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On Conditional Symmetries of Multidimensional Nonlinear Equations of Quantum Field Theory

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Abstract

We give a brief review of our results on investigating conditional (non-classical) symmetries of the multidimensional nonlinear wave Dirac and $SU(2)$ Yang-Mills equations.

Below we give a brief account of results of studying conditional symmetries of multidimensional nonlinear wave, Dirac and Yang-Mills equations obtained in collaboration with W.I. Fushchych in 1989–1995. It should be noted that till our papers on exact solutions of the nonlinear Dirac equation [1]–[4], where both symmetry and conditional symmetry reductions were used to obtain its exact solutions, only two-dimensional (scalar) partial differential equations (PDEs) were studied (for more detail, see, [5]). The principal reason for this is the well-known fact that the determining equations for conditional symmetries are nonlinear (we recall that determining equations for obtaining Lie symmetries are linear). Thus, to find a conditional symmetry of a multidimensional PDE, one has to find a solution of the nonlinear system of partial differential equations whose dimension is higher that the dimension of the equation under study! In paper [3], we have suggested a powerful method enabling one to obtain wide classes of conditional symmetries of multidimensional Poincaré-invariant PDEs. Later on it was extended in order to be applicable to Galilei-invariant equations [6] which yields a number of conditionally-invariant exact solutions of the nonlinear Levi-Leblond spinor equations [7]. The modern exposition of the above-mentioned results can be found in monograph [8].

Historically, the first physically relevant example of conditional symmetry for a multidimensional PDE was obtained for the nonlinear Dirac equation. However, in this paper, we will concentrate on the nonlinear wave equation which is easier for understanding the basic techniques used to construct its conditional symmetries.

As is well known, the maximal invariance group of the nonlinear wave equation

$$\Box u \equiv u_{x_0 x_0} - \Delta u = F_0(u), \quad F \in C^1(\mathbf{R}^1, \mathbf{R}^1) \tag{1}$$

is the 10-parameter Poincaré group having the generators

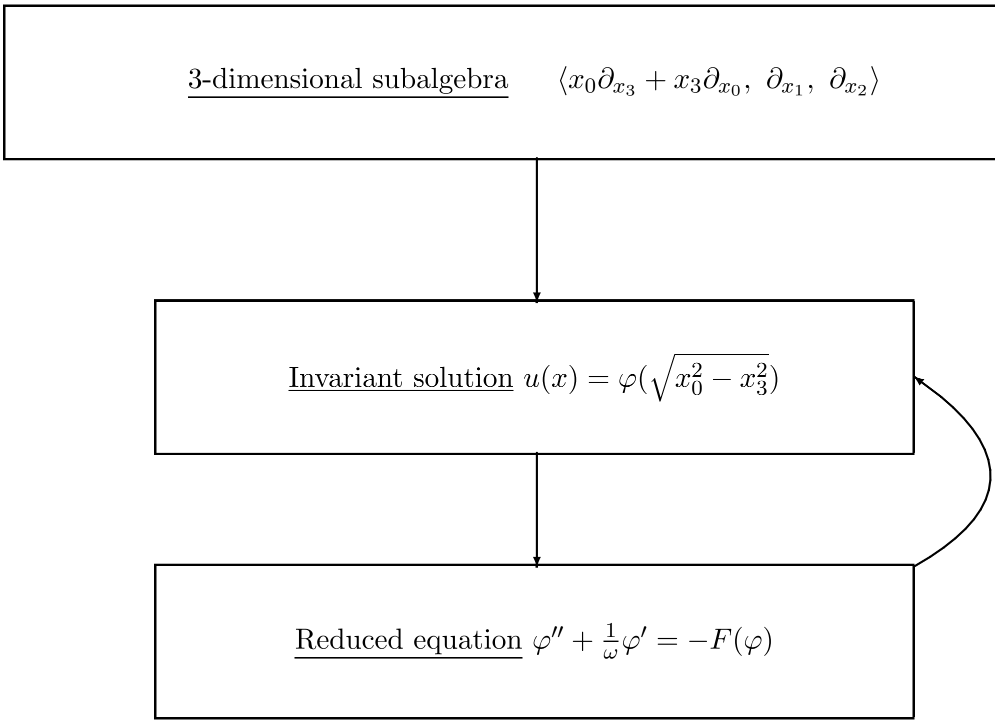
$$P_\mu = \partial_{x_\mu}, \quad J_{0a} = x_0 \partial_{x_a} + x_a \partial_{x_0}, \quad J_{ab} = x_a \partial_{x_b} - x_a \partial x_a,$$

where $\mu = 0, 1, 2, 3,\ a, b = 1, 2, 3.$

The problem of symmetry reduction for the nonlinear wave equation by subgroups of the Poincaré group in its classical setting has been solved in [9]. Within the framework of the symmetry reduction approach, a solution is looked for as a function

$$u(x) = \varphi(\omega) \tag{2}$$

Fig.1. Symmetry reduction scheme



of an invariant $\omega(x)$ of a subgroup of the Poincaré algebra. Then inserting the Ansatz $\varphi(\omega)$ into (1) yields an ordinary differential equation (ODE) for the function $\varphi(\omega)$. As an illustration, we give Fig.1, where ω is the invariant of the subalgebra $\langle J_{03}, P_1, P_2\rangle \in AP(1,3)$.

The principal idea of our approach to constructing conditionally-invariant Ansätze for the nonlinear wave equation was to preserve the form of Ansatz (2) but not to fix *a priori* the function $\omega(x)$. The latter is so chosen that inserting (2) should yield an ODE for the function $\omega(x)$. This requirement leads to the following intermediate problem: we have to integrate the system of two nonlinear PDEs in four independent variables called the d'Alembert-eikonal system

$$\omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \quad \Box\omega = F_2(\omega). \tag{3}$$

Hereafter, summation over repeated indices in the Minkowski space with the metric tensor $g_{\mu\nu} = \delta_{\mu\nu} \times (1, -1, -1, -1)$ is understood, i.e.,

$$\omega_{x_\mu}\omega_{x^\mu} \equiv \omega_{x_0}^2 - \omega_{x_1}^2 - \omega_{x_2}^2 - \omega_{x_3}^2.$$

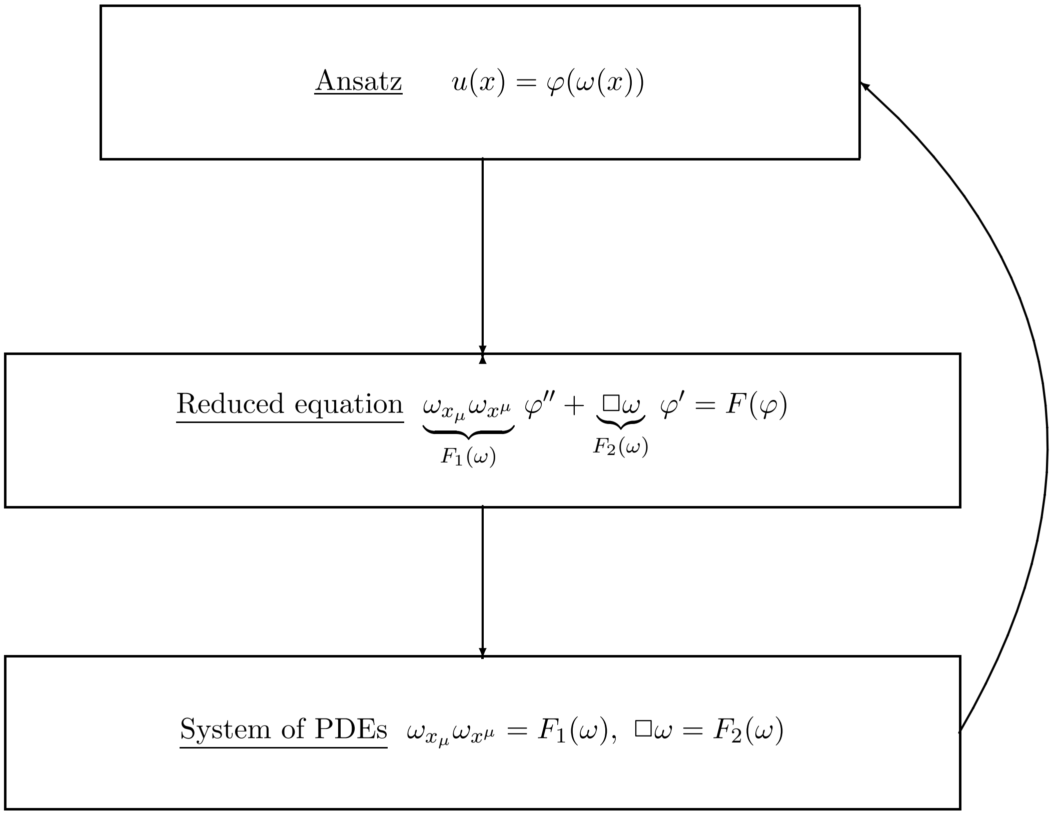
As an illustration, we give below Fig.2.

According to [10], the compatible system of PDEs (3) is equivalent to the following one:

$$\omega_{x_\mu}\omega_{x^\mu} = \lambda, \quad \Box\omega = \frac{\lambda N}{\omega}, \tag{4}$$

where $\lambda = 0, \pm 1$ and $N = 0, 1, 2, 3$. In papers [11, 12], we have constructed general solutions of the above system for all possible values λ, N. Here, we restrict ourselves to giving the general solution of the d'Alembert-eikonal system for the case $N = 3, \lambda = 1$.

Fig.2. Conditional symmetry reduction scheme



Theorem. *The general solution of the system of nonlinear PDEs*

$$\omega_{x_\mu}\omega_{x^\mu} = 1, \quad \Box\omega = \frac{3}{\omega} \tag{5}$$

is given by the following formula:

$$u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big),$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0$$

and the functions $A_\mu(\tau)$, $B_\mu(\tau)$ satisfy the relations

$$A'_\mu B^\mu = 0, \qquad B_\mu B^\mu = 0.$$

This solution contains **five** arbitrary functions of one variable. Choosing $A_\mu = C_\mu =$ const, $B_\mu = 0$, $\mu = 0, 1, 2, 3$, yields an invariant of the Poincaré group $\omega(x) = (x_\mu + C_\mu)(x^\mu + C^\mu)$. All other choices of the functions A_μ, B_μ lead to Ansätze that correspond to conditional symmetry of the nonlinear wave equation. Conditional symmetry generators can be constructed in explicit form, however, the resulting formulae are rather cumbersome. That is why we will consider a more simple example of a non-Lie Ansatz, namely

$$u(x) = \varphi\Big(x_1 + w(x_0 + x_3)\Big).$$

As a direct computation shows, the above function is the general solution of the system of linear PDEs

$$Q_a u(x) = 0, \quad a = 1, 2, 3,$$

where

$$Q_1 = \partial_{x_0} - \partial_{x_3}, \quad Q_2 = \partial_{x_0} + \partial_{x_3} - 2w'\partial_{x_1}, \quad Q_3 = \partial_{x_2}.$$

The operator Q_2 cannot be represented as a linear combination of the basis elements of the Lie algebra of the Poincaré group, since it contains an arbitrary function. Furthermore, the operators Q_1, Q_2, Q_3 are commuting and fulfill the relations

$$\widehat{Q}_1 L = 0, \quad \widehat{Q}_2 L = \underline{4w''\partial_{x_1}(Q_1 u)}, \quad \widehat{Q}_3 L = 0,$$

where $L = \Box u - F(u)$, whence it follows that the system

$$\Box u = F(u), \quad Q_1 u = 0, \quad Q_2 u = 0, \quad Q_3 u = 0$$

is invariant in Lie's sense with respect to the commutative Lie algebra $\mathcal{A} = \langle Q_1, Q_2, Q_3 \rangle$. This means, in its turn, that the nonlinear wave equation $\Box u = F(u)$ is conditionally-invariant with respect to the algebra $\mathcal{A}$. The geometric interpretation of these reasonings is given in Fig.3.

We recall that a PDE

$$U(x, u) = 0$$

is conditionally-invariant under the (involutive) set of Lie vector fields $\langle Q_1, \ldots, Q_n \rangle$ if there exist PDEs

$$U_1(x, u) = 0, \quad U_2(x, u) = 0, \ldots, \quad U_N(x, u) = 0$$

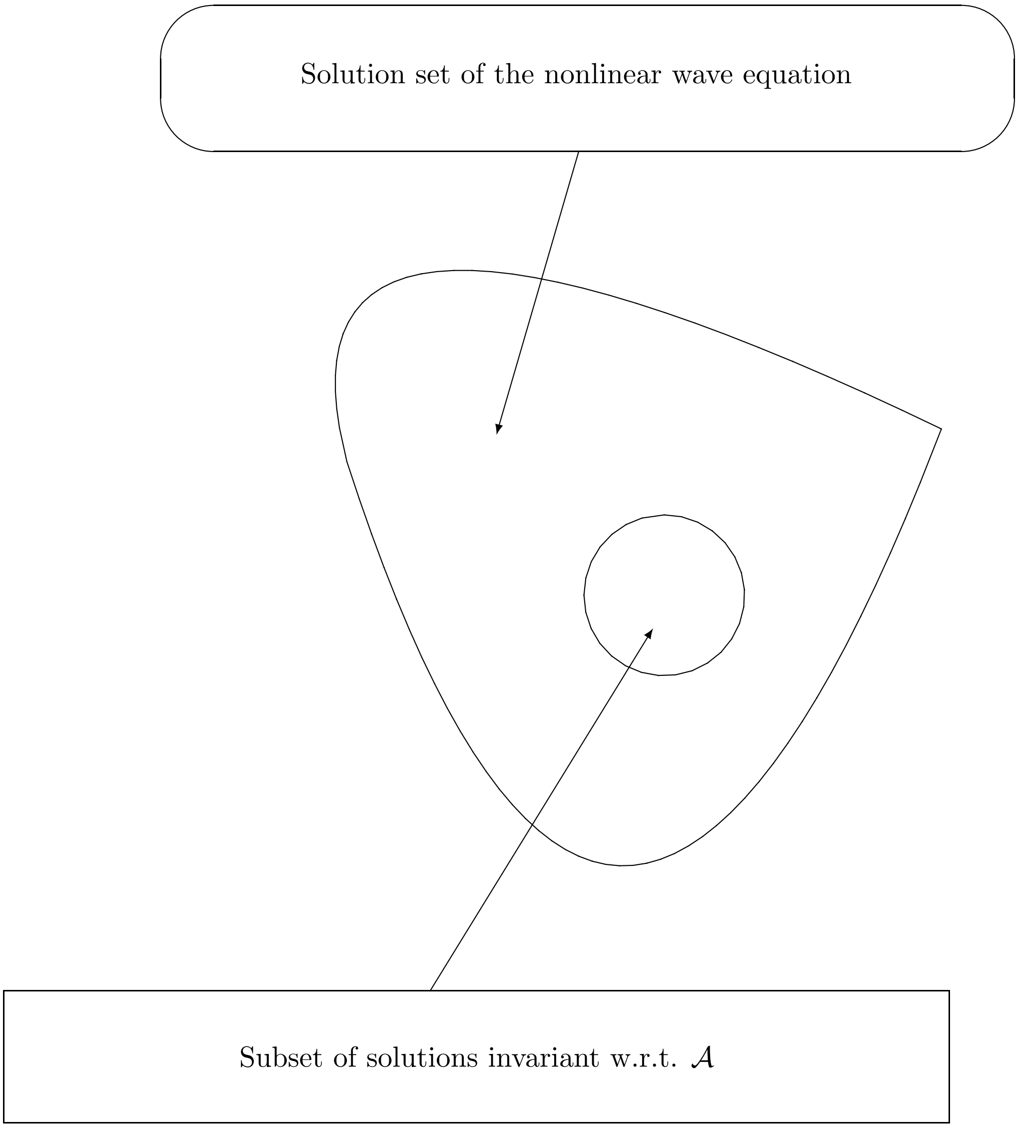
such that the system

$$\begin{cases} U(x, u) = 0, \\ U_1(x, u) = 0, \\ \ldots \\ U_N(x, u) = 0 \end{cases}$$

is invariant in Lie's sense with respect to the operators Q_a, $\forall a$.

In particular, we may take $n = N$, $U_i(x, u) = Q_i u$ which yields a particular form of the conditional symmetry called sometimes Q-conditional symmetry (for more detail, see [8, 14]).

Fig.3



Next we give without derivation examples of new conditionally-invariant solutions of the nonlinear wave equation with polynomial nonlinearities $F(u)$ obtained in [15].

1. $F(u) = \lambda u^3$

1) $u(x) = \dfrac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2} \tan\left\{ -\dfrac{\sqrt{2}}{4} \ln\Big(C(\omega)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)\right\}$,

where $\lambda = -2a^2 < 0$,

2) $u(x) = \dfrac{2\sqrt{2}}{a} C(\omega)\Big(1 \pm C^2(\omega)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)^{-1}$,

where $\lambda = \pm a^2$;

2. $F(u) = \lambda u^5$

1) $$u(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\Big\{\sin\ln\Big(C(\omega)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) + 1\Big\}^{1/2} \times \Big\{2\sin\ln\Big(C(\omega)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) - 4\Big\}^{-1/2},$$

where $\lambda = a^4 > 0$,

2) $$u(x) = \frac{3^{1/4}}{\sqrt{a}}C(\omega)\Big(1 \pm C^4(\omega)(x_1^2 + x_2^2 - x_0^2)\Big)^{-1/2},$$

where $\lambda = \pm a^2$.

In the above formulae, $C(\omega)$ is an arbitrary twice continuously differentiable function of

$$\omega(x) = \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

Let us turn now to the following class of Poincaré-invariant nonlinear generalizations of the Dirac equation:

$$\Big(i\gamma_\mu\partial_{x_\mu} - f_1(\bar\psi\psi, \bar\psi\gamma_4\psi) - f_2(\bar\psi\psi, \bar\psi\gamma_4\psi)\gamma_4\Big)\psi = 0. \tag{6}$$

Here, γ_μ are 4×4 Dirac matrices, $\mu = 0, 1, 2, 3$, $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$, ψ is the 4-component complex-valued function, $\bar\psi = (\psi^*)^T\gamma_0$, f_1, f_2 are arbitrary continuous functions.

The class of nonlinear Dirac equations (6) contains as particular cases the nonlinear spinor models suggested by Ivanenko, Heisenberg and Gürsey.

We have completely solved the problem of symmetry reduction of system (6) to systems of ODEs by subgroups of the Poincaré group. An analysis of invariant solutions obtained shows that the most general form of a Poincaré-invariant solution reads

$$\psi(x) = \exp\{\theta_j\gamma_j(\gamma_0 + \gamma_3)\}\exp\{(1/2)\theta_0\gamma_0\gamma_3 + (1/2)\theta_3\gamma_1\gamma_2\}\varphi(\omega), \tag{7}$$

where $\theta_0(x), \ldots, \theta_3(x), \omega(x)$ are some smooth functions and $\varphi(\omega)$ is a new unknown four-component function.

Now we make use of the same idea as above. Namely, we do not fix *a priori* the form of the functions θ_μ, ω. They are chosen in such a way that inserting Ansatz (7) into system (6) yields a system of ODEs for $\varphi(\omega)$. After some tedious calculations, we get a system of 12 nonlinear first-order PDEs for five functions θ_μ, ω. We have succeeded in constructing its general solution which gives rise to 11 classes of Ansätze (7) that reduce the system of PDEs (6) to ODEs. And what is more, only five of them correspond to the Lie symmetry of (6). Other six classes are due to the conditional symmetry of the nonlinear Dirac equation.

Below we give an example of a conditionally-invariant Ansatz for the nonlinear Dirac equation

$$\theta_j = \frac{1}{2}w_j' + \left(\frac{a\sqrt{z_1^2 + z_2^2}}{x_0 + x_3}\arctan\frac{z_1}{z_2} + w_3\right)\partial_{x_j}\left(\arctan\frac{z_1}{z_2}\right),$$

$$\theta_0 = \ln(x_0 + x_3), \quad \theta_3 = -\arctan\frac{z_1}{z_2}, \quad \omega = z_1^2 + z_2^2,$$

where $z_j = x_j + w_j,\ j = 1, 2,\ w_1, w_2, w_3$ are arbitrary smooth functions of $x_0 + x_3$ and a is an arbitrary real constant. Provided,

$$a = 0, \quad w_1 = \text{const}, \quad w_2 = \text{const}, \quad w_3 = 0$$

the above Ansatz reduces to a solution invariant under the 3-parameter subgroup of the Poincaré group. However if, at least, one of these restrictions is not satisfied, the Ansatz cannot in principle be obtained by the symmetry reduction method.

Next we will turn to the $SU(2)$ Yang-Mills equations

$$\begin{aligned}&\Box\vec{A}_\mu - \partial_{x^\mu}\partial_{x_\nu}\vec{A}_\nu + e\Big((\partial_{x_\nu}\vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_{x_\nu}\vec{A}_\mu)\times\vec{A}_\nu\\ &\quad +(\partial_{x^\mu}\vec{A}_\nu)\times\vec{A}^\nu\Big) + e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0}.\end{aligned} \tag{8}$$

Here, $\vec{A}_\mu = \vec{A}_\mu(x_0, x_1, x_2, x_3)$ is the three-component vector-potential of the Yang-Mills field, symbol $\times$ denotes vector product, e is a coupling constant.

It is a common knowledge that the maximal symmetry group of the Yang-Mills equations contains as a subgroup the ten-parameter Poincaré group $P(1,3)$. In our joint paper with Fushchych and Lahno, we have obtained an exhaustive description of the Poincaré-invariant Ansätze that reduce the Yang-Mills equations to ODEs [16]. An analysis of the results obtained shows that these Ansätze, being distinct at the first sight, have the same general structure, namely

$$\vec{A}_\mu = R_{\mu\nu}(x)\vec{B}_\nu(\omega(x)), \quad \mu = 0, 1, 2, 3, \tag{9}$$

where

$$\begin{aligned}R_{\mu\nu}(x) &= (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\sinh\theta_0\\ &+2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu\\ &+(\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3\\ &-(c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu).\end{aligned}$$

In (9), $\omega(x)$, $\theta_\mu(x)$ are some smooth functions and what is more

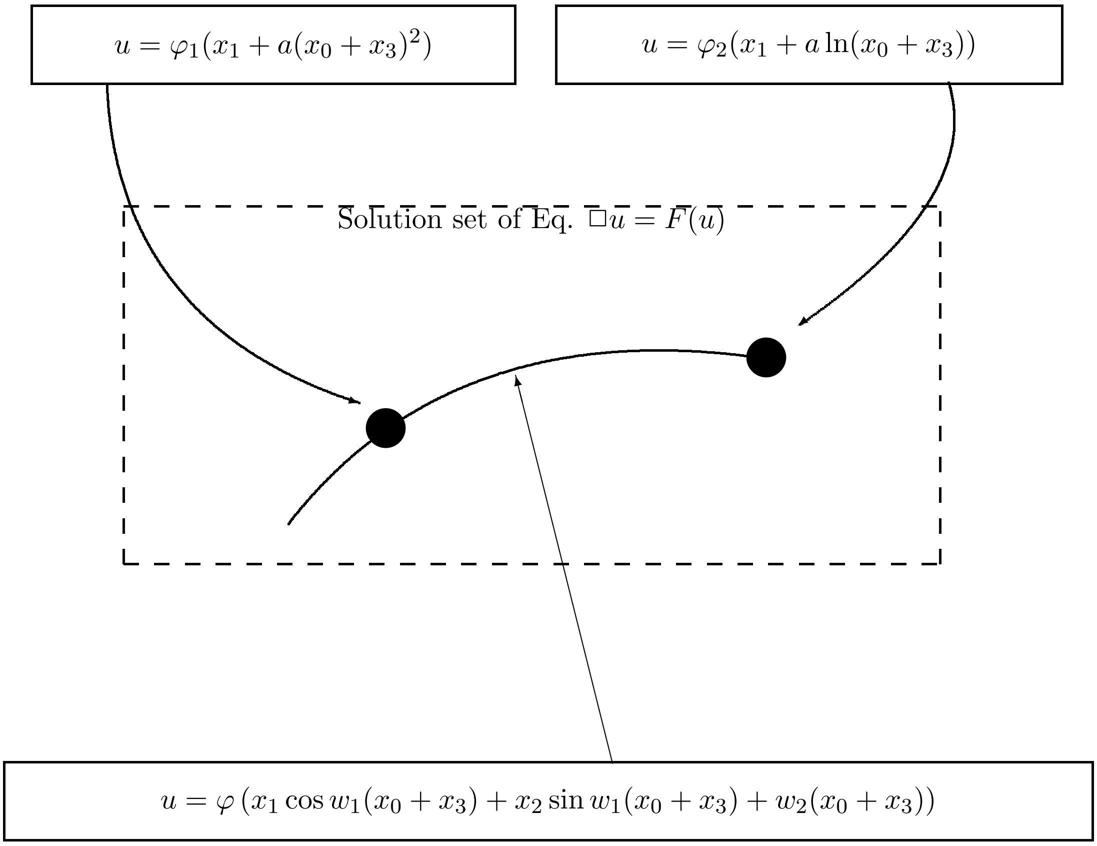
$$\theta_j = \theta_j(a_\mu x^\mu + d_\mu x^\mu,\ b_\mu x^\mu,\ c_\mu x^\mu), \quad j = 1, 2,$$

$a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ are arbitrary constants satisfying the following relations:

$$\begin{aligned}&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\ &a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.\end{aligned}$$

We have succeeded in constructing three classes of conditionally-invariant Ansätze of the form (9) which yield five new classes of exact solutions of the $SU(2)$ Yang-Mills equations [17].

Fig.4



In conclusion, we would like to point out a remarkable property of conditionally-invariant solutions obtained with the help of the above-presented approach. As noted in [15], a majority of solutions of the wave and Dirac equations constructed by virtue of the symmetry reduction routine are particular cases of the conditionally-invariant solutions. They are obtained by a proper specifying of arbitrary functions and constants contained in the latter. As an illustration, we give in Fig.4, where we demonstrate the correspondence between two invariant solutions $u(x) = \varphi_1(x_1 + a(x_0 + x_3)^2)$, $u(x) = \varphi_2(x_1 + a(x_0 + x_3)^2)$ and the more general conditionally-invariant solution of the form:

$$u(x) = \varphi\Big(x_1 \cos w_1(x_0 + x_3) + x_2 \sin w_1(x_0 + x_3) + w_2(x_0 + x_3)\Big),$$

where w_1, w_2 are arbitrary functions.

Acknowledgments

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References

[1] Fushchych W.I. and Zhdanov R.Z., Symmetry and exact solutions of the nonlinear Dirac equation, *Fizika Elementar. Chastits i Atom. Yadra*, 1988, V.19, N 5, 1154–1196.

[2] Fushchych W.I. and Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor equations, *Phys. Repts.*, 1989, V.172, N 4, 123–174.

[3] Fushchych W.I. and Zhdanov R.Z., Non-Lie Ansätze and exact solutions of the nonlinear spinor equation, *Ukrain. Math. J.*, 1990, V.42, N 7, 958–962.

[4] Fushchych W.I. and Zhdanov R.Z., On non-Lie reduction of the nonlinear Dirac equation, *J. Math. Phys.*, 1991, V.32, N 12, 3488–3490.

[5] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics, Naukova Dumka, Kyiv, 1989 (translated into English by Kluwer Academic Publishers, Dordrecht, 1993).

[6] Zhdanov R.Z. and Andreitsev A.Yu., On non-Lie reduction of Galilei-invariant spinor equations, *Dopovidi Akad. Nauk Ukr.*, 1990, N 7A, 8–11.

[7] Fushchych W.I. and Zhdanov R.Z., Nonlinear Spinor Equations: Symmetry and Exact Solutions, Kyiv, Naukova Dumka, 1992.

[8] Fushchych W.I. and Zhdanov R.Z., Symmetries and Exact Solutions of Nonlinear Dirac Equations, Kyiv, Mathematical Ukraina Publisher, 1997.

[9] Grundland A., Harnad J. and Winternitz P., Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, V.25, N 4, 791–806.

[10] Fushchych W.I., Zhdanov R.Z. and Revenko I.V., Compatibility and solutions of the nonlinear d'Alembert and Hamilton equations, Preprint, Inst. of Math. Acad. of Sci. Ukraine, N 90-39, Kyiv, 1990.

[11] Fushchych W.I., Zhdanov R.Z. and Revenko I.V., General solution of the nonlinear wave equation and eikonal equation, *Dopovidi Akad. Nauk Ukr.*, 1991, N 10A, 29–31.

[12] Fushchych W.I., Zhdanov R.Z. and Revenko I.V., General solutions of the nonlinear wave and eikonal equations, *Ukrain. Math. J.*, 1991, V.43, N 11, 1471–1486.

[13] Fushchych W.I. and Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukrain. Math. J.*, 1992, V.44, N 7, 970–982.

[14] Zhdanov R.Z. and Tsyfra I.M., Reduction of differential equations and conditional symmetry, *Ukr. Math. J.*, 1996, V.48, N 5, 595–602.

[15] Zhdanov R.Z., Revenko I.V. and Fushchych W.I., On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications, *J. Math. Phys.*, 1995, V.36, N 12, 7109–7127.

[16] Lahno V.I., Zhdanov R.Z. and Fushchych W.I., Symmetry reduction and exact solutions of the Yang-Mills equations, *J. Nonlin. Math. Phys.*, 1995, V.2, N 1, 51–72.

[17] Zhdanov R.Z. and Fushchych W.I., Conditional symmetry and new classical solutions of $SU(2)$ Yang-Mills equations, *J. Phys. A: Math. Gen.*, 1995, V.28, N 21, 6253–6263.