Symmetry in Nonlinear Mathematical Physics 1997, V.2, 353–358.

Three-Gap Elliptic Solutions of the KdV Equation

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1 Introduction

The theory of finite-gap integration of the KdV equation is interesting both in applied problems of mathematical physics and in connection with the development of general methods for the solving of integrable nonlinear partial differential equations. The finite-gap solutions are the coefficient functions of auxiliary linear differential equations which have the Baker-Akhiezer eigenfunctions associated with determined complex algebraic curves and the respective Riemann surfaces (see [1, 2, 3]). With the aid of the Baker-Akhiezer functions, these solutions are expressed via theta functions in an implicit form. As parameters, ones contain characteristics of the Riemann surfaces, the computation of which is a special algebraic geometry problem (see [3, 4]).

In this paper, a simple method is proposed for calculating the three-gap elliptic solutions of the KdV equations in the explicit form. One is based on the usage of a system of trace formulae and auxiliary time evolution equations in the representation of the elliptic Weierstrass function ($\wp$-representation) [5]. It is shown that the initial three-gap elliptic solutions (at $t = 0$) are the linear combinations of $\wp$-functions with shifted arguments which are determined by the trace formulae. In view of the evolution equations, the three-gap elliptic solutions of the KdV equation are a double sum of $\wp$-functions with the time dependent shifts of poles. The number of terms in this sum is determined by the condition of a coincidence of the general expression with the initial conditions at $t \to 0$. It is shown that the time evolution of the finite-gap elliptic solution is determined through X_i-functions which are determined by the trace formulae and which are roots of the algebraic equations of corresponding orders.

In distinct to the known methods (see [6, 7]), our method is characterized by a simple and general algorithm which is valid for the computation of finite-gap elliptic solutions of the KdV equation in cases of arbitrary finite-gap spectra of the auxiliary linear differential equations.

2 The finite-gap equations

The finite-gap solutions of integrable nonlinear equations, in particular the KdV equation, are solutions of the spectral problem for auxiliary linear differential equations. In so doing, the first motion integral of these equations must be the polynomial in their eigenvalues E.

In the case of the KdV equation, the finite-gap solutions ($U(x,t)$) are solutions of the spectral problem of the Schrödinger equation $[-\partial_x^2 + U(x,t)]\Psi = E\Psi$ with the eigenfunctions (Ψ-functions) which satisfy the condition

$$\sqrt{P(E)} = \Psi_-\partial_x\Psi_+ - \Psi_+\partial_x\Psi_-. \quad (1)$$

The right-hand side in (1) is the motion integral which follows from the known (see [8]) Ostrogradskii-Liouville formula $W(x) = W(x_0) \times \exp(\int\limits_{x_0}^{x} dta_1(t))$ for the Wronski determinant $W(x)$ of the fundamental solutions Ψ_-, Ψ_+ (a_1 is the coefficient at the $(n-1)$th derivative in a linear differential equation of nth order). The dependence of the Ψ-function on the time t is described by the auxiliary linear equation $\partial_t\Psi = A\Psi$, where $A = 4\partial_x^3 - 3[U, \partial_x + U, \partial_x]$ is the time evolution operator (see [1]). From equation (1) in accordance with the asymptotic relation $\Psi \to \exp i\sqrt{E}x$, $E \to \infty$, the finite-gap Ψ-function has the form

$$\Psi = \sqrt{\chi_R(x,t,E)} \exp i \int\limits_{x_0}^{x} dx\chi_R(x,t,E), \quad (2)$$

where

$$\chi_R(x,t,E) = \Psi_-\Psi_+ = \frac{\sqrt{P(E)}}{\prod\limits_{i=1}^{g}(E - \mu_i(x,t))}$$

is equivalent to

$$\begin{aligned} \chi_R &= \sqrt{E}\left(1 + \sum_{n=1}^{\infty} A_n E^{-n}\right), \\ A_n &= \frac{1}{n!}\partial_z^n \left(\frac{\sqrt{\sum\limits_{n=0}^{2g+1} a_n z^n}}{\sum\limits_{n=0}^{g} b_n z^n} \right)\Bigg|_{z=0}. \end{aligned} \quad (3)$$

Here a_i are symmetrized products of the spectrum boundaries E_j of ith order and b_i are coefficient functions of x changing in the spectrum gaps.

On the other hand, with the aid of substitution (2) into the auxiliary Schrödinger equation, we obtain

$$\begin{aligned} \chi_R &= \sqrt{E}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1} E^{-(n+1)}\right), \\ \chi_{n+1} &= \partial_x\chi_n + \sum_{k=1}^{n-1} \chi_k\chi_{n-k}, \ \chi_1 = -U(x), \end{aligned} \quad (4)$$

where the second recurrent equation determines the coefficient functions χ_n in the form of polynomials in U-functions and their derivatives.

The equalizing of the coefficient at similar powers of E of expressions (3) and (4) leads to the trace formulae

$$A_{n+1} = \frac{(-1)^n}{2^{2n+1}}\chi_{2n+1}, \tag{5}$$

which presents the system of equations which describe the finite-gap elliptic function $U(x,t)$.

The finite-gap elliptic solutions of the KdV equation admit the $\wp$-representation. Using this representation in the trace formulae (5) at the initial time $t = 0$ and comparing the Laurent expansion in $\wp$ of their left-hand and right-hand sides, we obtain the general expression

$$U(z) = \alpha_0\wp(z) + \sum_i(\alpha_i\wp(z+\omega_i) + \beta_i(\wp(z+\varphi_i) + \wp(z-\varphi_i)) + C, \tag{6}$$

($\wp(z) \equiv \wp(z|\omega,\omega')$, $\omega_i = (\delta_{i,1} - \delta_{i,2})\omega + (\delta_{i,3} - \delta_{i,2})\omega'$) describing even initial finite-gap elliptic solutions of the KdV equation (see [6]). Here α, β and φ_i are unknown parameters, ω, ω' or ω, $\tau = \omega'/\omega$ are independent parameters. The constant C is determined from the condition of vanishing a constant in the Laurent expansion in $\wp$ of function (6). This correspond to a vanishing shift of the spectrum the Schrödinger equation. Then the mentioned unknown parameters are determined by substitution (6) in the trace formulae and comparison of the Laurent expansion in $\wp$ of their left- and right-hand sides.

3 Initial three-gap elliptic solutions

In the case of the three-gap spectrum, the unknown parameters of expression (6) and spectrum parameters a_i are described by the system of five trace formulae (5). Index n in these formulae receives the values $n = (0,1,2,3)$ and $b_i = 0$, $i \geq 3$. These four trace formulae are reduced to the equation

$$\begin{aligned}&-16a_2^2 + 64a_4 + 32a_3U(x) + 24a_2U(x)^2 + 35 * U(x)^4 - \\ &\quad 70 * U(x)U'(x)^2 - 8 * a_2U''(x) - 70 * U(x)^2U''(x) + \\ &\quad 21U(x)''^2 + 28U'(x)U'''(x) + 14U(x)U^{(4)}(x) - U^{(6)}(x) = 0,\end{aligned} \tag{7}$$

determining the unknown parameters of initial three-gap elliptic solutions of the KdV equation. Under the condition of vanishing of coefficients of the Laurent expansion in $\wp$ of the left-hand side (7), we obtain a closed system of algebraic equations for the mentioned parameters α, α_i, β_i and φ_i. The general relation $\alpha \neq 0$ and *a*) $\alpha_i = \beta_i = 0$; *b*) $\alpha_i \neq 0$, $\beta_i = 0$, *c*) $\alpha_i = 0$, $\beta_i \neq 0$, following from these equations determine three kinds of initial three-gap elliptic solutions. In the case *a*), the substitution of expression (6) into equation (7) under the condition of vanishing the coefficients of its Laurent expansion in $\wp$ gives $\alpha = 12$. In so doing, from (6) we obtain the expression

$$U(z) = 12\wp(z) \tag{8}$$

for the well-known [5] three-gap Lamé potential.

In the case *b*), the parameters α_i have the form $\alpha_i = \sum\limits_{j=1}^{m}\delta_{i,j} \times \mathrm{const}$, $j = (1,2,3)$. In so doing, the substitution of expression (6) in equation (7) and nuliifying coefficients of the Laurent expansion of its left-hand side give a simple algebraic equation for the unknown parameter. Taking into account values of these parameters in (6), we obtain the expressions

$$U(z) = 12\wp(z) + 2\wp(z+\omega_i) - 2e_i, \qquad e_i = \wp(\omega_i) \tag{9}$$

$$U(z) = 12\wp(z) + 2(\wp(z+\omega_i) + \wp(z+\omega_j)) - 2(e_i + e_j) \tag{10}$$

($e_i = \wp(\omega_i)$) which describe the initial three-gap elliptic solutions of the KdV equation which are similar to the well-known [9] two-gap Treibich-Verdier potentials.

In the case of relations c), the parameters β_i have the form $\alpha_i = \sum\limits_{j=1}^{m} \delta_{i,j}\mathrm{const}$, $j = (1,2,3)$. Then the substitution of expression (6) into equation (7) and nullifying of coefficients of the Laurent expansion in $\wp$ of its left-hand side lead to $\alpha = 12$, $\beta = 2$. In so doing, the condition of vanishing the coefficient of the fifth order pole at the point $\wp = h$, $h = \wp(\varphi)$ determines the equation

$$h^6 + \frac{101}{196}g_2h^4 + \frac{29}{49}g_3h^3 - \frac{43}{784}g_2^2h^2 - \frac{23}{196}g_2g_3h - \left(\frac{1}{3136}g_2^3 + \frac{5}{98}g_3^2\right) = 0$$

for $h = h(\varphi)$. Six values of h determine six values of the parameter $\varphi = \wp^{-1}(h)$. The substitution of the obtained parameters into (6) leads to the expression

$$U(z) = 12\wp(z) + 2(\wp(z+\varphi) + \wp(z-\varphi)) - 4\wp(\varphi) \tag{11}$$

describing the initial three-gap elliptic solutions of the KdV equations similar to the two-gap potential in [6].

4 Dynamics of three-gap elliptic solutions

The time dependent three-gap elliptic solutions of the KdV equation are solutions of the system involving both the trace formulae (5) and the above-mentioned auxiliary evolution equation. Substitution (2) into the last equation and separation of the real and imaginary parts transform one to the form

$$\partial_t\chi_R(x,t,E) = \partial_x\{(\lambda\chi_R(x,t,E)\}, \quad \lambda = -2(U(x,t,E) + E)$$

Relation (3) reduces the last equality to

$$\partial_t b_n - 2\{b_n\partial_x U - U\partial_x b_n + 2\partial_x b_{n+1}\} = 0, \quad n = (1,\ldots,g) \tag{12}$$

(g means the number of spectral gaps), where b_n-functions in view of the trace formulae (5) are polynomials in U-functions and their derivatives. Substitution of the U-function in the $\wp$-representation into equation (12) and a comparison of the Laurent expansions of their left-hand and right-hand sides give simple algebraic equations determining the general form and the time dependence of the U-function. The general finite-gap elliptic solutions of the KdV equation have the form

$$U(z,t) = 2\sum_{i=1}^{N} \wp(z - \varphi_i(t)) + C, \tag{13}$$

where the number N and the constant C are determined by the condition of coincidence of (13) with the initial finite-gap elliptic solutions at $t \to 0$. In so doing, the dynamic equations (12) are reduced to the equations

$$\begin{gathered} \partial_t \varphi_i(t) = -12 X_i(t) + C,\ X_i(\varphi_i(t)) = \sum_{\substack{i \neq j \\ i=1}}^{N-1} \wp(\varphi_i(t) - \varphi_j(t)),\ (g \geq 2), \\ \sum_{n=1}^{N} \partial_t \wp(z - \varphi_i(t)) = 0, \quad (g = 1) \end{gathered} \tag{14}$$

describing the time evolution of poles φ_i. Here, the N functions $X_i(t) = X_i(h_i(\varphi_i(t)))$ are determined by the trace formula (5) with the index $n = N - 1$. A compariszon of the Laurent expansions in $\wp$ of the left-hand and right-hand sides of the last equation leads to the algebraic equation of Nth order for the unknown functions $X_i(h_i)$. Then (14) can be transformed to the equality

$$\int_{\varphi_{0i}}^{\varphi_i} \frac{d\varphi_i}{X_i(h_i(\varphi_i))} = -12t, \tag{15}$$

describing the dynamics of poles in expression (13). Here, initial values φ_{0i} are determined from the initial conditions describing by expressions (8)-(11) in the three-gap case. Thus, the problem of time evolution of three-gap elliptic solutions of the KdV equation is reduced to computation of the functions $X_i(h_i(\varphi_i))$.

In the three-gap case, the condition of coincidence of the general expression (13) with (8)–(11) at $t \to 0$ leads to $N = (6, 7, 8)$.

The values $N = 6$ and $N = 7$ determine two three-gap elliptic solutions of the KdV equation with the initial conditions (8) and (9), respectively. The substitution of expression (13) with $N = 6$ and $N = 7$ in the trace formulae with the index $n = 5$ and $n = 6$, respectively, and a usage of the Laurent expansion in $\wp$ lead to the algebraic equations

$$\sum_{i=0}^{6} c_i^{(6)}(h) X^i = 0, \qquad \text{and} \qquad \sum_{i=0}^{7} c_i^{(7)}(h) X^i = 0.$$

of the 6th and 7th orders, respectively. Here, the coefficients $c_i^{(6)}(h)$ and $c_i^{(7)}(h)$ are rational functions of $h(\varphi)$ and $h'(\varphi)$; $c_6^{(6)}(h) = c_7^{(7)}(h) = 1$. The roots of these equations coincide with the functions $X_i = \sum\limits_{j=1}^{6} \wp(\varphi_i - \varphi_j)$ and $X_i = \sum\limits_{j=1}^{7} \wp(\varphi_i - \varphi_j)$ which enter into the dynamic equation (15).

The substitution $N = 8$ into (13) gives the three-gap elliptic solution of the KdV equation with the possible initial conditions (10) and (11). In so doing, X_i-functions of the dynamic equation (15) are determined by the trace formula (5) with the index $n = 7$. In this case, a comparison of the Laurent expansions in $\wp$ of the left-hand and right-hand sides of the last equation leads to the algebraic equation of the 8th order

$$\sum_{i=0}^{8} c_i^{(8)}(h)^8 X^i = 0, \qquad c_8^{(8)}(h) = 1.$$

The solutions of this equation coincide with the functions $X_i = \sum_{j=1}^{8} \wp(\varphi_i - \varphi_j)$ entering into the dynamic equation (15).

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