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On a Construction Leading to Magri-Morosi-Gel'fand-Dorfman's Bi-Hamiltonian Systems

Roman G. SMIRNOV

Department of Mathematics and Statistics, Queen's University
Kingston, Ontario, Canada, K7L 3N6
Fax: (613)-545-2964,
E-mail: smirnovr@mast.queensu.ca

Abstract

We present a method of generating Magri-Morosi-Gel'fand-Dorfman's (MMGD) bi-Hamiltonian structure leading to complete integrability of the associated Hamiltonian system and discuss its applicability to study finite-dimensional Hamiltonian systems from the bi-Hamiltonian point of view. The approach is applied to the finite-dimensional, non-periodic Toda lattice.

1 Introduction

The generalization due to Lichnerowicz [1] of symplectic manifolds to Poisson ones gives a possibility to reconsider the basic notions of the Hamiltonian formalism accordingly. As a fundamental tool in this undertaking, we use the Schouten bracket [2] of two contravariant objects $Q^{i_1,\ldots,i_q}$ and $R^{j_1,\ldots,j_r}$, given on a local coordinate chart of a manifold M by

$$\begin{aligned}
[Q,R]^{i_1,\ldots,i_{q+r-1}} = {} & \left(\sum_{k=1}^{q} Q^{(i_1,\ldots,i_{k-1}|\mu|i_k,\ldots,i_{q-1}}\right)\partial_\mu R^{i_p,\ldots,i_{q+r-1})} + \\
& \left(\sum_{k=1}^{q}(-1)^k Q^{\{i_1,\ldots,i_{k-1}|\mu|i_k,\ldots,i_{q-1}}\right)\partial_\mu R^{i_q,\ldots,i_{q+r-1}\}} - \\
& \left(\sum_{l=1}^{r} R^{(i_1,\ldots,i_{l-1}|\mu|i_l,\ldots,i_{r-1}}\right)\partial_\mu Q^{i_r,\ldots,i_{q+r-1})} - \\
& \left(\sum_{l=1}^{r}(-1)^{qr+q+r+l} R^{\{i_1,\ldots,i_{l-1}|\mu|i_l\ldots i_{r-1}}\right)\partial_\mu Q^{i_r,\ldots,i_{q+r-1}\}}.
\end{aligned} \tag{1}$$

Here the brackets (,) and { , } denote the symmetric and skew-symmetric parts of these contravariant quantities respectively. We note that throughout this paper the bracket [,] is that of Schouten (1). As is follows from formula (1), even the usual commutator of two vector fields $[X,Y]$, $X,Y \in TM$, is a sub-case of the general formula (1).

Consider a Poisson manifold (M,P), i.e., a differential manifold M equipped with a Poisson bi-vector P, that is a skew-symmetric, 2-contravariant tensor satisfying the vanishing condition: $[P, P] = 0$. Then a general Hamiltonian vector field X_H defined on (M, P) takes the following form on a local coordinate chart $x^1, \ldots, x^{2n}$ of M:

$$x^i(t) = X^i(x) = P^{i\alpha}\frac{\partial H(x)}{\partial x^\alpha}.$$

Here H is the Hamiltonian function (total energy). We use the Einstein summation convention. Comparing this formula with (1), we come to the conclusion that it can be rewritten in the following coordinate-free form, employing the Schouten bracket:

$$X_H = [P, H] \tag{2}$$

Note that the dimension of M may be arbitrary: not necessarily even. An example of an odd-dimensional Hamiltonian system defined by (2) is the Volterra model (see, for example [3]). An alternative way of defining the basic notions of the Hamiltonian formalism is by employing the Poisson calculus (see, for example, [4, 5]).

Integrability of the Hamiltonian systems (2) defined on an even-dimensional manifold is the subject to the classical Arnol'd-Liouville's theorem [7, 6]. We study the bi-Hamiltonian approach to the Arnol'd-Liouville's integrability originated in works by Magri [8], Gel'fand & Dorfman [9] and Magri & Morosi [10]. It concerns the following systems. Given a Hamiltonian system (2), assume it admits two distinct bi-Hamiltonian representations, i.e.,

$$X_{H_1,H_2} = [P_1, H_1] = [P_2, H_2], \tag{3}$$

provided that P_1 and P_2 are compatible: $[P_1, P_2] = 0$. We call systems of the form (3), that is the quadruples $(M, P_1, P_2, X_{H_1,H_2})$ defined by compatible Poisson bi-vectors — *Magri-Morosi-Gel'fand-Dorfman's (MMGD) bi-Hamiltonian systems*. To distinguish them from the bi-Hamiltonian systems with incompatible Poisson bi-vectors, see Olver [11], Olver and Nutku [12] and Bogoyavlenskij [13]. The bi-Hamiltonian approach to integrability of the MMGD systems is the subject of the theorem that follows.

Theorem 1 (Magri-Morosi-Gel'fand-Dorfman) *Given an MMGD bi-Hamiltonian system: $(M^{2n}, P_1, P_2, X_{H_1,H_2})$. Then, if the linear operator $A := P_1P_2^{-1}$ (assuming P_2 is non-degenerate) has functionally independent eigenvalues of minimal degeneracy, i.e., — exactly n eigenvalues, the dynamical system determined by the vector field X_{H_1,H_2}*

$$\dot{x}(t) = X_{H_1,H_2}(x)$$

is completely integrable in Arnol'd-Liouville's sense.

Remark 1. This approach to integrability gives a constructive way to derive the set of n first integrals for the related MMGD bi-Hamiltonian system:

$$I_k := \frac{1}{k}Tr(A^k), \qquad k = 1, 2, \ldots \tag{4}$$

The result of Theorem 1 suggests to pose the following problem: *Given a Hamiltonian system (2). How to transform such a system into the MMGD bi-Hamiltonian form?* This representation definitely will allow us to study integrability of the related Hamiltonian vector field.

2 A constructive approach to the bi-Hamiltonian formalism

We note that the problem posed below can be easily solved in some instances, for example, if we study a system with two degrees of freedom. Then the bi-Hamiltonian construction may be defined by two Poisson bi-vectors P_1, P_2 with constants coefficients, which are easy to work with (see [14]).

In general, the problem is far from being simple. The compatibility imposes a complex condition on P_1 and P_2, and so it would be desirable to circumvent this difficulty.

To solve the problem, we introduce the following geometrical object.

Definition 1 *Given a Hamiltonian system (M^{2n}, P, X_H). A vector field $Y_P \in TM^{2n}$ is called a master locally Hamiltonian (MLH) vector field of the system if it is not locally Hamiltonian with respect to the Poisson bi-vector (i.e., $L_{Y_P}(P) \neq 0$), while the commutator $[Y_P, X_H]$ is:*

$$L_{[Y_P,X_H]}(P) = 0.$$

We note that the notion of an MLH vector field is in a way reminiscent of the notion of *master symmetry* (MS) introduced by Fuchssteiner [15]:

Definition 2 *A vector field $Z \in TM^{2n}$ is called the master symmetry (MS) of a vector field $X \in TM^{2n}$ if it satisfies the condition*

$$[[Z, X], X] = 0,$$

provided that $[Z, X] \neq 0$.

For a complete classification of master symmetries related to integrable Hamiltonian systems, see [16].

Theorem 2 *Let (M, P, X_H) be a Hamiltonian system defined on a non-degenerate Poisson (symplectic) manifold (M, P). Suppose, in addition, that there exists an MLH vector field $Y_P \in TM$ for X_H. Then, if X_H is a Hamiltonian vector field with respect to $\tilde{P} = L_{Y_P}(P) : X_H = [\tilde{P}, \tilde{H}]$ (it implies that $\tilde{P}$ is a Poisson bi-vector and there is a second Hamiltonian $\tilde{H}$), X_H is an MMGD bi-Hamiltonian vector field:*

$$X_{H,\tilde{H}} = [P, H] = [\tilde{P}, \tilde{H}].$$

We note that the proof of this theorem employs some basic properties of the Schouten bracket [5].

Using this theorem, we can circumvent difficulties connected with finding a second Poisson bi-vector for a given Hamiltonian system. It is definitely easier to find a suitable MLH vector field (non-Hamiltonian symmetry) generating a Poisson bi-vector compatible with the initial one, rather than a second Poisson bi-vector itself, for it is easier in general to find a "vector" (i.e., MLH vector field or non-Hamiltonian symmetry), than a "matrix" (i.e., Poisson bi-vector). Besides, in this case the pair of Poisson bi-vectors so constructed is automatically compatible; we do not need to verify the compatibility condition, which, as was already mentioned, is in general a very complicated problem (for instance, see [17]).

3 Application

As an example illustrating this approach, consider the non-periodic, finite-dimensional Toda lattice, i.e., the system of equations that describes the dynamics of a one-dimensional lattice of particles with exponential interaction of nearest neighbors. In terms of the canonical coordinates q^i and momenta p_i, $(i = 1, 2, \ldots, n)$ it is given by (with the understanding that $e^{q^0-q^1} = e^{q^n-q^{n+1}} = 0$)

$$\begin{aligned} \mathrm{d}q^i/\mathrm{d}t &= p_i, \\ \mathrm{d}p_i/\mathrm{d}t &= e^{q^{i-1}-q^i} - e^{q^i-q^{i+1}}, \end{aligned} \tag{5}$$

where $q^i(t)$ can be interpreted as the coordinate of the i-th particle in the lattice. This system takes the Hamiltonian form (2) with the Hamiltonian function H_0 defined by

$$H_0 := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i-q^{i+1}},$$

while the corresponding Poisson bi-vector P_0 is canonical. We obviously deal with the same Poisson manifold: $(\mathbb{R}^{2n}, P_0)$.

Consider the following vector field:

$$Y_P = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} p_j \frac{\partial}{\partial q^i} + \left(-\sum_{i=1}^{n-1} e^{q^i-q^{i+1}} + \frac{1}{2}\sum_{i=1}^{n} p_i^2\right)\frac{\partial}{\partial p_i}. \tag{6}$$

Direct verification shows that the vector field (6) is an MLH vector field for system (5). The result of its action on the canonical Poisson bi-vector through the Lie derivative is the following tensor: $P_1 = L_{Y_P}(P_0) = [Y_P, P_0] = \sigma_0 Y_P$, where

$$P_1 = \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{i<j} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q_i}.$$

We first observe that the Hamiltonian vector field X_H of (5) can be expressed by means of this tensor and the function $H_1 = \sum\limits_{i=1}^{n} p_i$, which is its first integral:

$$X_H = P_1^{i\alpha} H_{1,\alpha}.$$

The last expression is not quite yet the Hamiltonian representation (2), since at this point it is unclear whether the tensor P_1 is a Poisson bi-vector or not. The direct checking of the condition $[P, P] = 0$ on the tensor P_1 would involve a lot of computational stamina, and so we observe further that P_1 can be expressed as follows:

$$P_1 = P_0 \omega_1 P_0, \tag{7}$$

where P_0 is the canonical Poisson bi-vector, while ω_1:

$$\omega_1 := \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} \mathrm{d}q^i \wedge \mathrm{d}q^{i+1} + \sum_{i=1}^{n} p_i \mathrm{d}q^i \wedge \mathrm{d}p_i + \frac{1}{2}\sum_{i<j} \mathrm{d}p_i \wedge \mathrm{d}p_j$$

is the second symplectic structure for system (5) found in [17] and proved to be compatible with the canonical symplectic form $\omega_0 := P_0^{-1}$. Relation (7) is equivalent to the vanishing: $[P_1, P_1] = 0$, which means that P_1 is a Poisson bi-vector. This fact follows from the following formula, taking into account the compatibility in terms of the corresponding Nijenhuis tensor of P_0 and ω_1:

$$\begin{aligned}&[P_1, P_1](\alpha, \beta) = \\ &\quad A[P_0, P_0](A^t\alpha, \beta) - A[P_0, P_0](A^t\beta, \alpha) + N_A(P_0\alpha, \omega_1^{-1}\beta) - AP_0 \mathrm{d}\omega_1(P_0\alpha, P_0\beta),\end{aligned} \tag{8}$$

where $\alpha, \beta \in T^*(M), A := P_0\omega_1(A^t := \omega_1 P_0)$ and N_A is the corresponding Nijenhuis tensor. Clearly, $[P_1, P_1 = 0]$ holds in view of compatibility of ω_0 and ω_1 [17]. Note that for the first time a formula analogous to (8) for a pre-symplectic form appeared in [10].

Now applying Theorem 2, we draw the conclusion: P_0 and P_1 constitute a compatible Poisson pair, the non-periodic, finite-dimensional, Toda lattice is an MMGD bi-Hamiltonian system defined by the Poisson bi-vectors P_0, P_1 and the corresponding Hamiltonians H_0, H_1. Now one can construct the set of first integrals $I_1, ...I_n$(mutually in involution, according to Theorem 1) by employing the formula $I_i := \frac{1}{i}\mathrm{Tr}(\tilde{A}^i)$, where $\tilde{A} := P_1 P_0^{-1}$. We present here a few first integrals obtained by using this method.

$$\frac{1}{2}I_1 = \frac{1}{2}\mathrm{Tr}(A) = \sum_{i=1}^{n} p_i = H_1,$$

$$\frac{1}{2}I_2 = \frac{1}{4}\mathrm{Tr}(A^2) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} = H_0,$$

$$\frac{1}{2}I_3 = \frac{1}{4}\mathrm{Tr}(A^3) = \frac{1}{3}\sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1}(p_i + p_{i+1})e^{q_i - q_{i+1}},$$

$$\frac{1}{2}I_4 = \frac{1}{8}\mathrm{Tr}(A^4) = \frac{1}{4}\sum_{i=1}^{n} p_i^4 + \sum_{i=1}^{n-1}\Big((p_i^2 + p_{i+1}^2 + p_i p_{i+1})e^{q_i - q_{i+1}} + \frac{1}{2}e^{2(q_i - q_{i+1})} + e^{q_i - q_{i+2}}\Big),$$

where $A = P_1 P_0^{-1}$.

4 Concluding remarks

We have presented a method of transforming a Hamiltonian system into a bi-Hamiltonian system in the MMGD sense, which may (under some extra assumptions) lead to its complete integrability according to the Magri-Morosi-Gel'fand-Dorfman scheme (see Theorem 1). It has a certain attractiveness: the second Poisson bi-vector generated from the initial one is compatible with the latter. Thus, the condition of compatibility leading to complete integrability in Arnol'd-Liouville's sense of the corresponding Hamiltonian system in this case is assured. Besides, the concepts of Poisson calculus have proved to form a quite natural setting for the Hamiltonian formalism. This method was also employed in [18] to study the classical Kepler problem from the bi-Hamiltonian point of view. A similar problem was studied by Kosmann-Schwarzbach [19]: there was presented a method of transforming a Hamiltonian system into the bi-Hamiltonian form based on the existence of an appropriate Lax representation.

The bi-Hamiltonian formalism emerged in the theory of soliton equations [8], and so it is natural to apply this method to the soliton systems as well. The work in this direction is under way.

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