\partial_{\mu}\mathcal{E}(x) = 0, \quad \mathcal{E} = V\Psi^{III}. \tag{15}$$

Even from the explicit form of equations (5), (6), (8), (15), one can suppose that Eqs.(6) and (8) are equations for a vector field, Eqs.(5) are those for the system of tensor and scalar field and Eqs.(15) are for a spinor field. From the electrodynamical point of view, one can interpret Eqs.(5) as the Maxwell equations with fixed gradient-like sources. However, before going from assumptions to assertions, we must investigate the transformation properties of the object $\mathcal{E}$ and symmetry properties of equations (5), (6), (8), (15). The mathematically well-defined assertion that substitutions (4) transform the Dirac equation into the Maxwell equations (5) for the electromagnetic and scalar fields is impossible without the proof that the object $\mathcal{E}$ can be transformed as electromagnetic and scalar fields, i.e., we need the additional arguments in order to have the possibility to interpret the real and imaginary parts of spinor components from (4) as the components of electromagnetic and scalar fields – such arguments can be taken from the symmetry analysis of the corresponding equations.

3. Symmetries

For the infinitesimal transformations and generators of the conformal group $C(1,3) \supset P$, we use the notations:

$$f(x) \to f^{'}(x) \stackrel{i}{=} \left(1 - a^{\rho}\partial_{\rho} - \frac{1}{2}\omega^{\rho\sigma}\widehat{j}_{\rho\sigma} - \kappa\widehat{d} - b^{\rho}\widehat{k}_{\rho}\right) f(x) \tag{16}$$

$$\partial_{\rho} \equiv \frac{\partial}{\partial x^{\rho}}, \quad \widehat{j}_{\rho\sigma} = M_{\rho\sigma} + S_{\rho\sigma}, \quad \widehat{d} = d + \tau = x^{\mu}\partial_{\mu} + \tau,$$
$$\widehat{k}_{\rho} = k_{\rho} + 2S_{\rho\sigma}x^{\sigma} - 2\tau x_{\rho} \equiv 2x_{\rho}\widehat{d} - x^2\partial_{\rho} + 2S_{\rho\sigma}x^{\sigma}, \tag{17}$$

The generators (17) obey the commutation relations

$$\begin{aligned}
&[\partial_\mu, \partial_\nu] = 0, \quad \left[\partial_\mu, \widehat{j}_{\nu\sigma}\right] = g_{\mu\nu}\partial_\sigma - g_{\mu\sigma}\partial_\nu, && \text{(a)}\\
&\left[\widehat{j}_{\mu\nu}, \widehat{j}_{\lambda\sigma}\right] = -g_{\mu\lambda}\widehat{j}_{\nu\sigma} - g_{\nu\sigma}\widehat{j}_{\mu\lambda} + g_{\mu\sigma}\widehat{j}_{\nu\lambda} + g_{\nu\lambda}\widehat{j}_{\mu\sigma}, && \text{(b)}\\
&\left[\partial_\mu, \widehat{d}\right] = \partial_\mu, \quad \left[\partial_\rho, \widehat{k}_\sigma\right] = 2(g_{\rho\sigma}\widehat{d} - \widehat{j}_{\rho\sigma}), \quad \left[\partial_\mu, \widehat{j}_{\mu\sigma}\right] = 0, &&\\
&\left[\hat{k}_\rho, \hat{j}_{\sigma\nu}\right] = g_{\rho\sigma} \hat{k}_\nu - g_{\rho\nu} \hat{k}_\sigma, \quad \left[\hat{d}, \hat{k}_\rho\right] = \hat{k}_\rho, \quad \left[\hat{k}_\rho, \hat{k}_\sigma\right] = 0. &&
\end{aligned} \tag{18}$$

with an arbitrary number τ (the conformal number) and $S_{\rho\sigma}$ matrices being the generators of the 4-dimensional representation of the Lorentz group $SL(2,C)$, e.g., for symmetries of the Dirac (m=0) equation, the conformal number $\tau = 3/2$ and 4×4 $S_{\rho\sigma}$ matrices are chosen in the form

$$S^I_{\rho\sigma} \equiv -\frac{1}{4}\left[\gamma_\rho, \gamma_\sigma\right] \in D\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right), \tag{19}$$

where $S^I_{\rho\sigma}$ are generators of the spinor representation $D\left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},0\right)$ of the $SL(2,C)$ group. Let us define the following six operators from the Pauli-Gursey-Ibragimov symmetry [22]:

$$S^{II}_{\rho\sigma} = -S^{II}_{\sigma\rho}: \quad \left\{ \begin{array}{lll} S^{II}_{01} = \frac{i}{2}\gamma^2 C, & S^{II}_{02} = -\frac{1}{2}\gamma^2 C, & S^{II}_{03} = -\frac{i}{2}\gamma^4, \\ S^{II}_{12} = -\frac{i}{2}, & S^{II}_{31} = -\frac{1}{2}\gamma^2\gamma^4 C, & S^{II}_{23} = \frac{i}{2}\gamma^2\gamma^4 C \end{array} \right\}. \tag{20}$$

It is easy to verify that operators (20) obey same commutation relations as generators (19) and, as a consequence, they form an another realization of the same spinor representation $D\left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},0\right)$ of the $SL(2,C)$ group. But, in contradiction to operators (19), they are themselves (without any differential angular momentum part) the symmetry operators of the massless Dirac equation (1), i.e., they leave this equation being invariant.

We prefer to use the Dirac-like form (15) of the Maxwell equations with gradient-like sources for the symmetry analysis. The operator equality $V\widehat{Q}_\psi V^\dagger = q_E$ allows one to find the connections between the symmetries of the Dirac equation (1) and those of equation (15) for the field $\mathcal{E} = (\mathcal{E}^\mu)$. It was shown in [15, 16] that the massless Dirac equation is invariant (in addition to the standard spinor representation) with respect to two bosonic representations of Poincaré group being generated by the $D\left(\frac{1}{2},\frac{1}{2}\right)$ and $D\left(1,0\right) \oplus \left(0,0\right)$ representations of the Lorentz group (for the 8-component form of the Dirac equation, the similar representations of Poincar'e group being generated by the reducible $D\left(\frac{1}{2},\frac{1}{2}\right) \oplus \left(\frac{1}{2},\frac{1}{2}\right)$ and $D\left(1,0\right) \oplus \left(0,0\right) \oplus \left(0,0\right) \oplus \left(0,1\right)$ representations of the Lorentz group were found in [13]). Of course, Eqs.(15) for the field $\mathcal{E}$ (the Maxwell equations with gradient-like sources) are also invariant with respect to these three different representations of the Poincar'e group that is evident due to the unitary connection (9) between the fields ψ and $\mathcal{E}$. Let us prove this fact directly.

Let us write down the explicit form of the $S^I_{\rho\sigma}$ and $S^{II}_{\rho\sigma}$ operators (19) and (20) in the B-representation

$$\widetilde{S}^I_{\rho\sigma} = -\frac{1}{4}\left[\widetilde{\gamma}_\rho, \widetilde{\gamma}_\sigma\right]: \quad \widetilde{S}^I_{jk} = -i\varepsilon^{jkl}\widetilde{S}^I_{0l}, \quad \widetilde{S}^I_{0l} = -\frac{1}{2}\widetilde{\gamma}^{0l}, \tag{21}$$

$$\widetilde{S}^{II}_{\rho\sigma} \equiv V S^{II}_{\rho\sigma} V^{\dagger}: \quad \widetilde{S}^{II}_{jk} = -i\varepsilon^{jkl}\widetilde{S}^{II}_{0l}, \quad \widetilde{S}^{II}_{01} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$
$$\widetilde{S}^{II}_{02} = \frac{1}{2}\begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{II}_{03} = \frac{1}{2}\begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \tag{22}$$

We consider in addition to the sets of generators (19), (20) the following two sets of matrices $S_{\rho\sigma}$:

$$S^{III}_{0k} = S^{I}_{0k} - S^{II}_{0k}, \quad S^{III}_{mn} = S^{I}_{mn} + S^{II}_{mn}, \tag{23}$$

$$S^{IV}_{\rho\sigma} = S^{I}_{\rho\sigma} + S^{II}_{\rho\sigma}. \tag{24}$$

Theorem 1. *The commutation relations (18b) of the Lorentz group are valid for each set $S^{I-IV}_{\rho\sigma}$ of $S_{\rho\sigma}$ matrices. Sets (19), (20) (or (21), (22)) are the generators of the same (spinor) representation $D\left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},0\right)$ of the $SL(2,C)$ group, set (23) consists of generators of the $D\,(0,1) \oplus (0,0)$ representation and set (24) consists of the generators of the irreducible vector $D\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the same group.*

Proof. The fact that matrices (19) are the generators of the spinor representation of the $SL(2,C)$ group is well known (for matrices (21) this fact is a consequence of the operator equality $V\widehat{Q}_{\psi}V^{\dagger} = q_E$ which unitarily connects operators in the PD- and B-representations). It is better to fulfil the proof of nontrivial assertions of Theorem 1 in the B-representation where their validity can be seen directly from the explicit form of the operators $S_{\rho\sigma}$ even without the Casimir operators calculations. In fact, using the explicit forms of matrices (21), (22), we find

$$\widetilde{S}^{II}_{\rho\sigma} = C\widetilde{S}^{I}_{\rho\sigma}C \Longleftrightarrow \widetilde{S}^{I}_{\rho\sigma} = C\widetilde{S}^{II}_{\rho\sigma}C, \tag{25}$$

$$\widetilde{S}^{IV}_{01} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{IV}_{02} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{IV}_{03} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \tag{26}$$

$$\widetilde{S}^{IV}_{mn} = \begin{vmatrix} s_{mn} & 0 \\ 0 & 0 \end{vmatrix} = \widetilde{S}^{III}_{mn}; \quad \widetilde{S}^{III}_{0k} = \begin{vmatrix} s_{0k} & 0 \\ 0 & 0 \end{vmatrix}, \tag{27}$$

where

$$s_{0k} = \frac{i}{2}\varepsilon^{kmn}s_{mn} = -s_{k0}, \quad s_{12} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$
$$s_{23} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}, \quad s_{31} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}. \tag{28}$$

The unitarity of the C operator in relations (25), the direct calculation of correspondings commutators and the Casimir operators $S_\pm$ of the $SL(2,C)$ group

$$S_\pm^2 = \frac{1}{2}(\tau_1 \pm i\tau_2), \quad \tau_1 \equiv -\frac{1}{2}S^{\mu\nu}S_{\mu\nu}, \quad \tau_2 \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma} \tag{29}$$

accomplish the proof of the theorem. ■

It is interesting to mark the following. Despite the fact that the matrices $\widetilde{S}^{I}_{\rho\sigma}$ and $\widetilde{S}^{II}_{\rho\sigma}$ are unitarily interconnected according to formulae (25), which in the PD-representation have the form

$$S^{I}_{\rho\sigma} = \hat{C}\, S^{II}_{\rho\sigma}\, \hat{C}, \quad \hat{C} \equiv V^\dagger C V = \begin{vmatrix} C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{30}$$

the matrices $\widetilde{S}^{I}_{\rho\sigma}$ (19) (or (21)), as well as the matrices (23), (24) (or (26), (27)), in contradiction to the matrix operators (20) (or (22)) being taken themselves, are not the invariance transformations of equation (1) (or (15)). It is evident because the C operator does not commute (or anticommute) with the Diracian $\gamma^\mu\partial_\mu$. Nevertheless, due to the validity of relations

$$\left[S^{II}_{\mu\nu}, S^{I,III,IV}_{\rho\sigma}\right] = 0, \quad \mu,\nu,\rho,\sigma = (0,1,2,3), \tag{31}$$

not only the generators $(\partial, \widehat{j}^{I})$ of the well-known spinor representation P^S of the Poincar'e group, but also the following generators

$$j^{III,IV}_{\rho\sigma} = M_{\rho\sigma} + S^{III,IV}_{\rho\sigma}. \tag{32}$$

are the transformations of invariance of equation (1) (or (15)). It means the validity of the following assertion.

Theorem 2. *The Maxwell equations with gradient-like sources are invariant with respect to the three different local representations P^S P^T P^V of the Poincar'e group P given by the formula*

$$\Psi(x) \longrightarrow \Psi'(x) = F^{I-III}(\omega)\Psi(\Lambda^{-1}(x-a)), \tag{33}$$

where

$$\begin{aligned} &F^{I}(\omega) \in D\left(0,\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2},0\right), \quad (P = P^S), \\ &F^{II}(\omega) \in D\left(0,1\right) \oplus \left(0,0\right), \quad (P = P^{Ts}), \\ &F^{III}(\omega) \in D\left(\tfrac{1}{2},\tfrac{1}{2}\right) \quad (P = P^V). \end{aligned} \tag{34}$$

Proof of the theorem for equations (15) follows from Theorem 1 and the above-mentioned consideration. It is only a small technical problem to obtain the explicit form of corresponding symmetry operators for the form (5) of these equations, having their explicit form for equations (15). ■

It is easy to construct the corresponding local $C(1,3)$ representations of the conformal group, i.e., C^S, C^T and C^V, but only one of them (the well-known local spinor representation C^S) gives the transformations of invariance of the massless Dirac equation and, therefore, of the Maxwell equations with gradient-like sources.

The simplest of Lie-Bäcklund symmetries are the transformations of invariance generated by the first-order differential operators with matrix coefficients. The operators of the Maxwell and Dirac equations also belong to this class of operators. In order to complete the present consideration, we shall recall briefly our result [14, 15].

Theorem 3. *The 128-dimensional algebra* A_{128}*, whose generators are*

$$Q_a = (\partial, \widehat{j}^I, \widehat{d}^I, \widehat{k}^I), \quad Q_b = (\gamma^2 C, i\gamma^2 C, \gamma^2\gamma^4 C, i\gamma^2\gamma^4 C, i, i\gamma^4, \gamma^4, I), \tag{35}$$

where $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$*, and all their compositions* Q_aQ_b*,* $a = (1, 2, 3, ..., 15)$*,* $b = (1, ..., 8)$*, is the simplest algebra of invariance of the Maxwell equations with gradient-like sources in the class of first-order differential operators with matrix coefficients.*

Proof. For the details of the proof via the symmetries of Eqs.(15) see [14, 15]. ■

In the class of nonlocal operators, we are able to represent here a new result. In spite of the fact that the above-mentioned local representations C^T and C^V are not the symmetries of equation (15), the corresponding symmetries can be constructed in the class of nonlocal operators.

Theorem 4. *The Maxwell equations with gradient-like sources (15) (or (5)) are invariant with respect to the representations* $\widetilde{C}^T$*,* $\widetilde{C}^V$ *of the conformal group* $C(1,3)$*. The corresponding generators are* $(\partial, \widehat{j}^{III,IV})$ *together with the following nonlocal operators*

$$\begin{aligned} d^{III,IV} &= \frac{1}{2}\left\{\frac{\partial_0, \partial_k}{\Delta}, j_{0k}^{III,IV}\right\}, \\ k_0^{III,IV} &= \frac{1}{2}\left\{\frac{\partial_0}{\Delta}, (j_{0k}^{III,IV})^2 + \frac{1}{2}\right\}, \quad k_m^{III,IV} = \left[k_0^{III,IV}, j_{0m}^{III,IV}\right], \end{aligned} \tag{36}$$

where $\Delta = \partial_k^2$*.*

Proof. The validity of this theorem follows from the above-mentioned Theorem 2 and Theorem 4 in [23]. ■

4. Conclusions

We prove that the object $\mathcal{E}$ of Eqs.(8), (15) can be interpreted as either (i) the spinor field, or (ii) the complex vector field, or (iii) the tensor-scalar field. Moreover, each of equations (5), (6), (8), (15) can be interpreted as either (i) the Maxwell equations with an arbitrary fixed gradient-like 4-current $j_\mu = \partial_\mu \mathcal{E}^0(x)$, being determined by a scalar function $\mathcal{E}^0(x)$, or (ii) the Dirac equation in the bosonic representation, or (iii) the Maxwell equations for the tensor-scalar field, or (iV) the equations for the complex vector field.

Let us underline that the Maxwell equations with gradient-like sources (5), (6), (8), (15) are the kind of Maxwell equations with maximally wide possible symmetry properties – they have both Fermi and Bose symmetries and can describe both fermions and bosons, see their corresponding quantization in [18].

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