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The Symmetry of a Generalized Burgers System

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Abstract

The following system of equations in the form $u_0^a + u^b u_b^a = F^a\left(\triangle u^1, \triangle u^2\right)$, $a, b = 1, 2$ is considered as a generalization of the classical Burgers equation. The symmetry properties of this system are investigated.

In a medium with dissipation, the Burgers equation

$$u_0 + uu_1 = u_{11}, \tag{1}$$

where $u_0 = \dfrac{\partial u}{\partial x_0}$, $u_1 = \dfrac{\partial u}{\partial x_1}$, $u_{11} = \dfrac{\partial^2 u}{\partial x_1^2}$, $u = u(x_0, x_1)$, describes a quasisimple wake behaviour. It's worth noticing, that equation (1), as is well known, reduces to the linear heat equation [1] by a nonlocal Cole-Hopf substitution. The algebra of invariance of the heat equation is well known.

The nonlinear generalization of equation (1)

$$u_0 + uu_1 = F\left(u_{11}\right), \tag{2}$$

where $F\left(u_{11}\right)$ is an arbitrary smooth function, $F \neq \text{const}$, is investigated too. In paper [2], symmetry properties of equation (2) were studied in detail depending on a form of a function $F\left(u_{11}\right)$.

Let us generalize equation (2) to the case of two functions $u^1 = u^1\left(x_0, x_1, x_2\right)$, $u^2 = u^2\left(x_0, x_1, x_2\right)$ by the following system

$$\begin{cases} u_0^1 + u^1 u_1^1 + u^2 u_2^1 = F^1\left(\triangle u^1, \triangle u^2\right), \\ u_0^2 + u^1 u_1^2 + u^2 u_2^2 = F^2\left(\triangle u^1, \triangle u^2\right), \end{cases} \tag{3}$$

where $F^1\left(\triangle u^1, \triangle u^2\right)$, $F^2\left(\triangle u^1, \triangle u^2\right)$ are arbitrary smooth functions, $F^1 \neq \text{const}$, $F^2 \neq \text{const}$, $\triangle u^a = u_{11}^a + u_{22}^a$, $a = 1, 2$. We'll study a symmetry of system (3) depending on forms of the functions F^1, F^2.

Theorem 1. *The maximum algebra of invariance of system (3) is given by operators:*

1. $\langle P_\alpha = \partial_\alpha, G_a = x_0 \partial_a + \partial_{u^a} \rangle$, *when* $F^1\left(\triangle u^1, \triangle u^2\right)$, $F^2\left(\triangle u^1, \triangle u^2\right)$ *are arbitrary functions;*

2. $\langle P_\alpha, G_a, J_{12} = x_2 \partial_1 - x_1 \partial_2 + u^2 \partial_{u^1} - u^1 \partial_{u^2} \rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \triangle u^1 \varphi^1(\omega) - \triangle u^2 \varphi^2(\omega),$$

$$F^2\left(\triangle u^1, \triangle u^2\right) = \triangle u^1 \varphi^2(\omega) + \triangle u^2 \varphi^1(\omega);$$

3. $\langle P_\alpha, G_a, D_1 = (n+1)x_0\partial_0 + (2-n)x_a\partial_a + (1-2n)u^a\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \left(\triangle u^1\right)^n \varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = \left(\triangle u^2\right)^n \varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right), \qquad n \neq 1;$$

4. $\langle P_\alpha, G_a, D_2 = x_0\partial_0 + \left(2x_a - \frac{3}{2}m_a x_0^2\right)\partial_a + (u^a - 3m_a x_0)\partial_{u^a}\rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = m_1 \ln \triangle u^1 + \varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = m_2 \ln \triangle u^2 + \varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right),$$

m_a *are arbitrary constants, which must not be equal to zero at the same time;*

5. $\langle P_\alpha, G_a, J_{12}, D_3 = 2(n+1)x_0\partial_0 + (1-2n)x_a\partial_a + (4n+1)u^a\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \omega^n\left(\lambda_2\triangle u^1 - \lambda_1\triangle u^2\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = \omega^n\left(\lambda_1\triangle u^1 + \lambda_2\triangle u^2\right), \qquad n \neq 0;$$

6. $\langle P_\alpha, G_a, D = 2x_0\partial_0 + x_a\partial_a - u^a\partial_{u^a}, \Pi = x_0^2\partial_0 + x_0x_a\partial_a + (x_a - x_0u^a)\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \triangle u^1\varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right), \qquad F^2\left(\triangle u^1, \triangle u^2\right) = \triangle u^2\varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right);$$

7. $\langle P_\alpha, G_a, J_{12}, D, \Pi\rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \lambda_2\triangle u^1 - \lambda_1\triangle u^2, \qquad F^2\left(\triangle u^1, \triangle u^2\right) = \lambda_1\triangle u^1 + \lambda_2\triangle u^2,$$

under the condition of the theorem $\omega = \left(\triangle u^1\right)^2 + \left(\triangle u^2\right)^2$, φ^a *are arbitrary smooth functions,* n, λ_a *are arbitrary constants,* $\alpha = 0, 1, 2$.

Proof. The symmetry classification of system (3) is carried out in a class of the first order differential operators

$$X = \xi^0\left(x_0, \vec{x}, u^1, u^2\right)\partial_0 + \xi^a\left(x_0, \vec{x}, u^1, u^2\right)\partial_a + \eta^a\left(x_0, \vec{x}, u^1, u^2\right)\partial_{u^a}. \qquad (4)$$

As system (3) includes the second order equations, finding a generation of operator (4), we may write a condition of the Lie's invariance in the form:

$$\begin{cases} \tilde{X}\left[u_0^1 + u^1u_1^1 + u^2u_2^1 - F^1\left(\triangle u^1, \triangle u^2\right)\right]\Big|_{u_0^a + u^b u_b^a = F^a(\triangle u^1, \triangle u^2)} = 0, \\ \tilde{X}\left[u_0^2 + u^1u_1^2 + u^2u_2^2 - F^2\left(\triangle u^1, \triangle u^2\right)\right]\Big|_{u_0^a + u^b u_b^a = F^a(\triangle u^1, \triangle u^2)} = 0, \end{cases}$$

where $b = 1, 2$.

Using Lie's algorithm [3], we get a system of defining equations to find the functions ξ^0, ξ^a, η^a and F^1, F^2. Solving this system, we obtain:

$$\begin{cases} 3C_2\left(\triangle u^1 F^1_{\triangle u^1} + \triangle u^2 F^1_{\triangle u^2} - F^1\right) + C_6 = 0, \\ 3C_2\left(\triangle u^1 F^2_{\triangle u^1} + \triangle u^2 F^2_{\triangle u^2} - F^2\right) + C_{10} = 0, \\ w^1 F^1_{\triangle u^1} + w^2 F^1_{\triangle u^1} + (C_3 - 2C_4)F^1 + C_1 F^2 + C_7 = 0, \\ w^1 F^2_{\triangle u^1} + w^2 F^2_{\triangle u^1} + (C_3 - 2C_4)F^2 - C_1 F^1 + C_{11} = 0, \end{cases} \tag{5}$$

$$\xi^0 = C_2 x_0^2 + C_4 x_0 + C_5,$$

$$\xi^1 = (C_2 x_0 + C_3)\, x_1 + C_1 x_2 + \frac{C_6 x_0^3}{6} + \frac{C_7 x_0^2}{2} + C_8 x_0 + C_9,$$

$$\xi^2 = (C_2 x_0 + C_3)\, x_2 - C_1 x_1 + \frac{C_{10} x_0^3}{6} + \frac{C_{11} x_0^2}{2} + C_{12} x_0 + C_{13},$$

$$\eta^1 = (-C_2 x_0 + C_3 - C_4)\, u^1 + C_1 u^2 + C_2 x_1 + \frac{C_6 x_0^2}{2} + C_7 x_0 + C_8,$$

$$\eta^2 = (-C_2 x_0 + C_3 - C_4)\, u^2 - C_1 u^1 + C_2 x_2 + \frac{C_{10} x_0^2}{2} + C_{11} x_0 + C_{12},$$

where $w^1 = \left((C_3 + C_4)\triangle u^1 - C_1 \triangle u^2\right)$, $w^2 = \left(C_1 \triangle u^1 + (C_3 + C_4)\triangle u^2\right)$, C_i are arbitrary constants, $i = 1, \ldots, 13$.

Analyzing system (5), we receive the condition of Theorem.

Notice. Following Theorem, the system

$$\begin{cases} u^1_0 + u^1 u^1_1 + u^2 u^1_2 = \lambda_2 \triangle u^1 - \lambda_1 \triangle u^2, \\ u^2_0 + u^1 u^2_1 + u^2 u^2_2 = \lambda_1 \triangle u^1 + \lambda_2 \triangle u^2, \end{cases} \tag{6}$$

has the widest symmetry among all systems of the form (3). If $\lambda_1 = 0$, then system (6) transforms to the classical Burgers system

$$\begin{cases} u^1_0 + u^1 u^1_1 + u^2 u^1_2 = \lambda_2 \triangle u^1, \\ u^2_0 + u^1 u^2_1 + u^2 u^2_2 = \lambda_2 \triangle u^2, \end{cases} \tag{7}$$

which is invariant with respect to the Galilean algebra. This system is used to describe real physical processes. Since (6) and (7) have the same symmetry, system (6) may be also used to describe these processes.

References

[1] Whitham G., Linear and Nonlinear Waves, New York, Wiley, 1974.

[2] Fushchych W. and Boyko V., Galilei-invariant higher-order equations of the Burgers and Korteweg-de Vries types, *Ukrain. Math. J.*, 1996, V.48, N 12, 1489–1601.

[3] Ovsyannikov L.V., Group Analysis of Differential Equations, Moscow, Nauka, 1978.