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Symmetry Properties of a Generalized System of Burgers Equations

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Abstract

The classification of symmetry properties of a generalized system of Burgers equations $u_0^a + u_{11}^a = F^{ab}(\vec{u})u_1^a$ is investigated in those cases where the invariance with respect to the Galilean algebra retains.

The symmetry of the classical Burgers equation

$$u_0 + \lambda u u_1 + u_{11} = 0,$$

is well known (see, for example, [1]). Its widest algebra of invariance in the class of the Lie's operators is given by the following basic elements

$$\partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad G = x_0\partial_1 + \frac{1}{\lambda}\partial_u,$$
$$D = 2x_0\partial_0 + x_1\partial_1 - u\partial_u, \quad \Pi = x_0^2\partial_0 + x_0x_1\partial_1 + \left(\frac{x_1}{\lambda} - x_0u\right)\partial_u.$$

In the paper [2], the classification of symmetry properties of the generalized Burgers equation

$$u_0 + u_{11} = F(u, u_1), \tag{1}$$

was investigated depending on a nonlinear function $F(u, u_1)$ with which the invariance of equation (1) with respect to the Galilean algebra retains.

A problem of a wider generalization of equation (1) is stated. In this paper, the symmetry properties of the generalized system of the Burgers equations

$$u_0^a + u_{11}^a = F^{ab}(\vec{u})u_1^a. \tag{2}$$

are studied. In these equations, $\vec{u}(u^1, u^2)$, $\vec{u} = \vec{u}(x)$, $x = (x_0, x_1)$, $u_\mu = \partial_\mu u$, $\partial_\mu = \dfrac{\partial}{\partial x_\mu}$, $\mu = 0, 1$; F^{ab} are arbitrary differentiable functions.

The following statements are true.

Theorem 1. *System (2) is invariant with respect to the Galilean algebra $AG = \langle \partial_0, \partial_1, G\rangle$ in the next cases:*
a) If $G = G_1 = x_0\partial_1 + k_1\partial_{u^1} + k_2\partial_{u^2}$, then

$$F^{11} = -\frac{u^1}{k_1} + \varphi^{11}(\omega), \quad F^{12} = \varphi^{12}(\omega),$$
$$F^{21} = \varphi^{21}(\omega), \quad F^{22} = -\frac{u^2}{k_2} + \varphi^{22}(\omega),$$

where $\omega = k_2u^1 - k_1u^2$, φ^{ab} are arbitrary differentiable functions; k_i are arbitrary constants.

b) If $G = G_2 = x_0\partial_1 + (k_1u^1 + k_2)\partial_{u^1} + (k_3u^2 + k_4)\partial_{u^2}$*, then*

$$F^{11} = -\frac{1}{k_1}\ln\omega_1 + \varphi^{11}(\omega_2), \quad F^{12} = \omega_1^{1-k_3/k_1}\varphi^{12}(\omega_2),$$

$$F^{21} = \omega_1^{1-k_3/k_1}\varphi^{21}(\omega), \qquad F^{22} = -\frac{1}{k_1}\ln\omega_1 + \varphi^{22}(\omega),$$

where $\omega_1 = k_2u^1 + k_2$, $\omega_2 = \dfrac{(k_1u^1 + k_2)^{k_3/k_1}}{k_3u^2 + k_4}$, φ^{ab} *are arbitrary differentiable functions;* k_i *are arbitrary constants.*

c) If $G = G_3 = x_0\partial_1 + k_1u^2\partial_{u^1}$*, then*

$$F^{11} = \frac{u^1}{u^2}\left(\varphi^{21}(u^2) - \frac{1}{k_1}\right) + \varphi^{11}(u^2),$$

$$F^{12} = \frac{u^1}{u^2}\left(\varphi^{22}(u^2) - \varphi^{11}(u^2)\right) - \varphi^{21}(u^2)\left(\frac{u^1}{u^2}\right)^2 + \varphi^{12}(u^2),$$

$$F^{21} = \varphi^{21}(u^2), \qquad F^{22} = -\frac{u^1}{u^2}\left(\varphi^{21}(u^2) + \frac{1}{k_1}\right) + \varphi^{22}(u^2),$$

where φ^{ab} *are arbitrary differentiable functions;* k_1 *is an arbitrary constant.*

d) If $G = G_4 = x_0\partial_1 + k_2u^1\partial_{u^2}$*, then*

$$F^{11} = -\frac{u^2}{u^1}\left(\varphi^{12}(u^1) + \frac{1}{k_2}\right) + \varphi^{11}(u^1), \qquad F^{12} = \varphi^{12}(u^1),$$

$$F^{21} = \frac{u^2}{u^1}\left(\varphi^{11}(u^1) - \varphi^{22}(u^1)\right) - \varphi^{12}(u^1)\left(\frac{u^2}{u^1}\right)^2 + \varphi^{21}(u^1),$$

$$F^{22} = \frac{u^2}{u^1}\left(\varphi^{12}(u^1) - \frac{1}{k_1}\right) + \varphi^{22}(u^1),$$

where φ^{ab} *are arbitrary differentiable functions;* k_2 *is an arbitrary constant.*

Theorem 2. *System (2) is invariant with respect to the Galilean algebra* $AG_1 = \langle\partial_0, \partial_1, G, D\rangle$ *in the next cases:*
1) $G = G_1$, $\quad D = D_1 = 2x_0\partial_0 + x_1\partial_1 + \left(-u^1 + m_1\omega\right)\partial_{u^1} + \left(-u^2 + m_2\omega\right)\partial_{u^2}$;

$$F^{11} = F^{22} = m_1u^2 - m_2u^1, \qquad F^{12} = F^{21} = 0,$$

where $\omega = k_2u^1 - k_1u^2$, φ^{ab} *are arbitrary differentiable functions;* k_i, m_i *are arbitrary constants.*

2) $G = G_4$, $\quad D = D_2 = 2x_0\partial_0 + x_1\partial_1 + Au^1\partial_{u^1} + (A-1)u^2\partial_{u^2}$, *where* A *is an arbitrary constant (AC).*

a) $A \neq 0$

$$F^{11} = -\frac{u^2}{u^1}\left(\frac{1}{k_2} + c_{12}\right) + c_{11}(u^1)^{-1/A}, \qquad F^{12} = c_{12},$$

$$F^{21} = c_{21}(u^1)^{-2/A} + (c_{11} - c_{22})u^2(u^1)^{-1-1/A} - c_{12}\left(\frac{u^2}{u^1}\right)^2,$$

$$F^{22} = \frac{u^2}{u^1}\left(-\frac{1}{k_2} + c_{12}\right) + c_{22}(u^1)^{-1/A},$$

where c_{ab}, k_2 *are arbitrary constants.*

b) $A = 0$

$$F^{11} = -\frac{u^2}{u^1}\left(\frac{1}{k_2} + \varphi^{12}(u^1)\right), \quad F^{12} = \varphi^{12}(u^1),$$
$$F^{21} = -\varphi^{12}(u^1)\left(\frac{u^2}{u^1}\right)^2, \qquad F^{22} = \frac{u^2}{u^1}\left(-\frac{1}{k_2} + \varphi^{12}(u^1)\right),$$

where φ^{ab} are arbitrary differentiable functions.

3) $G = G_3$, $D = D_3 = 2x_0\partial_0 + x_1\partial_1 + (B-1)u^1\partial_{u^1} + Bu^2\partial_{u^2}$, *where B is an arbitrary constant.*

a) $B \neq 0$

$$F^{11} = \frac{u^1}{u^2}\left(-\frac{1}{k_1} + c_{12}\right) + c_{11}(u^2)^{-1/B},$$
$$F^{12} = c_{12}(u^2)^{-2/B} + (c_{22} - c_{11})u^1(u^2)^{-1-1/B} - c_{21}\left(\frac{u^1}{u^2}\right)^2,$$
$$F^{21} = c_{12}, \qquad F^{22} = -\frac{u^1}{u^2}\left(\frac{1}{k_1} + c_{12}\right) + c_{22}(u^2)^{-1/B},$$

where c_{ab}, k_1 are arbitrary constants.

b) $B = 0$

$$F^{11} = \frac{u^1}{u^2}\left(-\frac{1}{k_1} + \varphi^{21}(u^2)\right), \quad F^{12} = -\varphi^{21}(u^2)\left(\frac{u^1}{u^2}\right)^2,$$
$$F^{21} = \varphi^{21}(u^2), \qquad F^{22} = -\frac{u^1}{u^2}\left(\frac{1}{k_1} + \varphi^{21}(u^2)\right),$$

where φ^{ab} are arbitrary differentiable functions.

Theorem 3. *System (2) is invariant with respect to the Galilean algebra $AG_2 = \langle \partial_0, \partial_1, G, D, \Pi\rangle$, when*

$$G = G_1, \quad D = D_1, \quad \Pi = x_0^2\partial_0 + x_0x_1\partial_1 - u^a\partial_{u^a} + x_0\omega(\alpha_2\partial_{u^1} - \alpha_1\partial_{u^2}) - x_1(\beta_2\partial_{u^1} - \beta_1\partial_{u^2}),$$

where $\omega = \vec{\beta}\vec{u}$, $F^{ab} = \delta_{ab}\vec{\alpha}\vec{u}$, α_i, β_i are arbitrary constants, and $\begin{vmatrix}\beta_1 & \beta_2\\ \alpha_1 & \alpha_2\end{vmatrix} = 1.$

Proof of Theorems 1–3. Let an infinitesimal operator X (see [3]) be

$$X = \xi^0(x,\vec{u})\,\partial_0 + \xi^1(x,u)\,\partial_1 + \eta^1(x,u)\,\partial_{u^1} + \eta^2(x,u)\,\partial_{u^2}. \tag{3}$$

Using the invariance condition of equation (2) with respect to operator (3), we obtain defining equations for the coordinates of operator (3) and functions $F^{ab}(u^1,u^2)$:

$$\begin{gathered}
\xi^0_1 = \xi^0_{u^a} = 0, \quad \xi^1_{u^a} = 0, \quad \xi^0_0 = 2\xi^1_1, \quad \eta^a_{u^bu^c} = 0,\\
-F^{11}\eta^1_1 - F^{21}\eta^2_1 + \eta^1_0 + \eta^1_{11} = 0, \quad -F^{21}\eta^1_1 - F^{22}\eta^2_1 + \eta^2_0 + \eta^2_{11} = 0,\\
F^{11}_{u^1}\eta^1 + F^{11}_{u^2}\eta^2 + F^{12}\eta^1_{u^1} + \xi^1_0 - 2\eta^1_{1u^1} + F^{11}\xi^1_1 - F^{21}\eta^1_{u^2} = 0,\\
F^{22}_{u^1}\eta^1 + F^{22}_{u^2}\eta^2 + F^{21}\eta^1_{u^22} + \xi^1_0 - 2\eta^2_{1u^2} + F^{22}\xi^1_1 - F^{12}\eta^2_{u^1} = 0,\\
F^{12}_{u^1}\eta^1 + F^{12}_{u^2}\eta^2 + F^{11}\eta^1_{u^2} - F^{12}(\eta^1_{u^1} - \eta^2_{u^2} - \xi^1_1) - 2\eta^1_{1u^2} - F^{22}\eta^1_{u^2} = 0,\\
F^{21}_{u^1}\eta^1 + F^{21}_{u^2}\eta^2 + F^{22}\eta^2_{u^1} - F^{21}(\eta^2_{u^2} - \eta^1_{u^1} - \xi^1_1) - 2\eta^2_{1u^1} - F^{11}\eta^2_{u^1} = 0.
\end{gathered} \tag{4}$$

Solutions of system (4) define a form of the nonlinear functions F^{ab} and invariance algebra.

Theorem 4. *The widest invariance algebra of the equation*

$$\vec{u}_0 + (\vec{\lambda}\vec{u})\vec{u}_1 + \vec{u}_{11} = 0 \tag{5}$$

consists of the operators

$$\begin{gathered}\partial_0, \quad \partial_1, \quad G_a = \lambda_a x_0 \partial_1 + \partial_{u^a}, \quad D_a = \lambda_a(2x_0\partial_0 + x_1\partial_1) - \vec{\lambda}\vec{u}\partial_{u^a}, \\ \Pi_a = \lambda_a(x_0^2\partial_0 + x_0x_1\partial_1 +) + \left(x_1 - x_0\vec{\lambda}\vec{u}\right)\partial_{u^a}, \quad Q_{ij} = u^i(\lambda_n\partial_{u^j} - \lambda_j\partial_{u^n}),\end{gathered} \tag{6}$$

where $a = 1, \ldots, n$*;* $i, j = 1, \ldots, n-1$.

This theorem is proved by the standard Lie's method.

Commutation relations of the operators of algebra (6) are as follows:

$$\begin{array}{lll} [\partial_0, \partial_1] = 0; & [\partial_0, G_a] = \lambda_a\partial_1; & [\partial_0, D_a] = 2\lambda_a\partial_0; \\ [\partial_0, \Pi_a] = D_a; & [\partial_0, Q_{ij}] = 0; & [\partial_1, G_a] = 0; \\ [\partial_1, D_a] = \lambda_a\partial_1; & [\partial_1, \Pi_a] = G_a; & [\partial_1, Q_{ij}] = 0; \\ [Q_{ij}, Q_{ab}] = 0; & [D_a, D_b] = \lambda_a D_b - \lambda_b D_a; & [G_a, D_b] = -\lambda_a G_b; \\ [G_a, \Pi_b] = 0; & [G_a, Q_{ij}] = \delta_{ai}(\lambda_n G_j - \lambda_j G_n); & [G_a, G_b] = 0; \\ [D_a, \Pi_b] = \lambda_a\Pi_b + \lambda_b\Pi_a; & [D_a, Q_{ij}] = \delta_{ai}(\lambda_j D_n - \lambda_n D_j); & [\Pi_a, \Pi_b] = 0; \\ [\Pi_a, Q_{ij}] = \delta_{ai}(\lambda_n\Pi_j - \lambda_j\Pi_n). & & \end{array}$$

References

[1] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

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[3] Ovsyannikov L.V., Group Analysis of Differential Equations, Moscow, Nauka, 1978.