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A Galilei-Invariant Generalization of the Shiff Equations

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Abstract

A Galilei-invariant generalization of the Shiff rotating frame electrodynamics is suggested.

To describe the charge and electromagnetic field distributions in a conductor rotating with a constant angular velocity (the reference frame is in rest with respect to the conductor), one uses the system of Shiff equations (see, for example, [1])

$$
\begin{gathered}
\operatorname{rot}\mathbf{E}+\mathbf{B}_t=\mathbf{0}, \qquad \operatorname{div}\mathbf{B}=0,\\
\operatorname{rot}\mathbf{B}-\mathbf{E}_t=\mu_0\left\{\mathbf{v}\times\operatorname{rot}\mathbf{E}+\frac{1}{\varepsilon_0}\operatorname{rot}\left[\mathbf{v}\times(\mathbf{E}-\mathbf{v}\times\mathbf{B})\right]+\mathbf{j}\right\}, \qquad (1)\\
\operatorname{div}\mathbf{E}=\frac{1}{\varepsilon_0}(\operatorname{div}(\mathbf{v}\times\mathbf{B})+j_0).
\end{gathered}
$$

Here,

$$\mathbf{v}=\omega\times\mathbf{x}, \qquad (2)$$

$\mathbf{E}=\mathbf{E}(t,\mathbf{x})$, $\mathbf{H}=\mathbf{H}(t,\mathbf{x})$ are three-vectors of electromagnetic field; $\omega=(\omega_1,\omega_2,\omega_3)$ is the vector of angular velocity, $\omega_a=\text{const}$; ε_0,μ_0 are the electric and magnetic permittivities; $j_0,\mathbf{j}$ are the charge and current densities, respectively.

In a sequel, we will consider system (1) under $j_0=0$, $\mathbf{j}=\mathbf{0}$.

As is well known, the system of Shiff equations (1), (2) is not invariant with respect to the Galilei group $G(1,3)$. However, if we suppose that $\mathbf{v}(t,\mathbf{x})$ is an arbitrary vector field, then system (1) admits the Galilei group having the following generators:

$$
\begin{gathered}
P_0=\frac{\partial}{\partial t}, \qquad P_a=\frac{\partial}{\partial x_a},\\
J_{ab}=x_a\frac{\partial}{\partial x_b}-x_b\frac{\partial}{\partial x_a}+E_a\frac{\partial}{\partial E_b}-E_b\frac{\partial}{\partial E_a}+B_a\frac{\partial}{\partial B_b}-B_b\frac{\partial}{\partial B_a}+v_a\frac{\partial}{\partial v_b}-v_b\frac{\partial}{\partial v_a}, \qquad (3)\\
G_a=t\frac{\partial}{\partial x_a}-\varepsilon_{abc}B_b\frac{\partial}{\partial B_c}-\varepsilon_0\frac{\partial}{\partial x_a},
\end{gathered}
$$

where $a,b,c=1,2,3$ and

$$
\varepsilon_{abc}=\begin{cases} 1, & (a,b,c)=\text{cycle}(1,2,3),\\ -1, & (a,b,c)=\text{cycle}(2,1,3),\\ 0, & \text{for other cases.}\end{cases}
$$

Consequently, the additional condition (2) breaks the Galilei invariance of system (1). The principal aim of the present paper is to suggest a generalization of system (1), (2) such that

- its solution set includes the set of solutions of system (1), (2) as a subset, and
- it is Galilei-invariant.

To this end, we replace the additional condition (2) by the following equation:

$$F(\operatorname{div}\mathbf{v}, \mathbf{v}^2) = 0, \tag{4}$$

where F is a smooth function of absolute invariants of the group $O(3)$ which is the subgroup of the Galilei group $G(1,3)$.

Applying the infinitesimal Lie approach (see, e.g., [2]), we obtain the following assertion.

Theorem. *The system of partial differential equations (1), (4) is invariant with respect to the group $G(1,3)$ having generators (3) if and only if*

$$F = \operatorname{div}\mathbf{v} + C, \tag{5}$$

where C is an arbitrary real constant.

Consequently, equation (4) now reads as $\operatorname{div}\mathbf{v} + C = 0$. Inserting into this equation $\mathbf{v} = \omega \times \mathbf{x}$ yields $C = 0$.

Thus, we have proved that system (1) considered together with the equation

$$\operatorname{div}\mathbf{v} = 0 \tag{6}$$

fulfills the Galilei relativity principle. Furthermore, its set of solutions contains all solutions of the Shiff system (1), (2).

References

[1] Gron O., Applications of Shiff's rotating frame electrodynamics, *Intern. J. Theor. Phys.*, 1984, V.23, N 5, 441–448.

[2] Ovsyannikov L.V., Group Analysis of Differential Equations, Academic Press, New York, 1982.