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Representation of Real Forms of Witten's First Deformation

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Abstract

Special star structures on Witten's first deformation are found. Description of all irreducible representations in the category of Hilbert spaces of the found $*$-algebras in both bounded and unbounded operators is obtained.

1 Introduction

Studying Jone's polynomials in node theories, their generalization and their connections with "vertex models" in two-dimensional statistical mechanics, Witten presents Hoph algebra deformations of the universal enveloping algebra $su(2)$. There is a family of associative algebras depending on a real parameter p. These algebras are given by the generators E_0, E_+, E_- and relations [2]:

$$pE_0E_+ - \frac{1}{p}E_+E_0 = E_+, \tag{1}$$

$$[E_+, E_-] = E_0 - \left(p - \frac{1}{p}\right) E_0^2, \tag{2}$$

$$pE_-E_0 - \frac{1}{p}E_0E_- = E_-. \tag{3}$$

In Section 1, we write down all (but from a certain class) star algebra structures on Witten's first deformation. In Section 3, we give description of all irreducible representations in bounded or unbounded operators of the found $*$-algebras in the category of Hilbert's spaces. We widely use the "dynamical relations" method developed in [3].

2 Real forms

Witten's first deformation is a family of associative algebras A_p given by generators and relations (1)–(3), with the parameter p in $(0,1)$. Extreme cases are $p = 1$, which is $su(2)$, and $p = 0$, with relations $E_+E_0 = E_0E_+ = E_0^2 = 0$. We consider stars on A_p, which are obtained from involutions on the free algebra with the invariant lineal subspace generated by relations (1)–(3).

Stars are equivalent if the corresponding real forms are isomorphic.

Lemma 1 *There are only two unequivalent stars on Witten's first deformation:*

$$E_0^* = E_0, \qquad E_+^* = E_-, \tag{4}$$

$$E_0^* = E_0, \qquad E_+^* = -E_-. \tag{5}$$

3 Dynamical relations.

Relations (1)–(3) with star (4) or (5) form a dynamic relation system [3]. The corresponding dynamic system on $\mathbb{R}^2$:

$$F(x,y) = \Big(p^{-1}(1+p^{-1}x), g(gy - x + (p - p^{-1})x^2)\Big),$$

where $g = 1$ ($g = -1$) for the first (second) real form.

It's a quite difficult task to find positive orbits of a two-dimensional nonlinear dynamic system, to avoid these difficulties, we use Casimir element:

$$C_p = p^{-1}E_+E_- + pE_-E_+ + E_0^2.$$

For any irreducible representation T: $T(C_p) = \mu I$, where μ is complex.

We will be working with the following system:

$$E_0E_+ = E_+f(E_0), \tag{6}$$

$$E_+^*E_+ = G_\nu(E_0), \tag{7}$$

where $\nu = \mu p$,

$$f(X) = p^{-1}(1+p^{-1}x),$$
$$G_\nu(y) = \frac{g}{1+p^2}(-y - p^{-1}y^2 + \nu I).$$

Lemma 2 *For any irreducible representation T of the real form A_p, there is a unique ν ($\nu \geq 0$ for the first real form) such that T is the representation of (6), (7).*

For an arbitrary ν ($\nu \geq 0$ for the first real form), every irreducible representation T of (6), (7) with $dim T > 1$ is a representation of A_p.

The dynamic system corresponding to relations (6), (7) is actually one-dimensional and linear depending on one real parameter.

Below we compile some basic facts from [3]. Every irreducible representation of a dynamic system is determined by the subset $\mathbb{R}^2 \supset \Delta$:

$$\Delta = \{(\lambda_k, \mu_k), j < k < J\},$$

where $\lambda_{k+1} = f(\lambda_k)$, $\mu_{k+1} = G_\nu(\lambda_k)$, $\mu_k \geq 0$, $\mu_k = 0$ for extreme k; j, J are integer or infinite; $l_2(\Delta)$ is a Hilbert space with the orthonormed base: $\{e_{(\lambda_k,\mu_k)} : (\lambda_k,\mu_k) \in \Delta\}$,

$$T(E_0)e_{(\lambda_k,\mu_k)} = \lambda_k e_{(\lambda_k,\mu_k)},$$
$$T(E_+)e_{(\lambda_k,\mu_k)} = \mu_{k+1}^{1/2} e_{(\lambda_{k+1},\mu_{k+1})}, \qquad j < k+1 < J,$$
$$T(E_-)e_{(\lambda_k,\mu_k)} = \mu_k^{1/2} e_{(\lambda_{k-1},\mu_{k-1})}, \qquad j < k-1,$$
$$T(E_+)e_{(\lambda_{J-1},\mu_{J-1})} = 0, \qquad T(E_-)e_{(\lambda_{j+1},\mu_{j+1})} = 0.$$

4 Classification of representations

Theorem 1 *Every irreducible representation of the first real form is bounded*

1. For every nonnegative integer m, there is a representation of dimension m + 1, with $\nu = \frac{p}{4}\left((\frac{(1-p^{2m})(1+p^2)}{(1+p^{2m})(1-p^2)})^2 - 1\right)$, $\Delta_\nu = \{f(k, x_1), -1 < k \le m+1\}$;

2. There is a family of one-dimensional representations $E_0 = \frac{p}{(p^2-1)}$; $E_+ = \lambda$; $E_- = \bar{\lambda}$; *λ is complex,* $\nu = \frac{p^3}{(1-p^2)^2}$;

3. For every $\nu \in \left[\frac{p^3}{(1-p^2)^2}; +\infty\right)$, *there is a representation with the upper weight* $\Delta_\nu = \{f(k, x_2),\ k < 1\}$;

4. For every nu $\in \left[\frac{p^3}{(1-p^2)^2}; +\infty\right)$, *there is a representation with the lower weight:* $\Delta_\nu = \{f(k, x_1),\ k < 1\}$.

In the theorem, we have used notation

$$f(k, x) = \frac{1}{p^{2k}}\left(x + \frac{p^{2k}-1}{p^2-1}p\right); \qquad g_\nu(x) = -x - p^{-1}x^2 + \nu,$$

were $x_1 < x_2$ are roots of the equation $g_\nu(x) = 0$.

Theorem 2 *There are bounded and unbounded irreducible representations of the second real form in an infinite-dimensional Hilbert space, except for a one-dimensional representation.*

1. *All bounded irreducible representations have the upper weight*

$$\nu \in \left[-\frac{p}{4}; \frac{p^3}{(1-p^2)^2}\right), \qquad \Delta_\nu = \{f(k, x_1), k < 1\};$$

2. *Unbounded irreducible representations:*

   (a) *There are two families with the upper weight*
   first family:

$$\nu \in \left(-\frac{p}{4}\ ;\ 0\right), \qquad \Delta_\nu = \{f(k, x_2), k < 1\}$$

   second family:

$$\nu \in \left(\frac{p^3}{(1-p^2)^2}; +\infty\right), \qquad \Delta_\nu = \{f(k, x_1), k > -1\}$$

   (b) *There are two families with the lower weight*
   first family:

$$\nu \in \left[\frac{p}{4}\ ; 0\right), \qquad \Delta_\nu = \{f(k, x_2), k > -1\}$$

   second family:

$$\nu \in \left(\frac{p^3}{(1-p^2)^2}; +\infty\right), \qquad \Delta_\nu = \{f(k, x_1), k > -1\}$$

(c) There are two families without the upper and lower weights:
first family is numerated by the set

$$\tau = \{(\lambda, \nu) : \lambda \in \left(-p\ ; -\frac{p}{2}\right) \cup \left(-\frac{p}{2}\ ; 0\right], \nu \in (-\infty\ ; \lambda(\lambda + p)p^{-1})\}$$

$$\Delta_{(\lambda,\nu)} = \{f(k, \lambda), k \in \mathbb{Z}\}$$

second family is numerated by the set

$$\epsilon = [-p^{-2}\ ; -p - 1) \times (-\infty\ ; p^3(1 - p^2)^{-1}), \quad \Delta_{(\lambda,\nu)} = \{f(k, \lambda), k \in \mathbb{Z}\}.$$

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