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On Some Integrable System of Hyperbolic Type

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Abstract

The example of an integrable hyperbolic system with exponential nonlinearities, which somewhat differs from known integrable systems of the Toda type as well as its local conservation laws and reductions is presented. A wide class of exact solutions in some particular case of the system is found.

1 Introduction

A large class of hyperbolic systems with exponential nonlinearities of the type

$$u^i_{xt} = \sum_{j=1}^{m} a^i_j \exp\left(\sum_{k=1}^{n} b^j_k u^k\right), \quad i = 1, \ldots, n, \tag{1}$$

as is known, can be written as the zero-curvature condition (see, for example, [1])

$$P_t - Q_x + [P, Q] = 0, \tag{2}$$

where $P = P(x, t, \zeta)$ and $Q = Q(x, t, \zeta)$ are two matrix-functions having a simple pole at $\zeta = \infty$ and $\zeta = 0$, respectevely. Representation (2) is one of the distinguished features of partial differential equations in one spatial and one temporal dimensions, which have the infinite sequences of symmetries and local conservation laws.

The aim of this work is to present an example of hyperbolic systems, which has representation (2) and, in the same time, somewhat differs from systems of the type (1). Also we present possible reductions of this system, which can be interesting from the physical point of view.

2 Auxiliary linear problem

To begin with, we consider a linear problem in the form of the first-order system

$$\Psi_x = P(x, \zeta)\Psi, \tag{3}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ is the column-vector depending upon the variable $x \in \mathbf{R}^1$ and spectral parameter $\zeta \in \mathbf{C}^1$. Matrix P is written down in its explicit form as follows:

$$P(x, \zeta) = \begin{pmatrix} -i\zeta + r^1 & 1 & r^3 \\ i\zeta r^2 & -2r^1 & -i\zeta r^2 \\ r^3 & 1 & i\zeta + r^1 \end{pmatrix}. \tag{4}$$

Thus, the dependence of matrix elements of $P(x,\zeta)$ on the spatial variable $x \in \mathbf{R}^1$ is defined by the collection of complex-valued functions $\{r^i = r^i(x),\ i = 1,2,3\}$, which are assumed to be smooth everywhere in some domain.

Let us give some remarks about notations. In what follows, we shall omit notations to indicate the evolution parameters dependence. Any vector-function $(r^1, r^2, r^3)^T$ will be denoted by r. We denote a ring of differential functions of r by A_r and a ring of matrix differential operators with coefficients from A_r by $A_r[\partial_x]$.

The linear system (3) is intimately linked with other linear problem. Let us consider the linear equation

$$Ly = (i\zeta)^2 My, \tag{5}$$

where $L = \partial_x^3 + q^1(x)\partial_x + q^2(x)$ and $M = \partial_x + q^3(x)$ are two linear differential operators. Eq.(5) can be used as an auxiliary linear problem for bi-Hamiltonian evolution equations [6]

$$q_{\tau_n} = \mathcal{E}\mathrm{grad}_q H_n = \mathcal{D}\mathrm{grad}_q H_{n+2} \tag{6}$$

with the sequence of Hamiltonians $H_n = \int\limits_{-\infty}^{+\infty} h_n dx$. The sequence of Hamiltonian densities h_n, in one's turn, can be calculated as logarithmic derivative coefficients

$$\psi^{-1}\psi_x = -i\zeta + \sum_{k=0}^{\infty} \frac{h_k[q]}{(i\zeta)^k}$$

of the formal solution of Eq.(5)

$$\psi(x,\zeta) = \mathrm{e}^{-i\zeta x} \sum_{j=0}^{\infty} \frac{\psi_j(x)}{(i\zeta)^j}.$$

Several first Hamiltonian densities $h_n \in A_q$ read as

$$h_0 = \frac{1}{2}q^3, \quad h_1 = \frac{1}{2}q^1 + \frac{3}{8}(q^3)^2, \quad h_2 = -\frac{1}{2}q^2 + \frac{1}{2}q^1q^3 + \frac{1}{2}(q^3)^3,$$

$$h_3 = \frac{1}{8}(q^1)^2 + \frac{105}{128}(q^3)^4 + \frac{15}{16}q^1(q^3)^2 - \frac{3}{4}q^2q^3 - \frac{3}{8}q^1q^3_x - \frac{15}{32}(q^3_x)^2, \text{ etc.}$$

Now we look for the Hamiltonian Miura map linking with the 'second' Hamiltonian structure $\mathcal{E} \in A_q[\partial_x]$, which is explicitly given by

$$\mathcal{E} = \begin{pmatrix} 4\partial_x^3 + 4q^1\partial_x + 2q^1_x & -2\partial_x^4 - 2q^1\partial_x^2 + (6q^2 - 4q^1_x)\partial_x + (4q^2_x - 2q^1_{xx}) & 0 \\ 2\partial_x^4 + 2q^1\partial_x^2 + 6q^2\partial_x + 2q^2_x & -\frac{4}{3}\partial_x^5 - \frac{8}{3}q^1\partial_x^3 - 4q^1_x\partial_x^2 + (4q^2_x - 4q^1_{xx} - \frac{4}{3}(q^1)^2)\partial_x + (2q^2_{xx} - \frac{4}{3}q^1_{xxx} - \frac{4}{3}q^1q^1_x) & 0 \\ 0 & 0 & \frac{4}{3}\partial_x \end{pmatrix}. \tag{7}$$

By definition [2], the noninvertible differential relationship $q = F[r]$ is a Miura map for a certain Hamiltonian operator $\tilde{\mathcal{E}} \in A_r[\partial_x]$ if one generates the transformation $A_r[\partial] \to A_q[\partial]$ by virtue of the relation

$$\mathcal{E}|_{q=F[r]} = F'[r]\tilde{\mathcal{E}}(F'[r])^{\dagger}, \tag{8}$$

where $F'[r] \in A_r[\partial_x]$ is a Fréchet derivative. To get one of the posible solutions of (8), we try out the factorization approach. If we require, for example, $L = \left(\partial_x + 2r^1\right) \times \left(\partial_x - r^1 - r^3\right)\left(\partial_x - r^1 + r^3\right)$, while $q^3 = 2(r^1 - r^2)$, the linear equation (5) may be rewritten equivalently as (3).

To write more suitable variables in a more symmetric form, the modifications of (6) and the hyperbolic system associated with (1) can be introduced as follows: $r^1 = \frac{1}{12}v^1 + \frac{1}{6}v^2 + \frac{1}{6}v^3$, $r^2 = \frac{1}{4}v^1$, $r^3 = -\frac{1}{4}v^1 + \frac{1}{2}v^2$. Then, as can be checked by direct calculation, the operator

$$\tilde{\mathcal{E}} = \begin{pmatrix} 0 & 0 & -8 \\ 0 & -4 & 0 \\ -8 & 0 & 0 \end{pmatrix} \partial_x \tag{9}$$

solves relation (8) with the corresponding ansatz $q = F[v]$. It is obvious that this operator is Hamiltonian [3]. Also, we can calculate Hamiltonian densities for the modified evolution equations

$$\tilde{h}_n = h_n|_{q=F[v]} \in A_v \ (\mathrm{mod}\ \mathrm{Im}\,\partial_x)$$

to obtain

$$\tilde{h}_0 = \frac{1}{6}(-v^1 + v^2 + v^3),\ \ \tilde{h}_1 = -\frac{1}{8}\left(v^1v^3 + (v^2)^2\right),\ \ \tilde{h}_2 = \frac{1}{16}v^1v^3\left(v^1 - 2v^2 - v^3\right) -$$

$$-\frac{1}{16}\left(v^1v^3_x - v^3v^1_x\right),\ \ \tilde{h}_3 = \frac{1}{32}\left(4v^1_xv^3_x + (v^2_x)^2\right) + \frac{3}{16}v^1v^3\left(v^1_x + v^3_x\right) + \frac{3}{32}v^2\left(v^1v^3_x - v^3v^1_x\right) +$$

$$+\frac{1}{128}\left(9(v^1v^3)^2 + (v^2)^4\right) - \frac{1}{32}v^1v^3\left((v^1)^2 + (v^3)^2\right) + \frac{3}{32}v^1v^2v^3\left(v^1 - \frac{1}{2}v^2 - v^3\right), \text{ etc.} \tag{10}$$

From the form of the Hamiltonian structure $\mathcal{E}$ given by (9), it is evident that every system of the modified hierarchy $v_{\tau_n} = \tilde{\mathcal{E}}\mathrm{grad}_v\tilde{H}_n$ can be written in potential form. After introducing potentials $v_i = u_{ix}$, the first nontrivial system in these variables is explicitly given by

$$\begin{cases} u^1_{\tau_2} = -u^1_{xx} - \frac{1}{2}(u^1_x)^2 + u^1_xu^2_x + u^1_xu^3_x, \\ u^2_{\tau_2} = \frac{1}{2}u^1_xu^3_x, \\ u^3_{\tau_2} = u^3_{xx} + \frac{1}{2}(u^3_x)^2 + u^2_xu^3_x - u^1_xu^3_x. \end{cases} \tag{11}$$

Now we observe the following fact. Let us write Eqs.(11) as one evolution equation

$$u_{\tau_2} = K(u_{xx}) + u_x \circ u_x \tag{12}$$

on the element $u = \sum_{i=1}^{3} u^i e_i$ of some commutative algebra $\mathcal{A}$, where $K : \mathcal{A} \to \mathcal{A}$ is the endomorphism of $\mathcal{A}$ and $\circ$ denote multiplication in this algebra defining by Eqs. (11). Then we can state that the algebra $\mathcal{A}$ is Jordan [4] (compare with [5]).

3 Hyperbolic integrable system

The following question will be of interest: what systems of partial differential equations can be written as the zero-curvature condition

$$P_t - Q_x + [P, Q] = 0 \tag{13}$$

if $Q(x,\zeta) = Q_{-1}(x)(i\zeta)^{-1} + Q_0(x) \in \mathrm{sl}(3,\mathbf{C})$ and matrix elements of Q_{-1} and Q_0 are differential functions of variables u^i. In such circumstances, it is easy to get

$$Q_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} c_1 \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \frac{c_3}{4} \exp\left(-\frac{1}{2}u^1 + u^2\right),$$

$$Q_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{c_1}{3} \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \frac{c_2}{4} \exp\left(-u^2 - \frac{1}{2}u^3\right),$$

while the system of equations, in general form, having representation (13) reads

$$\begin{cases} u^1_{xt} = c_1 u^1_x \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right), \\ u^2_{xt} = c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right) - c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right), \\ u^3_{xt} = -c_1 u^3_x \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right). \end{cases} \tag{14}$$

Here $c_1,\ c_2,\ c_3$, in general case, are arbitrary complex constants. By direct calculation, it can be checked that Eqs.(11) present the one-parameter Lie-Bäcklund group for the hyperbolic system (14). Using an explicit form for the Hamiltonian densities of polynomial flows, we can calculate any densities-fluxes of the conservation laws $\partial_t q_k + \partial_x p_k = 0,\ q_k,\ p_k \in A_u,\ k = 0, 1, ...$ of system (11). We have

$$q_0 = -u^1_x + u^2_x + u^3_x, \ \ p_0 = 2c_1 \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right),$$

$$q_1 = u^1_x u^3_x + (u^2_x)^2, \ \ p_1 = 2c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right) + 2c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right), \ \text{ etc.}$$

Consider different particular cases of system (14) and possible reductions. Choose, for example, $c_2 = c_3 = -\frac{1}{2}$ and $c_1 = ic$, where c is a real number. Putting in (14) $u^2 = in$, where $n = n(x,t)$ is a real-valued function and $u^1 = u^{3*} = \varphi$, where $\varphi = \varphi(x,t)$ is a complex-valued function, Eqs.(14) becomes

$$\begin{cases} n_{xt} = \exp\left(-\frac{1}{2}\mathrm{Re}\,\varphi\right) \sin\left(n - \frac{1}{2}\mathrm{Im}\,\varphi\right), \\ \varphi_{xt} = ic\varphi_x \exp\left(\mathrm{Re}\,\varphi\right) + \frac{1}{2}\exp\left(-in - \frac{1}{2}\varphi^*\right), \end{cases} \tag{15}$$

For the case $c_1 = 0$, Eqs.(14) reduce to the system of Toda type [7]. Putting $u^2 = u^1 - u^3$, $c_2 = c_3 = -2$ in (14), we get

$$\begin{cases} v^1_{xt} = \exp\left(2v^1 - v^2\right), \\ v^2_{xt} = \exp\left(-v^1 + 2v^2\right), \end{cases} \tag{16}$$

where $v^1 = -\frac{1}{2}u^1$ and $v^2 = -\frac{1}{2}u^3$.

Finally, let us consider the case $c_2 = c_3 = 0$. Without loss of generality, we can put $c_1 = -1$. In this particular case, we have

$$\begin{cases} u^1_{xt} = -u^1_x \exp\left(\dfrac{1}{2}u^1 + \dfrac{1}{2}u^3\right), \\ u^2_{xt} = 0, \\ u^3_{xt} = u^3_x \exp\left(\dfrac{1}{2}u^1 + \dfrac{1}{2}u^3\right). \end{cases} \tag{17}$$

For this case, a large class of exact solutions can be found in the form

$$u^1 = F^1(\xi) + \ln(\xi_t) + g_3(t), \;\; u^2 = \eta,$$

$$u^3 = F^2(\xi) + \ln(\xi_t) - g_3(t),$$

where $\xi(x,t) = f_1(x) + g_1(t)$, $\eta(x,t) = f_2(x) + g_2(t)$ and $f_i(x)$, $i = 1, 2$, $g_i(t)$, $i = 1, 2, 3$ are arbitrary smooth functions of variables $x \in \mathbf{R}^1$ and $t \in \mathbf{R}^1$, respectively. Putting this ansatz in (17), we lead to the system of ordinary differential equations

$$\begin{cases} F^{1\prime\prime} = -F^{1\prime} \exp\left(\dfrac{1}{2}F^1 + \dfrac{1}{2}F^2\right), \\ F^{2\prime\prime} = F^{2\prime} \exp\left(\dfrac{1}{2}F^1 + \dfrac{1}{2}F^2\right). \end{cases} \tag{18}$$

System (18) has a solution in the form

$$F^1(\xi) = \ln\left\{\frac{2k_1k_3}{\left(\cos(k_1\xi + k_2) + k_3\sin(k_1\xi + k_2)\right)^2}\right\} + \frac{k_1}{k_3}\left(k_3^2 - 1\right)\xi + k_4,$$

$$F^2(\xi) = \ln\left\{\frac{2k_1k_3}{\left(\cos(k_1\xi + k_2) - k_3\sin(k_1\xi + k_2)\right)^2}\right\} - \frac{k_1}{k_3}\left(k_3^2 - 1\right)\xi - k_4, \;\; k_1, \;\; k_3 \neq 0.$$

As $k_1\zeta + k_2$ also can be written in the form $f(x) + g(t)$, then, without loss of generality, we can put $k_1 = 1$, $k_2 = 0$. So, for system (17) we have a solution in the form

$$u^1 = \ln\left\{\frac{2p\xi_t}{\left(\cos\xi + p\sin\xi\right)^2}\right\} + \left(p - p^{-1}\right)\xi + g_3(t), \;\; u^2 = \eta,$$

$$u^3 = \ln\left\{\frac{2p\xi_t}{\left(\cos\xi - p\sin\xi\right)^2}\right\} + \left(p^{-1} - p\right)\xi - g_3(t),$$

where $p = k_3 \neq 0$.

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