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An Orbit Structure for Integrable Equations of Homogeneous and Principal Hierarchies

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Abstract

A construction of integrable Hamiltonian systems associated with different graded realizations of untwisted loop algebras is proposed. These systems have the form of Euler-Arnold equations on orbits of certain subalgebras of loop algebras and coincide with hierarchies of higher stationary equations for some nonlinear partial differential equations integrable by the inverse scattering method. We apply the general scheme for the loop algebras sl(3) $\otimes\mathcal{P}\left(\lambda,\lambda^{-1}\right)$. The corresponding equations on the orbit are interpreted as a two-(or three-)component nonlinear Schrödinger-type equation, an SU(3) – Heisenberg magnet type equation for the homogeneous realization, and the Boussinesq's equation for the principal realization.

We dedicate this work to the memory of W.I. Fushchych.

1. Introduction

In this paper, we deal with hierarchies of nonlinear evolutionary equations which are integrable Hamiltonian systems in the phase space of functions of one variable "x" and can be written in the form of a "zero-curvature" equation

$$\frac{\partial\Lambda}{\partial\tau_n}+\frac{\partial A_n}{\partial x}+[\Lambda,A_n]=0,\qquad n=1,2,\ldots. \tag{1}$$

Here, Λ and A_n are elements of $\tilde{\tilde{g}}$, an algebra of Laurent polynomials with coefficients belonging to the central extention of a current algebra on the circle.

Let $\widetilde{g}$ denote an algebra of Laurent polynomials with coefficients in a semisimple Lie algebra g, with σ being an internal finite-order automorphism of g. The automorphism σ lifted up to $\widetilde{g}$ fixes a gradation. The gradation in $\widetilde{g}$ and the symmetry of the elements Λ and A_n with respect to σ determine the hierarchy type.

It is known that classes of equivalent finite-order automorphisms coincide with conjugate classes of the Weyl group of a semisimple Lie algebra g [1, 2]. In the case of the identity automorphism, one has the homogeneous gradation and, accordingly, the homogeneous hierarchy of equations (1) involving the many-component nonlinear Schrödinger equation, the Heisenberg magnet type equation as well as many other equations. The maximal order automorphism is related to the principal gradation, with corresponding equations such as KdV type equations, the Boussinesq's equation and others.

The dressing procedure is used in the standard approach [3, 4] to find the matrices A_n for hierarchy (1) . Given the operator $L = \dfrac{\partial}{\partial x} - \Lambda$, the dressing operator K is determined by the equation:

$$L = K^{-1}\left(\frac{\partial}{\partial x} - \Lambda_0\right) K,$$

where Λ_0 is a constant σ-invariant element of Heisenberg subalgebras in $\widetilde{g}$.

In the approach developed earlier by one of the authors [5, 6], the objects A_n were interpreted as points of a coadjoint orbit of the subalgebra of Laurent series with positive or negative powers only and it may be obtained by restriction onto a coadjoint orbit of the general position element. The stationary (finite-zone) equations $\partial_x A_n = [\Lambda, A_n]$ form a hierarchy of finite-dimensional Hamiltonian systems on the orbit, with the standard set of commuting integrals. The number of these integrals suffies for integrability in the case of a homogeneous hierarchy. A Hamiltonian reduction of orbits is needed in other cases (the principal and intermediate hierarchies). As to the principal hierarchy, this reduction is the Drinfeld-Sokolov reduction restricted onto a finite-dimensional phase-space. From the point of view of the orbit approach, the dressing procedure is nothing but a restriction onto orbits as algebraic manifolds in the case of the homogeneous gradation or, for other gradations, onto a reduced phase space. In this paper, we develop this approach and expand it for higher range algebras as well as for nonhomogeneous gradations.

2. Constructing integrable Hamiltonian equations on coadjoint orbits

1. Let g be a semisimple finite-dimensional Lie algebra of rank r and $\mathcal{P}(\lambda, \lambda^{-1})$ the associative algebra of Laurent polynomials in a complex parameter λ . Let us consider the loop algebra $\widetilde{g} = g \otimes \mathcal{P}(\lambda, \lambda^{-1})$ with the commutator [7]:

$$\left[\sum A_i\lambda^i, \sum B_j\lambda^j\right] = \sum\, [A_i, B_j]\, \lambda^{i+j}.$$

Define the family of Ad-invariant nondegenerate forms on $\widetilde{g}$:$\langle A, B\rangle_k = \sum_{i+j=k}(A_i, B_j)$, $k \in \mathbf{Z}$, where (,) denotes the Killing form in g. Decompose $\widetilde{g}$ in the sum of two subspaces, $\widetilde{g} = \widetilde{g}_- + \widetilde{g}_+$, where

$$\widetilde{g}_+ = \left\{\sum_{i\geq 0} A_i\lambda^i\right\}, \qquad \widetilde{g}_- = \left\{\sum_{i<0} A_i\lambda^i\right\}.$$

Then $\widetilde{g}_+$ and $\widetilde{g}_-$ are subalgebras of $\widetilde{g}$ and form the dual pair relative to $\langle,\rangle_{-1}$ with the coadjoint action

$$ad^*_A\mu = \mathcal{P}_+[\mu, A], \qquad A \in \widetilde{g}_-, \qquad \mu \in \widetilde{g}_-{}^* \simeq \widetilde{g}_+, \tag{2}$$

where $\mathcal{P}_+$ denotes the projector onto $\widetilde{g}_+$. Let $\{X_i\}_1^{\dim g}$ be a basis in g. Keeping in mind our further purposes, it is more convenient to fix a dual basis $\{X_i\}_1^{\dim g}$ determined by $(X_i^*, X_j) = \delta_{ij}$. The finite-dimensional subspaces

$$M^{N+1} = \left\{\mu \in \widetilde{g}_-^* :\ \ \mu = \sum_{\ell=0}^{N+1}\sum_{i=1}^{\dim g} \mu_i^\ell X_i^* \lambda^\ell\right\} \subset \widetilde{g}_+, \quad N = 0, 1, 2, \ldots < \infty,$$

where $\mu_i^\ell = \langle \mu, X_i^{-\ell-1}\rangle_{-1}$ are the coordinates on M^{N+1}, are invariant under the coadjoint action of $\widetilde{g}_-$. The coadjoint action of $\widetilde{g}_+$ is defined on M^{N+1} also. In this case, $\mu(A) = \langle \mu, A\rangle_{N+1}$, $A \in \widetilde{g}_+$, and we identify $\widetilde{g}_-^*$ with the subspace $\widetilde{g}_- \otimes M^{N+1}$. The coordinates on M^{N+1} can be written down as $\mu_i^\ell = \langle \mu, X_i^{-\ell+N+1}\rangle_{N+1}$, and M^{N+1} is invariant under the action of $\widetilde{g}_+$.

The coadjoint actions induce the family of Lie-Poisson brackets on M^{N+1}:

$$\{f_1, f_2\}_n = \sum_{\ell=0}^{N+1}\sum_{i=1}^{\dim g} W_{ij}^{\ell s}(n)\frac{\partial f_1}{\partial \mu_i^\ell}\frac{\partial f_2}{\partial \mu_j^s}, \quad \forall f_1, f_2 \in \mathbf{C}^\infty(M^{N+1}), \tag{3}$$

where

$$W_{ij}^{\ell s}(n) = \langle \mu, [X_i^{-\ell+n}, X_j^{-s+n}]\rangle_n, \qquad n \in \mathbf{Z}. \tag{4}$$

Symplectic leaves of the Lie-Poisson structures $W(n)$ are called generic orbits of the corresponding loop subalgebras acting on M^{N+1} . Two following cases are important: $\mathcal{O}_-^{gen}$ denotes the generic orbit for $n = -1$, and the generic orbit for $n = N+1$ will be denoted $\mathcal{O}_+^{gen}$.

Let H^ν, $\nu = 2, 3, \ldots, r+1$, be Casimir functions in the enveloping algebra of g. They are polynomials in the variables $\mu_k = (\mu, X_k)$ on the dual g^* of g. The substitution $\mu_k \mapsto \mu_k(\lambda) = \sum_{\ell=0}^{N+1} \mu_k^\ell \lambda^\ell$ provides the continuation of H^ν to $\mathbf{C}^\infty(M^{N+1})$:

$$H^\nu = \sum_{\alpha=0}^{\nu(N+1)} h_\alpha^\nu \lambda^\alpha, \qquad h_\alpha^\nu \in \mathbf{C}^\infty(M^{N+1}). \tag{5}$$

Theorem 1. *1. The functions $\{h_\alpha^\nu\}$ constitute an involutive collection in $\mathbf{C}^\infty(M^{N+1})$, relative to the Lie-Poisson brackets $W(-1)$ and $W(N+1)$.*
2. The functions $\{h_\alpha^\nu\}$, $\alpha \geq (\nu-1)(N+1)$, annihilate the Lie-Poisson brackets $W(-1)$.
3. The functions $\{h_\alpha^\nu\}$, $\alpha = 0, 1, \ldots, N+1$, annihilate the Lie-Poisson brackets $W(N+1)$.

Proof. Let $\widetilde{X}_i^{-\ell+n}(n)$ be the tangent vector field corresponding to the basis element $X_i^{-\ell+n}$ and the coadjoint action (2) of $\widetilde{g}_-$ (for $n=-1$) or $\widetilde{g}_+$ (for $n = N+1$). Then one can show that

$$\widetilde{X}_i^{-\ell+n}(n) = \sum_{k=1}^{\dim g}\sum_{r=0}^{N+1} W_{ik}^{\ell r}(n)\frac{\partial}{\partial \mu_k^r}. \tag{6}$$

Note that

$$\frac{\partial}{\partial \mu_k^r} = \frac{\partial \mu_k(\lambda)}{\partial \mu_k^r}\frac{\partial}{\partial \mu_k(\lambda)} = \lambda^r \frac{\partial}{\partial \mu_k(\lambda)}.$$

By (6) and (4),

$$\sum_\ell \left(\lambda^{\ell+1}\widetilde{X}_i^{-\ell-1}(-1) + \lambda^{\ell-N-1}\widetilde{X}_i^{-\ell+N+1}(N+1)\right) = \sum_{j,k} C_{ik}^j \mu_j(\lambda)\frac{\partial}{\partial \mu_k(\lambda)},$$

where C_{ik^j} denotes the structure constants of g. The ad^*-invariance of H^ν means that

$$\sum_{j,k} C^j_{ik}\mu_j(\lambda)\frac{\partial}{\partial\mu_k(\lambda)}H^\nu = 0,$$

or, by the previous formula,

$$\sum_\ell \left(\lambda^{\ell+1}\widetilde{X}_i^{-\ell-1}(-1) + \lambda^{\ell-N-1}\widetilde{X}_i^{-\ell+N+1}(N+1)\right) H^\nu = 0.$$

Substitute (5) herein and equate the coefficients at the same powers of λ. Then the following relations arise:

$$\widetilde{X}_i^{-\ell-1}(-1)h^\nu_\alpha = 0, \qquad \alpha \geq (\nu-1)(N+1); \tag{7}$$

$$\widetilde{X}_i^{-\ell+N+1}(N+1)h^\nu_\alpha = 0, \qquad 0 \leq \alpha \leq N+1; \tag{8}$$

$$\widetilde{X}_i^{-\ell-1}(-1)h^\nu_\alpha + \widetilde{X}_i^{-\ell+N+1}(N+1)h^\nu_\alpha = 0, \qquad N+1 < \alpha < (\nu-1)(N+1). \tag{9}$$

The second and third assertions of the theorem follow immediately from (7) and (8), respectively.

The first assertion is clear if $\nu \neq \mu$. Let $\nu = \mu$. Then (9) leads to the sequence of equations:

$$\{h^\nu_\alpha, h^\nu_\beta\}_{-1} = \{h^\nu_{\alpha-N-2}, h^\nu_{\beta+N+2}\}_{-1} = \cdots = \{h^\nu_{\alpha-m(N+2)}, h^\nu_{\beta+m(N+2)}\}_{-1},$$

where m is a natural number. For any pair of nonnegative integer numbers α and β that are less than $(\nu-1)(N+1)$, there exists a number m such that one of the following inequalities holds: $\alpha - m(N+2) < 0$, or $\beta + m(N+2) \geq (\nu-1)(N+1)$. The first one implies that $h^\nu_{\alpha-m(N+2)} \equiv 0$, and the second that $h^\nu_{\beta-m(N+2)}$ annihilates the Poisson structure $W(-1)$. In the both cases, the vanishing of the brackets $\{h^\nu_\alpha, h^\nu_\beta\}_{-1}$ is guaranteed. The involutivity of the functions h^ν_α and h^ν_β with respect to $\{,\}_{N+1}$ are proved in the same way.

Remark. This proof is a realization of the "Adler scheme" [7, 8] and carried out in a gradation invariant way. A.G.Reyman and M.A.Semenov-Tian-Shansky applied the R-matrix technique to prove an analogue of Theorem 1 for the homogeneous gradation [10, 11]. By Theorem 1, the generic orbit $\mathcal{O}^{gen}_-$ is a real algebraic manifold embedded into M^{N+1} by the constraints h^ν_α, $\alpha \geq (\nu-1)(N+1)$. Fixing the functions h^ν_α, $\alpha = 0, 1, \ldots, N+1$, determines the real algebraic manifold being the generic orbit $\mathcal{O}^{gen}_+$. Set Hamiltonian flows of the form

$$\frac{d\mu}{d\tau^\nu_\alpha} = \{\mu, h^\nu_\alpha\}_n = ad^*_{dh^\nu_\alpha}\mu = [\mu, dh^\nu_\alpha], \tag{10}$$

on $\mathcal{O}^{gen}_-$ (resp., $\mathcal{O}^{gen}_+$), where $\alpha < (\nu-1)(N+1)$ (resp., $\alpha > N+1$) and dh^ν_α is the differential of the Hamiltonian h^ν_α (τ^ν_α is the corresponding trajectory parameter).

In case $g \simeq sl(n)$, one checks easily that:

1. The dimensions of the generic orbits $\mathcal{O}^{gen}_-$ and $\mathcal{O}^{gen}_+$ are equal to (N+1)(n-1)n.
2. The number of nonannihilators on the generic orbits is equal to (N+1)(n-1)n/2.

3a. The functions h^{ν}_{α}, $(\nu-1)(N+1) \le \alpha < \nu(N+1)$, are functionally independent almost everywhere on M^N.

3b. The functions h^{ν}_{α}, $0 \le \alpha < (\nu-1)(N+1)$, are functionally independent almost everywhere on $\mathcal{O}^{gen}_{-}$.

4a. The functions h^{ν}_{α}, $0 < \alpha \le N+1$, are functionally independent on M^{N^ℓ}.

4b. The functions h^{ν}_{α}, $\alpha > N+1$, are functionally independent on $\mathcal{O}^{gen}_{+}$.

A consequence of Theorem 1 and propositions 1–4 is that the Hamiltonian flows (10) are integrable in the Liouville sense.

3. Integrable Hamiltonian systems on orbits of the algebra $su(3) \otimes \mathcal{P}(\lambda^{-1})$

Let $\widetilde{g} = sl(3,\mathbf{C}) \otimes \mathcal{P}(\lambda,\lambda^{-1})$ be an algebra of polynomial loops with values in the Lie algebra $sl(3,\mathcal{C})$ and let $\widetilde{g} = \widetilde{g}_+ + \widetilde{g}_-$ be the decomposition into a direct sum of two subspaces, as was described in the previuos section. Let $\{H_1, H_2, E_{\pm\alpha i} \equiv E_i, i = 1,2,3\}$ be the standard root basis in $sl(3,\mathbf{C})$. If A_i is a basis element in g, then $A^l_i = \lambda^l A_i, l \in \mathbf{Z}$ is a basis element in $\widetilde{g}$. Fix the following finite-dimensional subspace in $\widetilde{g}_+$:

$$M^{N+1} = \left\{ \mu \in \widetilde{g}^*_- |_\mu = \sum_{\ell=0}^{N+1} \left[\alpha^\ell_1 H^\ell_1 + \alpha^\ell_2 H^\ell_2 + \sum_{i=1}^{3} \left(E^\ell_{+i}\beta^\ell_i + E^\ell_{-i}\gamma^\ell_i \right) \right] \right\}.$$

M^{N+1} is manifestly invariant under the coadjoint action of $\widetilde{g}_-$. Define two Lie-Poisson brackets on M^{N+1}, putting $n = -1$ and $n = N+1$ in (3). We indentify the coadjoint orbits of $\widetilde{g}_-$ and $\widetilde{g}_+$ in the subspace M^{N+1} with the symplectic leaves of the corresponding Lie-Poisson brackets.

It follows from the explicit form of the first brackets $\{f_1, f_2\}_{-1}$ that the variables α^{N+1}_1, α^{N+1}_2, β^{N+1}_i, γ^{N+1}_i, are its annihilators. Thus, we may set

$$\beta^{N+1}_i = \gamma^{N+1}_i = 0, \qquad \alpha^{N+1}_{1,2} = \text{const} \neq 0. \tag{11}$$

This means that symplectic leaves are fully embedded into the subspace $M^N \subset M^{N+1}$, which location is fixed by the constants α^{N+1}_1 and α^{N+1}_2.

The ad^*-invariant functions $I_2(\mu(\lambda)) = \frac{1}{2} Tr\mu^2(\lambda)$ and $I_3(\mu(\lambda)) = \frac{1}{3} Tr\mu^3(\lambda)$ are polynomials in λ of order $2(N+1)$ and $3(N+1)$, respectively:

$$I_2(\mu(\lambda)) = h_0 + \lambda h_1 + \cdots + \lambda^{2(N+1)} h_{2(N+1)},$$
$$I_3(\mu(\lambda)) = f_0 + \lambda f_1 + \cdots + \lambda^{3(N+1)} f_{3(N+1)}.$$

The coefficient functions h_α and f_β can be found easily. By Theorem 1, they are in involution with respect to the both Lie-Poisson brackets. The same holds for the restrictions of h_α and f_β onto the subspace M^N. Under this restriction, the functions $h_{2(N+1)}$ and $f_{3(N+1)}$ become fixed constants expressed via $\alpha^{N+1}_{1,2}$ and, hence, should be left out of the consideration.

The functions $h_{N+1}, h_{N+2}, \ldots, h_{2N+1}$ and $f_{2N+2}, f_{2N+3}, \ldots, h_{3N+2}$ ($2(N+1)$ functions) are Casimir functions with respect to the first brackets. By fixing their values, one obtains

$2(N+1)$ algebraic equations which determine a coadjoint orbit of the algebra $\widetilde{g}_-$ in the subspace M^N. The dimension of such a generic orbit is equal to $6(N+1)$. The $3(N+1)$ remaining functions give rise to nontrivial commuting flows on the orbit and provide the integrability in the sense of the Liouville theorem.

Choose the function h_{N-1} as the Hamiltonian and write down the corresponding Hamiltonian equations in terms of the coordinates $\{\mu_a^\ell\} \equiv \{\alpha_1^\ell, \alpha_2^\ell, \beta_1^\ell, \beta_2^\ell, \beta_3^\ell, \gamma_1^\ell, \gamma_2^\ell, \gamma_3^\ell\}$, $a = \overline{1,8}$,

$$\frac{d\mu_a^\ell}{d\tau_{N-1}} = \left\{\mu_a^\ell, h_{N-1}\right\}_{-1}. \tag{12}$$

The algebraic equations $h_\alpha = c_\alpha$, $\alpha = \overline{N+1, 2N+1}$, and $f_\beta = c_\beta$, $\beta = \overline{2N+2, 3N+2}$, can be solved for the variables α_1^ℓ and α_2^ℓ, $\ell = N, N-1, \ldots, 0$, expressing them via β_i^ℓ, γ_i^ℓ, $i = 1,2,3$. This implies that the coadjoint orbit of $\widetilde{g}_-$ is diffeomorphic to a flat space, with the variables β_i^ℓ, γ_i^ℓ being its global coordinates.

The structure of equations (12) allows us to express the variables $\beta_i^{\ell-1}$, $\gamma_i^{\ell-1}$ through β_i^ℓ, γ_i^ℓ and their first derivatives and, therefore, to reduce the system of $6(N+1)$ first order equations to six equations of order $N+1$ for the variables β_i^N, γ_i^N. These equations are interpreted as the higher stationary equations for a many-component nonlinear Shrodinger equation.

In order to ground this interpretation, reduce system (12) onto the orbit of the algebra $su(3)\otimes\mathcal{P}(\lambda^{-\infty})$. To this end, set $\alpha_{1,2}^\ell = \sqrt{-1}a_{1,2}^\ell$, $a_{1,2}^\ell$ being real numbers, and $\gamma_i^\ell = -\beta_i^{\ell *}$. Consider the Hamiltonian flow generated by h_{N-2}, $N > 1$, on the phase space reduced in the above sense:

$$\frac{d\beta_i^\ell}{dt} = \left\{\beta_i^\ell, h_{N-2}\right\}_{-1}. \tag{13}$$

Let the variables β_i^ℓ lie on trajectories of (12). This amounts to the following. First, β_i^ℓ depend on a "parameter" $\tau_{N-1} \equiv x$ and, second, they are expressed through β_i^N and their derivatives. Then the system (13) takes on the form:

$$\begin{aligned}
i\frac{d\beta_1}{dt} &= -\frac{\partial^2\beta_1}{\partial x^2} - \left(2|\beta_1|^2 - |\beta_2|^2 + \frac{1}{2}|\beta_3|^2\right)\beta_1 - \beta_3\frac{\partial\beta_3^*}{\partial x} - \frac{1}{2}\beta_2^*\frac{\partial\beta_3}{\partial x},\\
i\frac{\partial\beta_2}{\partial t} &= -\frac{\partial^2\beta_2}{\partial x^2} - \left(2|\beta_2|^2 - |\beta_1|^2 + \frac{1}{2}|\beta_3|^2\right)\beta_2 - \beta_3\frac{\partial\beta_1^*}{\partial x} - \frac{1}{2}\beta_1^*\frac{\partial\beta_3}{\partial x},\\
i\frac{\partial\beta_3}{\partial t} &= -\frac{\partial^2\beta_3}{\partial x^2} - \left(|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2\right)\beta_3 + \beta_2\frac{\partial\beta_1}{\partial x} - \beta_1\frac{\partial\beta_2}{\partial x},
\end{aligned} \tag{14}$$

where we denote $\beta_i^N \equiv \beta_i$. Equations (12) are higher stationary equations for the evolutionary system (14).

Let us focus on the degenerate orbit passing through the point $\{\mu_a^\ell\} \in M^N$ for which $\alpha_2^{N+1} = 2\alpha_1^{N+1}$, and consider the restriction of (12) onto this orbit. In the Lie algebra $g \simeq su(3)$, the point with the coordinates $\alpha_2 = 2\alpha_1$ remains invariant under the coadjoint action of the subgroup $SU(2)\times U(1)$. The corresponding orbit going through this point is the quotient $SU(3)/SU(2)\times U(1) \simeq \mathbf{CP}^2$.

The analogue of this in the space M^N is a $4(N+1)$-dimensional complex manifold with the structure of the vector fibre over $\mathbf{CP}^2$. The local coordinates on this manifold are $\beta_2^0, \beta_3^0, \beta_2^1, \beta_3^1, \ldots, \beta_2^N, \beta_3^N$. On the degenerate orbit, equation (14) takes the form:

$$\begin{aligned} i\frac{\partial\beta_2^N}{\partial t} &= -\frac{\partial^2\beta_2^N}{\partial x^2} + 2\left(|\beta_2^N|^2 + |\beta_3^N|^2\right)\beta_2^N - 3ia_1^N\frac{\partial\beta_2^N}{\partial x}, \\ i\frac{\partial\beta_3^N}{\partial t} &= -\frac{\partial^2\beta_3^N}{\partial x^2} + 2\left(|\beta_2^N|^2 + |\beta_3^N|^2\right)\beta_3^N - 3ia_1^N\frac{\partial\beta_3^N}{\partial x}. \end{aligned} \tag{15}$$

This system coincides with the two-component nonlinear Schrödinger equation.

4. Systems on orbits of the algebra $\widetilde{g}_+ \simeq su(3) \otimes \mathcal{P}(\lambda)$. The $SU(3)$-Heisenberg magnet equation

As is mentioned in the second section, there is the coadjoint action of the subalgebra $\widetilde{g}_+$ defined on the space M^{N+1}, apart from that of $\widetilde{g}_-$. The corresponding Lie-Poisson brackets have the form:

$$\{f_1, f_2\}_{N+1} = \sum\langle\mu, \left[X_i^{-\ell+N+1}, X_j^{-s+N+1}\right]\rangle_{N+1}\frac{\partial f_1}{\partial\mu_i^\ell}\frac{\partial f_2}{\partial\mu_j^s}. \tag{16}$$

In order to fix the relative orbit (symplectic leaf of the brackets (16)) $\mathcal{O}_2^N$, one equalizes $2(N+2)$ functions $h_0, h_1, \ldots, h_{N+1}$, $f_0, f_1, \ldots, f_{N+1}$, to constans.

The functions $h_{N+2}, \ldots, h_{2N+1}, f_{N+2}, \ldots, f_{3N+2}$ are independent almost everywhere on $\mathcal{O}_2^N \cap M^N$ and form an involutive collection relative to the Lie-Poisson brackets, in accordance with Theorem 1. This implies the complete integrability of the Hamiltonian systems generated by $h_{N+2}, \ldots, h_{2N+1}$, $f_{N+2}, \ldots, f_{3N+2}$. We pay special attention to the geometry of the phase space $\mathcal{O}_2^N$ and the systems generated by h_{N+2} and h_{N+3}.

To choose the dynamical variables $\mu_a = (\mu, X_a)$ conveniently, fix the following orthonormalized basis in the Lie algebra $su(3) : \{X_a,\ a = 1, 2, \ldots, 8\}$, where X_a are anti-Hermitian matrices, with the orthonormalization performed according to the scalar product $(A, B) = -2TrAB$. The commutation relations $[X_a, X_b] = C_{abc}X_c$, are fulfilled by matrices X_a, where the structure constants C_{abc} are real and antisymmetric. The polynomial $I_2 = \sum_{a=1}^{8}\mu_a^2$ and $I_3 = \sum_{a,b,c} d_{abc}\mu_a\mu_b\mu_c$ are independent ad^*-invariant Casimir functions on $su^*(3)$, where $d_{abc} = -2Tr(X_aX_bX_c + X_bX_aX_c)$. They are continued to functions on M^{N+1} by the substitution $\mu_a \longrightarrow \mu_a(\lambda) = \mu_a^0 + \lambda\mu_a^1 + \ldots + \lambda^{N+1}\mu_a^{N+1}$. Some of the coefficient functions h_α, f_β (see the previous subsection) have the form: $h_0 = \mu_a^0\mu_a^0$, $h_1 = 2\mu_a^1\mu_a^0, \ldots, f_0 = d_{abs}\mu_a^0\mu_b^0\mu_c^0$, $f_1 = 3d_{abc}\mu_a^1\mu_b^0\mu_c^0, \ldots$, where the summation over repeated indices is assumed.

The orbit $\mathcal{O}_+^N$ is fixed by the set of algebraic equations

$$h_\alpha = R_\alpha^2, \qquad f_\beta = C_\beta^3, \qquad \alpha, \beta = \overline{0, N+1}. \tag{17}$$

In particular, the equations $h_0 = \mu_a^0\mu_a^0 = R_0^2$, $f_0 = d_{abc}\mu_a^0\mu_b^0\mu_c^0 = C_0^3$ determine the generic coadjoint orbit in $su^*(3)$, which is diffeomorphic to the factor-space $SU(3)/U(1)\times U(1)$. The whole orbit $\mathcal{O}_+^N$ has the structure of the vector bundle over $SU(3)/U(1)\times U(1)$ (see eq.(17)).

Write down the Hamiltonian equations generated by h_{N+r} on the orbit $\mathcal{O}_+^N$:

$$\frac{d\mu_a^\ell}{dx} = \{\mu_a^\ell, h_{N+2}\}_{N+1} = f_{abc}\mu_c^0\mu_b^{\ell+1}, \tag{18}$$

that coincides with the higher stationary equations for the evolutionary system considered earlier in [14]. The latter can be obtained when reducing the Hamiltonian equations

$$\frac{d\mu_a^0}{dt} = \{\mu_a^0, h_{N+3}\}_{N+1} = f_{abc}\mu_c^0\mu_b^{\ell+2} \tag{19}$$

onto trajectories of system (18).

Degenerate orbits. Let the initial point $\mu_0 \in su^*(3)$ have the coordinates $(\mu_0, x_a) = 0$, $a = 1, 2, \ldots, 7$, $(\mu_0, x_8) = R_0$. The stationary subgroup of this point is isomorphic to $SU(2) \times U(1)$. Then the orbit passing through μ_0 is diffeomorphic to the factor-space $SU(3)/SU(2) \times U(1) \simeq \mathbf{CP}^2$ which is a two-dimensional complex projective space. There is a degenerate four-dimensional orbit that can be fixed by the system of quadrics in the space $su^*(3)$:

$$d_{abc}\mu_b\mu_c + \frac{R_0}{\sqrt{3}}\mu_a = 0,$$

amongst which there are only four independent. The whole degenerate orbit has the structure of a vector bundle over $\mathbf{CP}^2$.

The variables μ_a^ℓ, $\ell = 1, 2, \ldots, N$, are expressed via μ_a^0 and the derivatives $\mu_{a_x}^0$, $\mu_{a_{xx}}^0, \ldots$, on trajectories of system (18). In view of this, equation (19) reduced on the degenerate orbit renders the form:

$$\frac{\partial\mu_a^0}{\partial t} = \frac{4}{3I_2} f_{abc}\mu_b^0\mu_{c-xx}^0 \tag{20}$$

which represents an $SU(3)$-generalization of the dynamical equation for the continuous Heisenberg magnet [14].

5. Higher stationary equations of the principal hierarchy. The Boussinesq's equation

Let σ be a third order automorphism of the algebra $g = sl(3, \mathbf{C})$, and g_1, g_ω, g_{ω^2} be the eigenspaces of σ with the eigenvalues 1, ω, ω^2, respectively. Here, $\omega = \exp\dfrac{2\pi i}{3}$. Let $\widetilde{g}$ be a loop algebra with the principal gradation. This means that the following decomposition is given:

$$\widetilde{g} = \sum_l \oplus g_1^l \oplus g_\omega^l \oplus g_{\omega^2}^l,$$

where $g^l_{1,\ \omega,\ \omega^2} = \mathbf{C}X^l_{1,\ \omega,\ \omega^2}$ are the 1-dim. eigenspaces of the gradiation operator $d' = 3\lambda\dfrac{d}{d\lambda} + ad\ H$, $H = \mathrm{diag}(1, 0, -1)$. It is not difficult to see that:

$$X_\omega^{3k} = \lambda^k \mathrm{diag}(\omega^2, \omega, 1), \qquad X_{\omega^2}^{3k} = \lambda^k \mathrm{diag}(\omega, \omega^2, 1)$$

$$X_1^{1+3k} = \lambda^k \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{bmatrix}, \qquad X_\omega^{1+3k} = \lambda^k \begin{bmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ \lambda & 0 & 0 \end{bmatrix},$$

$$X_{\omega^2}^{1+3k} = \lambda^k \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ \lambda & 0 & 0 \end{bmatrix}, \qquad X_1^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$X_\omega^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}, \qquad X_{\omega^2}^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}.$$

Elements X_1^{1+3k}, X_1^{2+3k} generate a maximaly commutative subalgebra in $\widetilde{g}$, which turns into Heisenberg subalgebra after central extension.

Let's consider coadjoint action of the subalgebra$\widetilde{g}_-(\sigma) = \mathrm{span}_{\mathbf{C}}\{X_1^{-1},\ X_1^{-2},\ X_\omega^{-2},\ X_{\omega^2}^{-2},\ X_\omega^{-3},\ X_{\omega^2}^{-3},\ \ldots\}$ in the dual space $\widetilde{g}_-^*(\sigma) \sim \widetilde{g}_+(\sigma)$. The subspace

$$M^N = \{\mu \in \widetilde{g}_+(\sigma); \qquad \mu = \sum_{k=-2}^{3N-1} (\mu_1^k X_\omega^{*k} + \mu_\omega^k X_\omega^{*k} + \mu_{\omega^2}^k X_{\omega^2}^{*k})\},$$

is invariant with respect to this action. We define the Lie-Poisson bracket on this subspace according to formulas (3) and (4) and putting $n = -1$. The symplectic leaf (orbit) of this bracket is fixed by equations:

$$\mu_\omega^{3N-1} = \mu_{\omega^2}^{3N-1} = 0;$$

$$\mu_1^{3N-1} \neq 0,\ h_\alpha^2 = c_\alpha^2,\ \alpha = N, \ldots, 2N-2;\ h_\beta^3 = c_\beta^3,\ \beta = 2N-1, \ldots, 3N-2$$

The next step is reduction of the orbit over the Hamiltonian flow of h_{N-1}^2. On the reduced orbit, $\omega\mu_\omega^{3(N-1)} + \omega^2\mu_{\omega^2}^{3(N-1)} = \mu_\omega^{3(N-1)} + \mu_{\omega^2}^{3(N-1)} = \dfrac{1}{3}u$ and we can define the integrable system with the Hamiltonian h_{2N-2}^3 on it. It's integrability is provided by integrals: $h_0^2,\ h_1^2, \ldots,\ h_{N-2}^2;\ h_0^3,\ h_1^3, \ldots,\ h_{2N-2}^3$. This system could be written as higher-order differential equation for the function $u(x)$ and coincides with the higher stationary Boussinesq's equation.

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