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Exact Solutions of the Nonlinear Diffusion Equation $u_0 + \nabla\left[u^{-\frac{4}{5}}\nabla u\right] = 0$

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Abstract

The symmetry reduction of the equation $u_0+\nabla\left[u^{-\frac{4}{5}}\nabla u\right]=0$ to ordinary differential equations with respect to all subalgebras of rank three of the algebra $A\widetilde{E}(1)\oplus AC(3)$ is carried out. New invariant solutions are constructed for this equation.

1 Introduction

This paper is concerned with some invariant solutions to the three-dimensional nonlinear diffusion equation

$$\frac{\partial u}{\partial x_0} + \nabla\left[u^{-\frac{4}{5}}\nabla u\right] = 0. \tag{1}$$

Its symmetry properties are know [1, 2]. The maximal Lie point symmetry algebra F of equation (1) has the basis

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = -\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a},$$
$$D_1 = x_0\frac{\partial}{\partial x_0} + \frac{5}{4}u\frac{\partial}{\partial u}, \quad D_2 = \sum_{a=1}^{3} x_a\frac{\partial}{\partial x_a} - \frac{5}{2}u\frac{\partial}{\partial u},$$
$$K_a = \left(2x_a^2 - \vec{x}^2\right)\frac{\partial}{\partial x_a} + 2\sum_{b\neq a} x_a x_b\frac{\partial}{\partial x_b} - 5x_a u\frac{\partial}{\partial u}$$

with $a,b = 1,2,3$. It is easy to see that $F = A\widetilde{E}(1)\oplus AC(3)$, where $A\widetilde{E}(1) = \langle P_0, D_1\rangle$ is an extended Euclidean algebra and $AC(3) = \langle P_a, K_a, J_{ab}, D_2 : a,b = 1,2,3\rangle$ is a conformal algebra. If we make the transformation $u = v^5$, we obtain the equation

$$\frac{\partial v}{\partial x_0} + v^{-4}\triangle v = 0. \tag{2}$$

Clearly, we see that

$$5u\frac{\partial}{\partial u} = v\frac{\partial}{\partial v}.$$

In the present article, the symmetry reduction of equation (2) is carried out with respect to all subalgebras of rank three of the algebra F, up to conjugacy with respect to the group of inner automorphisms. Some exact solutions of equation (1) are obtained by means of this reduction (for the concepts and results used here, see also [3, 4]).

2 Classification of subalgebras of the invariance algebra

If $v = v(x_1, x_2, x_3)$ is a solution of equation (2), then v is a solution of the Laplace equation $\triangle v = 0$. In this connection, let us restrict ourselves to those subalgebras of the algebra F that do not contain P_0. Among subalgebras possessing the same invariants, there exists a subalgebra containing all the other subalgebras. We call it I-maximal.

Theorem 1 *Up to the conjugation under to the group of inner automorphisms, the algebra F has 18 I-maximal subalgebras of rank three which do not contain P_0:*

$$L_1 = \langle P_1, P_2, P_3, J_{12}, J_{13}, J_{23}\rangle,\ L_2 = \langle P_0 - P_1, P_2, P_3, J_{23}\rangle,$$

$$L_3 = \langle P_2, P_3, J_{23}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),\quad L_4 = \langle P_0 - P_1, P_3, D_1 + D_2\rangle,$$

$$L_5 = \langle P_3, J_{12}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),\quad L_6 = \langle P_0 - P_3, J_{12}, D_1 + D_2\rangle,$$

$$L_7 = \langle P_3, J_{12} + \alpha P_0, D_2 + \beta P_0\rangle\ (\alpha = 1,\ \beta \in \mathbb{R}\ \textit{or}\ \alpha = 0\ \textit{and}\ \beta = 0, \pm 1),$$

$$L_8 = \langle P_2, P_3, J_{23}, D_1 - P_1\rangle,\ L_9 = \langle P_2, P_3, J_{23}, D_2 + \alpha P_0\rangle\ \ (\alpha = 0, \pm 1),$$

$$L_{10} = \langle P_3, D_1 + \alpha J_{12}, D_2 + \beta J_{12}\rangle\ \ (\alpha > 0,\ \beta \in \mathbb{R}\ \textit{or}\ \alpha = 0,\ \beta \geq 0),$$

$$L_{11} = \langle J_{12}, D_1, D_2\rangle,\ L_{12} = \langle J_{12}, J_{13}, J_{23}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),$$

$$L_{13} = \langle J_{12}, J_{13}, J_{23}, D_2 + \alpha P_0\rangle\ (\alpha = 0, \pm 1),\ L_{14} = \langle J_{12}, K_3 - P_3, D_1\rangle,$$

$$L_{15} = \langle P_1 + K_1, P_2 + K_2, J_{12}, D_1\rangle,$$

$$L_{16} = \langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23}\rangle,$$

$$L_{17} = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + \alpha D_1\rangle\ (\alpha \in \mathbb{R}),$$

$$L_{18} = \langle P_1 + K_1, P_2 + K_2, P_3 + K_3, J_{12}, J_{13}, J_{23}\rangle.$$

Proof. Let

$$\Omega_{0a} = \frac{1}{2}(P_a + K_a),\quad \Omega_{a4} = \frac{1}{2}(K_a - P_a),\ \Omega_{ab} = -J_{ab},\quad \Omega_{04} = D$$

with $a, b = 1, 2, 3$. They satisfy the following commutation relations:

$$[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] = g_{\alpha\delta}\Omega_{\beta\gamma} + g_{\beta\gamma}\Omega_{\alpha\delta} - g_{\alpha\gamma}\Omega_{\beta\delta} - g_{\beta\delta}\Omega_{\alpha\gamma},$$

where $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 4$ and $(g_{\alpha\beta}) = \mathrm{diag}\,[1, -1, -1, -1, -1]$. It follows from this that $AC\,(3) \cong AO\,(1, 4)$.

The classification of subalgebras of the algebra $AO\,(1, 4)$ up to $O\,(1, 4)$-conjugacy is done in [5]. Let L is an I-maximal subalgebra of rank three of the algebra F and $P_0 \notin L$. Denote by K the projection of L onto $AO\,(1, 4)$. If K has an invariant isotropic subspace in the Minkowski space $\mathbb{R}_{1,4}$, then K is conjugate to a subalgebra of the extended Euclidean algebra $A\widetilde{E}(3)$ with the basis P_a, J_{ab}, D_2 $(a, b = 1, 2, 3)$. In this case, on the basis of Theorem 1 [6], L is conjugate with one of the subalgebras $L_1, \ldots, L_{13}$. If K has no invariant isotropic subspace in the space $\mathbb{R}_{1,4}$, then K is conjugate with one of the following algebras [5]:

$$N_1=\langle\Omega_{12},\Omega_{34}\rangle,\quad N_2=AO(1,2)=\langle\Omega_{01},\Omega_{02},\Omega_{12}\rangle,$$

$$N_3=\langle\Omega_{12}+\Omega_{34},\Omega_{13}-\Omega_{24},\Omega_{23}+\Omega_{14}\rangle,$$

$$N_4=\langle\Omega_{12}+\Omega_{34},\Omega_{13}-\Omega_{24},\Omega_{23}+\Omega_{14}\rangle\oplus\langle\Omega_{12}-\Omega_{34}\rangle,$$

$$N_5=AO(1,2)\oplus AO(2)=\langle\Omega_{01},\Omega_{02},\Omega_{12},\Omega_{34}\rangle,$$

$$N_6=AO(4)=\langle\Omega_{ab}:\ a,b=1,2,3,4\rangle,\quad N_7=AO(1,3)=\langle\Omega_{\alpha\beta}:\ \alpha,\beta=0,1,2,3\rangle,$$

$$N_8=AO(1,4)=\langle\Omega_{\alpha\beta}:\ \alpha,\beta=0,1,2,3,4\rangle.$$

The rank of N_j equals 2 for $j=1,2$; 3 for $j=3,\ldots,7$; 4 for $j=8$. From this we conclude that L is conjugate with one of the subalgebras $L_{14},\ldots,L_{18}$. The theorem is proved.

3 Reduction of the diffusion equation to the ordinary differential equations

For each of the subalgebras $L_1,\ldots,L_{18}$ obtained in Theorem 1, we point out the corresponding ansatz $\omega'=\varphi(\omega)$ solved for v, where ω and ω' are functionally independent invariants of a subalgebra, as well as the reduced equation which is obtained by means of this ansatz. The numbering of the considered cases corresponds to the numbering of the subalgebras $L_1,\ldots,L_{18}$.

3.1. $v=\varphi(\omega)$, $\omega=x_0$, $\varphi'=0$. The corresponding exact solution of the diffusion equation (1) is trivial: $u=C$.
3.2. $v=\varphi(\omega)$, $\omega=x_0-x_1$, then

$$\varphi''+\varphi^4\varphi'=0. \tag{3}$$

The general solution of equation (3) is of the form

$$\int\frac{d\varphi}{C_1-\varphi^5}=\frac{1}{5}\omega+C_2,$$

where C_1 and C_2 are arbitrary constants. If $C_1=0$, then

$$\varphi=\left(\frac{5}{4\omega+C}\right)^{\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is of the form

$$u=\left[\frac{5}{4\,(x_0-x_1)+C}\right]^{\frac{5}{4}}.$$

3.3. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = x_1 x_0^{-\alpha}$, then

$$\varphi'' - \alpha\omega\varphi^4\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0. \tag{4}$$

The nonzero function $\varphi = \left(A\omega^2 + B\omega + C\right)^{-\frac{1}{4}}$ is a solution of equation (4) if and only if one of the following conditions is satisfied:

1. $\alpha = 1$, $A = 0$, $C = \frac{5}{4}B^2$.
2. $\alpha = 0$, $A = -\frac{1}{3}$, $C = -\frac{3}{4}B^2$.
3. $A = -\frac{1}{3}$, $B = C = 0$.
4. $\alpha = \frac{5}{6}$, $A = -\frac{1}{3}$, $B = 0$.
5. $\alpha = \frac{1}{2}$, $A = 0$, $B = 0$.

By means of φ obtained above, we find such invariant solutions of equation (1):

$$u = \left(Bx_1 + \frac{5}{4}B^2 x_0\right)^{-\frac{5}{4}}, \quad u = \left[-\frac{12x_0}{(2x_1+3B)^2}\right]^{\frac{5}{4}}, \quad u = \left(-\frac{x_1^2}{3x_0} + Bx_0^{\frac{2}{3}}\right)^{-\frac{5}{4}},$$

where B is an arbitrary constant.

3.4. $v = (x_0 - x_1)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = x_2 (x_0 - x_1)^{-1}$,

$$\left(1+\omega^2\right)\varphi'' + \left(\frac{5}{2} - \varphi^4\right)\omega\varphi' + \frac{5}{16}\varphi - \frac{1}{4}\varphi^5 = 0. \tag{5}$$

The function $\varphi = (A\omega + B)^{-\frac{1}{4}}$, where A and B are constants, satisfies equation (5) if and only if $A^2 = -B\left(B - \frac{4}{5}\right)$. The corresponding invariant solution of equation (1) is of the form $u = [Ax_2 + B(x_0 - x_1)]^{-\frac{5}{4}}$. It is easy to see that the coefficient B can take on any value from the interval $\left(0; \frac{4}{5}\right)$.

3.5. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2\right) x_0^{-2\alpha}$,

$$4\omega\varphi'' + \left(4 - 2\alpha\omega\varphi^4\right)\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0.$$

Integrating this equation for $\alpha = \frac{5}{2}$, we obtain the following equation:

$$4w\varphi' - \omega\varphi^5 = C,$$

where C is an arbitrary constant. If $C = 0$, then

$$\varphi = (-\omega + B)^{-\frac{1}{4}}.$$

Thus, we find the exact solution

$$u = \left(Bx_0^4 - \frac{x_1^2 + x_2^2}{x_0}\right)^{-\frac{5}{4}}.$$

3.6. $v = (x_0 - x_3)^{-\frac{1}{4}}\varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2\right)(x_0 - x_3)^{-2}$, then

$$4\omega(1+\omega)\varphi'' + \left(7\omega + 4 - 2\omega\varphi^4\right)\varphi' + \frac{5}{16}\varphi - \frac{1}{4}\varphi^5 = 0.$$

3.7. $v = \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}}\varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2\right)^{-\frac{\beta}{2}}\exp\left(x_0 + \alpha\arctan\frac{x_1}{x_2}\right)$, then

$$\left(\alpha^2 + \beta^2\right)\omega^2\varphi'' + \left(\varphi^4 + \alpha^2 + \beta^2 + \beta\right)\omega\varphi' + \frac{1}{4}\varphi = 0.$$

For $\alpha = \beta = 0$, we have $\varphi = (-\ln\omega - C)^{\frac{1}{4}}$. Consequently, we find the exact solution

$$u = \left(-\frac{x_1^2 + x_2^2}{x_0 + C}\right)^{-\frac{5}{4}}.$$

3.8. $v = x_0^{\frac{1}{4}}\varphi(\omega)$, $\omega = x_1 - \ln x_0$,

$$\varphi'' - \varphi^4\varphi' + \frac{1}{4}\varphi^5 = 0.$$

3.9. $v = x_1^{-\frac{1}{2}}\varphi(\omega)$, $\omega = x_1^{-\alpha}\exp(x_0)$, then

$$\alpha^2\omega^2\varphi'' + \left(\varphi^4 + \alpha^2 + 2\alpha\right)\omega\varphi' + \frac{3}{4}\varphi = 0.$$

If $\alpha = 0$, then

$$\varphi = \left(-\frac{1}{3\ln\omega + C}\right)^{-\frac{1}{4}}.$$

Whence we obtain the exact solution of (1):

$$u = \left(-\frac{3x_0 + C}{x_1^2}\right)^{\frac{5}{4}}.$$

3.10. $v = x_0^{\frac{1}{4}}\left(x_1^2 + x_2^2\right)^{-\frac{1}{4}}\varphi(\omega)$, $\omega = \arctan\frac{x_1}{x_2} + \alpha\ln x_0 + \frac{\beta}{2}\ln\left(x_1^2 + x_2^2\right)$,

$$\left(1 + \beta^2\right)\varphi'' + \left(\alpha\varphi^4 - \beta\right)\varphi' + \frac{1}{4}\left(\varphi + \varphi^5\right) = 0.$$

3.11. $v = x_0^{\frac{1}{4}}\left(x_1^2 + x_2^2\right)^{-\frac{1}{4}}\varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2\right)x_3^{-2}$, then

$$4\omega^2(1+\omega)\varphi'' + \left(6\omega^2 + 2\omega\right)\varphi' + \frac{1}{4}\left(\varphi + \varphi^5\right) = 0.$$

3.12. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2 + x_3^2\right) x_0^{-2\alpha}$, then

$$4\omega\varphi'' + \left(6 - 2\alpha\omega\varphi^4\right)\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0. \tag{6}$$

Integrating this reduced equation for $\alpha = \frac{5}{2}$, we find the equation

$$4\omega\varphi' + 2\varphi - \omega\varphi^5 = \widetilde{C},$$

where $\widetilde{C}$ is an arbitrary constant. If $\widetilde{C} = 0$, then the general solution of this reduced equation is of the form

$$\varphi = \left(C\omega^2 + \omega\right)^{-\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is a function

$$u = \left[\frac{x_0^6}{C\left(x_1^2 + x_2^2 + x_3^2\right)^2 + x_0^5\left(x_1^2 + x_2^2 + x_3^2\right)}\right]^{\frac{5}{4}}.$$

For $\alpha = -\frac{5}{2}$ equation (6) has a solution given by

$$\varphi = (\omega + C)^{-\frac{1}{4}}.$$

Thus, we find the exact solution

$$u = \left(\frac{x_0^6}{\left(x_1^2 + x_2^2 + x_3^2\right)x_0^5 + C}\right)^{\frac{5}{4}}.$$

3.13. $v = \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = 2x_0 - \alpha \ln\left(x_1^2 + x_2^2 + x_3^2\right)$, then

$$2\alpha^2\varphi'' + \varphi^4\varphi' - \frac{1}{8}\varphi = 0.$$

For $\alpha = 0$ we have $\varphi = \left(\frac{1}{2}\omega + C\right)^{\frac{1}{4}}$. Whence we obtain the exact solution

$$u = \left(\frac{x_0 + C}{x_1^2 + x_2^2 + x_3^2}\right)^{\frac{5}{4}}.$$

3.14. $v = x_0^{\frac{1}{4}} \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = \dfrac{x_1^2 + x_2^2 + x_3^2 + 1}{\left(x_1^2 + x_2^2\right)^{\frac{1}{2}}}$,

$$\left(\omega^2 - 4\right)\varphi'' + 2\omega\varphi' + \frac{1}{4}\varphi\left(1 + \varphi^4\right) = 0.$$

3.15. $v = x_0^{\frac{1}{4}} x_3^{-\frac{1}{2}} \varphi(\omega)$, $\omega = \dfrac{x_1^2 + x_2^2 + x_3^2 - 1}{x_3}$,

$$\left(\omega^2 + 4\right)\varphi'' + 3\omega\varphi' + \frac{1}{4}\varphi\left(3 + \varphi^4\right) = 0.$$

3.16. $v = \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^{-\frac{1}{2}} \varphi(\omega), \ \omega = x_0$, then

$$\varphi^3\varphi' - 3 = 0.$$

In this case, $\varphi = (12\omega + C)^{\frac{1}{4}}$, and therefore

$$u = \left[\frac{12x_0 + C}{\left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2}\right]^{\frac{5}{4}}.$$

3.17. $v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right]^{-\frac{1}{4}} \exp\left[\frac{\alpha}{8} \arctan\frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3}\right] \varphi(\omega)$,

$$\omega = \alpha \arctan\frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - 2\ln x_0,$$

$$4\alpha^2\varphi'' + \left[\alpha^2 - 2\varphi^4 \exp\left(\frac{\omega}{2}\right)\right]\varphi' + \frac{\alpha^2 + 16}{16}\varphi = 0.$$

We have the exact solution

$$u = \left[\frac{C - 4x_0}{\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2}\right]^{\frac{5}{4}}.$$

3.18. $v = \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^{-\frac{1}{2}} \varphi(\omega), \ \omega = x_0$, then

$$\varphi^3\varphi' + 3 = 0.$$

Integrating this equation, we obtain

$$\varphi = (-12\omega + C)^{\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is of the form

$$u = \left[\frac{C - 12x_0}{\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2}\right]^{\frac{5}{4}}.$$

4 Multiplication of solutions

Solving the Lie equations corresponding to the basis elements of the algebra F, we obtain a one-parameter group of transformations $(x_0, x_1, x_2, x_3, u) \to (x_0', x_1', x_2', x_3', u')$ generated by these vector fields:

$$\exp(\Theta P_0): \quad x_0' = x_0 + \Theta, \quad x_a' = x_a \ (a = 1, 2, 3), \quad u' = u;$$

$$\exp(\Theta P_a): \quad x_0' = x_0, \quad x_a' = x_a - \Theta, \quad x_c' = x_c \text{ for } c \neq a, \quad u' = u;$$

$$\exp(\Theta J_{ab}): \quad x_0' = x_0, \quad x_a' = x_a \cos\Theta - x_b \sin\Theta,$$

$$x_b' = x_a \sin\Theta + x_b \cos\Theta, \ x_c' = x_c, \ u' = u, \ \text{where } \{a, b, c\} = \{1, 2, 3\};$$

$$\exp(\Theta D_1): \quad x_0' = x_0 \exp(\Theta), \quad x_a' = x_a \, (a = 1, 2, 3), \quad u' = u \exp\left(\frac{5\Theta}{4}\right);$$

$$\exp(\Theta D_2): \quad x_0' = x_0, \quad x_a' = x_a \exp(\Theta) \, (a = 1, 2, 3), \quad u' = u \exp\left(-\frac{5\Theta}{2}\right);$$

$$\exp(\Theta K_a): \quad x_0' = x_0, \quad x_c' = \frac{x_c - \delta_{ac}\Theta \vec{x}^2}{1 - 2\Theta x_a + \Theta^2 \vec{x}^2} \quad (c = 1, 2, 3),$$

$$u' = u \left(1 - 2\Theta x_a + \Theta^2 \vec{x}^2\right)^{\frac{5}{2}}, \quad \text{where} \quad \vec{x}^2 = x_1^2 + x_2^2 + x_3^2.$$

References

[1] Dorodnitsyn W.A., Knyazeva I.W. and Svirschewskij S.R., Group properties of the heat conduction equation with source in the two- and three-dimensional cases, *Differential Equations*, 1983, V.19, N 7, 1215–1223 (in Russian).

[2] Fushchych W.I., Shtelen V.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Kyiv, Naukova Dumka, 1989 (in Russian; English version published by Kluwer, Dordrecht, 1993).

[3] Ovsyannikov L.V., Group Analysis of Differential Equations, New York, Academic Press, 1982.

[4] Olver P., Applications of Lie Groups to Differential Equations, New York, Springer, 1987.

[5] Patera J., Winternitz P. and Zassenhaus H., Quantum numbers for particles in de Sitter space, *J. Math. Phys.*, 1976, V.17, N 5, 717–728.

[6] Barannik L.F. and Lahno H.O., Symmetry reduction of the Boussinesq equation to ordinary differential equations, *Reports on Math. Phys.*, 1996, V.38, N 1, 1–9.