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Deformed Oscillators with Interaction

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Abstract

Deformation of the Heisenberg-Weyl algebra W^s of creation-annihilation operators is studied and the problem of eigenvalues of the Hamiltonian for s deformed oscillators with interaction is solved within this algebra. At first, types of deformation are found for which solutions could be presented analytically and a simple $q-$deformation is considered by the means of the perturbation theory. Cases of reducing Hamiltonians which do not preserve the total particle number to that studied here are indicated.

Investigations of new types of symmetries in different areas of mathematical physics by using the Inverse Scattering Method (ISM) and nonlinear differential equations, led to the appearance of notions "Quantum Group", "Quantum Algebra" [1]. Nowadays, the deformed quantum statistics, parastatistics and corresponding algebras of creation-annihilation operators are widely studied.

The common Heisenberg-Weyl algebra W^s consists of generators $\widehat{a}_i$, $\widehat{a}_i^+$, $\widehat{n}_i$, $i = 1, \ldots, s$, satisfying commutative relations:

$$\begin{aligned} &[\widehat{a}_i, \widehat{a}_j^+] = \delta_{ij}, \qquad [\widehat{n}_i, \widehat{a}_j^\pm] = \pm \widehat{a}_j^\pm \delta_{ij}, \\ &[\widehat{a}_i, \widehat{a}_j] = [\widehat{a}_i^+, \widehat{a}_j^+] = 0, \qquad \widehat{n}_i^+ = \widehat{n}_i. \end{aligned} \tag{1}$$

In this paper, the deformed algebra $A_d(s)$ with generators $\widehat{a}_i$, $\widehat{a}_i^+$, $\widehat{n}_i$, $i = 1, \ldots, s$, satisfying:

$$\begin{aligned} &[\widehat{a}_i, \widehat{a}_j^+] = f_i(\widehat{n}_1, \ldots, \widehat{n}_s)\delta_{ij}, \\ &\text{the rest relations are the same as in (1)} \end{aligned} \tag{2}$$

is considered. If we assume $f_i(\widehat{n}_1, \ldots, \widehat{n}_s) = 1$, we return to the algebra W^s (1). One-dimensional deformed oscillators were studied in different papers [2, 3, 10].

Here, we study the problem of eigenvalues of the Hamiltonian

$$\widehat{H} = \sum_{i,j=1}^{s} w_{ij} \widehat{a}_i^+ \widehat{a}_j \tag{3}$$

which gives us s deformed oscillators with interaction.

Algebraically solvable Case

By means of the linear transformation

$$\widehat{b}_i = \sum_k \alpha_{ik} \widehat{a}_k, \tag{4}$$

where $||\alpha_{ik}||$ is unitary matrix with

$$\sum_k \alpha_{ik} \alpha^*_{jk} = \delta_{ij}, \tag{5}$$

one can receive a new set of generators $b_i, b_i^+, N_i, \quad i = 1, \ldots, s,$ which determine a new algebra $B_d(s)$.

Now we can rewrite (3) using the generators of the algebra $B_d(s)$:

$$\widehat{H} = \sum_{ij} w'_{ij} \widehat{b}_i^+ \widehat{b}_j, \tag{6}$$

where $w'_{ij} = \sum\limits_{kl} w_{kl} \alpha_{ik} \alpha^*_{jl}$.

Choosing the appropriate matrix $||\alpha_{ik}||$, we can diagonalize $||w'_{ij}||$ and obtain

$$\widehat{H} = \sum_i w'_i \widehat{b}_i^+ \widehat{b}_i. \tag{7}$$

This Hamiltonian gives us a set of noninteracting deformed oscillators. To find eigenvalues of (7), one should know the commutative relations on $B_d(s)$:

$$\begin{aligned}
&[\widehat{b}_i, \widehat{b}_j^+] = F_{ij}(\widehat{n}_1, \ldots, \widehat{n}_s), \\
&\text{the rest are the same as in (1)} \\
&\text{(considering the substitution } \widehat{a}_i \to \widehat{b}_i,\ \widehat{n}_i \to \widehat{N}_i),
\end{aligned} \tag{8}$$

where

$$F_{ij}(\widehat{n}_1, \ldots, \widehat{n}_s) = \sum \alpha_{ik} \alpha^*_{jk} f_i(\widehat{n}_1, \ldots, \widehat{n}_s) \tag{9}$$

are the functions of "old" generators $\widehat{n}_1, \ldots, \widehat{n}_s$, but we should express the right side of the first equation (8) in the terms of "new" generators $\widehat{N}_1, \ldots, \widehat{N}_s$

The operators $\widehat{N}_i$ cannot be represented as functions of operators $\widehat{n}_i$ generally. Therefore, in the case of the arbitrary algebra $A_d(s)$, the transition from $\widehat{n}_i$ to $\widehat{N}_i$ on the right side of the first equation (8) cannot be fulfilled.

One can introduce operators

$$\widehat{n} = \widehat{n}_1 + \cdots + \widehat{n}_s, \qquad \widehat{N} = \widehat{N}_1 + \cdots + \widehat{N}_s, \tag{10}$$

of the total particle number for the old ($A_d(s)$) and new ($B_d(s)$) algebras, respectively.

It can be easily proved that

$$\widehat{n} = \widehat{N}. \tag{11}$$

So, if we have deformed the algebra $A_d(s)$ with functions f_i satisfying

$$f_i(\widehat{n}_1,\dots,\widehat{n}_s) = f_i(\widehat{n}_1+\cdots+\widehat{n}_s) = f_i(\widehat{n}), \tag{12}$$

then the first relation (8) will read due to (11) as

$$[\widehat{b}_i,\widehat{b}_j^+] = F_{ij}(\widehat{N}), \tag{13}$$

where

$$F_{ij}(\widehat{N}) = \sum_k \alpha_{ik}\alpha_{jk}^* f_i(\widehat{N}). \tag{14}$$

Eigenvalues of (7) can be immediately found now:

$$E_{N_1\dots N_S} = \sum_{i=1}^{s} w_i'\beta_i(N,N_i) \tag{15}$$

with the corresponding eigenfunctions $|N_1,\dots,N_s>$, where

$$\beta_i(N,N_i) = \sum_{j=1}^{N_i} F_{ii}(N-j), \qquad N = N_1+\cdots+N_s.$$

These deformed algebras $A_d(s)$ (2), (12), for which the problem of eigenvalues is solvable, don't factorize, i.e., cannot be represented as sets of s independent deformed algebras.

q-Deformation

Let us consider now the deformed algebra $A_q(2)$ consisting of 2 one-dimensional q-deformed algebras with generators $\widehat{a}$, $\widehat{b}$ ($\widehat{a}=\widehat{a}_1$, $\widehat{b}=\widehat{a}_2$) and standard particle number operators $\widehat{n}_a$, $\widehat{n}_b$ satisfying:

$$\widehat{a}\widehat{a}^+ - q\widehat{a}^+\widehat{a} = 1, \qquad \widehat{b}\widehat{b}^+ - q\widehat{b}^+\widehat{b} = 1. \tag{16}$$

After introducing new operators

$$\widehat{S}^+ = \widehat{a}^+ b, \qquad \widehat{S}^- = \widehat{b}^+ a, \tag{17}$$

Hamiltonian (3) will read:

$$\widehat{H} = w_a\widehat{n}_a + w_b\widehat{n}_b + v(\widehat{S}^+ + \widehat{S}^-). \tag{18}$$

The set of generators $\{\widehat{n}_a,\widehat{b}_a,\widehat{S}^+,\widehat{S}^-\}$ satisfies:

$$[\widehat{n}_a,\widehat{S}^\pm] = \pm\widehat{S}^\pm, \quad [\widehat{n}_b,\widehat{S}^\pm] = \mp\widehat{S}^\pm, \quad [\widehat{n}_a,\widehat{n}_b] = 0, \quad [\widehat{S}^+,\widehat{S}^-] = [\widehat{n}_a]-[\widehat{n}_b], \tag{19}$$

where $[x]$ means the function $[x] = \dfrac{q^x-1}{q-1}$.

The Casimir operator of the algebra $\{\widehat{n}_a,\widehat{b}_a,\widehat{S}^+,\widehat{S}^-\}$ is

$$\widehat{K} = \widehat{n} = \widehat{n}_a + \widehat{n}_b. \tag{20}$$

Consider the problem of eigenvalues of the reduced Hamiltonian (18) on invariant subspaces $K_n = L(|i, n-i> \ | \ i = 1, \ldots, n)$. One can obtain basis vectors in K_n by using the vector $|0, n>$:

$$|i, n-i>= \sqrt{\frac{[n-1]!}{[n]![i]!}}(S^{+})^{i}|0, n>, \tag{21}$$

where $[n]! = [n][n-1]\cdots[1]$.

Now let us take the representation where operators $\{\widehat{S}^{+}|_{K_n}, \widehat{S}^{-}|_{K_n}, \widehat{n}_a|_{K_n}, \widehat{n}_b|_{K_n}\}$ reduced on the subspaces K_n are given by:

$$\begin{aligned}
&\widehat{S}^{+}|_{K_n} = t,\\
&\widehat{S}^{+}|_{K_n} = \frac{1}{t}\left[t\frac{t}{dt}\right]\left[n+1-t\frac{t}{dt}\right],\\
&\widehat{n}_a|_{K_n} = t\frac{t}{dt},\\
&\widehat{n}_b|_{K_n} = n - t\frac{t}{dt}.
\end{aligned} \tag{22}$$

Vectors $\psi \in \widehat{K}_n$ read:

$$\psi = \sum_{i=1}^{n} C_i t^i. \tag{23}$$

In view of (21), one can write $\psi = \left(\sum_{i=1}^{n} C_i S^{+i}\right)\Big|0, n>$ and, taking into account the first equation (22), we receive (23). Now we obtain the equation

$$\widehat{T}_n\psi(t) = E\psi(t), \tag{24}$$

where

$$\widehat{T}_n = v\cdot\left(t + \frac{1}{t}\left[t\frac{t}{dt}\right]\left[n+1-\frac{t}{dt}\right]\right) + w_1\cdot\left[t\frac{t}{dt}\right] + w_2\cdot\left[n - t\frac{t}{dt}\right], \tag{25}$$

on the condition that $\psi(t)$ is an analytic function. In the nondeformed case ($q = 1$), equation (24) can be transformed into a degenerated hypergeometric equation and easily solved:

$$E_{nm} = n\frac{w_1+w_2}{2} + (2m-n)\sqrt{\left(\frac{w_1-w_2}{2}\right)^2 + v^2}. \tag{26}$$

In the deformed case, equation (24) was solved by the means of perturbation theory, using the potential of interaction v between oscillators as a small parameter. Energy eigenvalues were found up to the 5-th order of perturbation theory:

$$\begin{aligned}
&E_0^{(k)} = w_1\cdot[k] + w_2\cdot[n-k], \qquad E_2^{(k)} = \frac{B(k-1)}{\Gamma(k-1)} + \frac{B(k)}{\Gamma(k+1)},\\
&E_4^{(k)} = \frac{B(k-1)}{\Gamma(k-1)^2}\left\{\frac{B(k-2)}{\Gamma(k-2)} - \frac{B(k-1)}{\Gamma(k-1)} - \frac{B(k)}{\Gamma(k+1)}\right\} +\\
&\qquad + \frac{B(k)}{\Gamma(k-1)^2}\left\{\frac{B(k+1)}{\Gamma(k+2)} - \frac{B(k-1)}{\Gamma(k-1)} - \frac{B(k)}{\Gamma(k+1)}\right\},
\end{aligned} \tag{27}$$

where $B(p) = [p+1][n-p]$, $\Gamma(p) = w\left([k]-[p]\right) + w\left([n-k]+[n-p]\right).$

All odd approximations are equal to zero:

$$E_1 = E_3 = \cdots = E_{2p+1} = 0. \tag{28}$$

Generalized Weyl Shift

A Weyl shift of the nondeformed algebra W^s reads:

$$\widehat{b}_i = \widehat{a}_i + \alpha_i. \tag{29}$$

For the deformed algebra $A_q(s)$, it can be assumed in the form proposed in [4]:

$$\widehat{b}_i = F_i(\widehat{n}_i)\widehat{a}_i - f_i(\widehat{n}_i), \qquad f_i = w_i q^{\widehat{n}_i}, \qquad F_i = \sqrt{1-|w_i|^2(1-q)q^{\widehat{n}_i}}. \tag{30}$$

Hamiltonian (3) preserves the total particle number $n = n_1 + \cdots + n_s$, but if we make the generalized Weyl shift (30), it will not. There will be some terms in a shifted Hamiltonian, which will increase or decrease the particle number by 1.

Our aim is to find "shifted" Hamiltonians (which violate the particle number at 1 unit) which can be reduced to "neutral" Hamiltonians (preserving the particle number n). The general "neutral" Hamiltonian which covers case (3) reads:

$$\widehat{H} = \sum_{ij} \phi_{ij}(\widehat{b}_i^+\widehat{b}_i, \widehat{b}_j^+\widehat{b}_j, \widehat{b}_i^+\widehat{b}_j, \widehat{b}_j^+\widehat{b}_i). \tag{31}$$

After shift (30), arguments of (31) will read as

$$\begin{aligned} \widehat{b}_i^+\widehat{b}_j &= \left(\widehat{a}_i^+F_i^+ - f_i^+\right)\left(F_j\widehat{a}_j - f_j\right) = \\ &= \widehat{a}_i^+\widehat{\sigma}_{ij}(\widehat{n}_i,\widehat{n}_j)\widehat{a}_j + \widehat{a}_i^+\widehat{\pi}_{ij}(\widehat{n}_i,\widehat{n}_j) + \widehat{\pi}^+ij(\widehat{n}_i,\widehat{n}_j)\widehat{a}_j + \widehat{\omega}ij(\widehat{n}_i,\widehat{n}_j), \end{aligned} \tag{32}$$

where $\widehat{\sigma}^{ij}, \widehat{\pi}^{ij}, \widehat{\pi}^{ij}$ are some "neutral" functions. Developing (31) into the series in new shifted arguments (right side of (32)), we will receive some "shifted" Hamiltonian $\widehat{H}'$. If we demand that $\widehat{H}'$ violate n not more than by 1 unit, we should consider ϕ_{ij} to be a linear function of its arguments, because these arguments (32) do violate n by one unit, and higher powers of (32) will violate n more than by one unit. ϕ_{ij} reads:

$$\phi_{ij}(x,y,z,t) = \alpha_{ij}x + \beta_{ij}y + \gamma_{ij}z + \zeta_{ij}t, \tag{33}$$

and we receive Hamiltonian (3):

$$\widehat{H} = \sum_{i,j=1}^{s} w_{ij}\widehat{b}_i^+\widehat{b}_j, \tag{34}$$

where

$$w_{ij} = \gamma_{ij} + \zeta_{ij} + \delta_{ij}\sum_k(\alpha_{ik} + \beta_{ik}).$$

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