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Discrete Integrable Systems and the Moyal Symmetry

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Abstract

The Hirota bilinear difference equation plays the central role in the study of integrable nonlinear systems. A direct correspondence of the large symmetry characterizing this classical system to the Moyal algebra, a quantum deformation of the Poisson bracket algebra, is shown.

1 Introduction

In the study of nonlinear systems, completely integrable systems play special roles. They are not independent at all, but are strongly correlated with each other owing to the large symmetries shared among themselves. We like to know how far we can extend such systems without loosing integrability. The question could be answered if we know how much we can deform the symmetries characterizing the integrable systems.

In this note, we would like to discuss the Moyal bracket algebra [1], a quantum deformation of the Poisson bracket algebra, as a scheme which should describe a large class of integrable systems. Here, however, we focus our attention to the Hirota bilinear difference equation (HBDE) [2], a difference analogue of the two-dimensional Toda lattice.

This paper is organized as follows. We first explain HBDE and what is the Moyal quantum algebra in the following two sections. Using the Miwa transformation [3] of soliton variables, the correspondence of the shift operator appeared in HBDE to the Moyal quantum operator is shown in sections 4 and 5. The large symmetry possessed by the universal Grassmannian of the KP hierarchy [4, 5] is explained in section 6 within our framework, and the connection of their generators to the Moyal quantum operators is shown in the last section.

2 Hirota bilinear difference equation

The Hirota bilinear difference eqution (HBDE) is a simple single equation which is given by [2]

$$\begin{aligned}&\alpha f(k_1+1,k_2,k_3)f(k_1,k_2+1,k_3+1)+\beta f(k_1,k_2+1,k_2)f(k_1+1,k_2,k_3+1)\\&\qquad+\gamma f(k_1,k_2,k_3+1)f(k_1+1,k_2+1,k_3)=0,\end{aligned} \tag{1}$$

This paper is dedicated to the memory of Professor Wilhelm Fushchych where $k_j \in \boldsymbol{Z}$ are discrete variables and $\alpha,\ \beta,\ \gamma \in \boldsymbol{C}$ are parameters subject to the constraint $\alpha+\beta+\gamma=0$.

The contents of this equation is, however, very large. In fact, this single difference equation

1. is equivalent to the soliton equations of the KP-hierarchy [2, 4, 5],
2. characterizes algebraic curves (Fay's trisecant formula) [4],
3. is a consistency relation for the Laplace maps on a discrete surface [6],
4. is satisfied by string correlation functions in the particle physics [7], and
5. by transfer matrices of certain solvable lattice models [8, 9, 10, 11, 12],

etc..

We note here that HBDE (1) is a collection of infinitely many 'classical' soliton equations [2, 3]. For the purpose of describing the symmetric nature of this equation, it will be convenient to rewrite it as

$$\begin{aligned}\Big(\alpha \exp\Big[\partial_{k_1}+\partial_{k_2'}+\partial_{k_3'}\Big]+\beta \exp\Big[\partial_{k_1'}+\partial_{k_2}+\partial_{k_3'}\Big]+\gamma \exp\Big[\partial_{k_1'}+\partial_{k_2'}+\partial_{k_3}\Big]\Big)\\ \times\, f(k_1,k_2,k_3)f(k_1',k_2',k_3')|_{k_j'=k_j}\,.\end{aligned} \tag{2}$$

As we will see later, it is this shift operator $\exp\partial_{k_j}$ that generates the symmetry of a system associated with the Moyal 'quantum' algebra. Classical soliton equations are obtained by expanding the shift operators in (2) into power series of derivatives [4, 5].

The space of solutions to this equation, hence to the KP hierarchy, is called the Universal Grassmannian [4]. Every point on this space corresponds to a solution of HBDE, which can be given explicitly. Starting from one solution, we can obtain other solutions via a sequence of Bäcklund transformations. The transformations generate a large symmetry which characterizes this particular integrable system.

3 Moyal quantum algebra

Let us explain what is the Moyal algebra[1]. We will show, in other sections, its relation to the integrable systems characterized by HBDE.

The Moyal bracket is a quantum deformation of the Poisson bracket and is given by [1]

$$i\{f,g\}_M := \frac{1}{\lambda}\sin\left\{\lambda\left(\boldsymbol{\partial}_x\boldsymbol{\partial}_{p'}-\boldsymbol{\partial}_{x'}\boldsymbol{\partial}_p\right)\right\} f(\boldsymbol{x},\boldsymbol{p})g(\boldsymbol{x'},\boldsymbol{p'})\Big|_{\boldsymbol{x'}=\boldsymbol{x},\boldsymbol{p'}=\boldsymbol{p}} \tag{3}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ are the coordinates and momenta in $\boldsymbol{R}^N$ with N being the number of degres of freedom. It turns to the Poisson bracket in the small λ limit. In this particular limit, the symmetry is well described by the language of differential geometry. Therefore, our question is whether there exists a proper language which can describe concepts, in the case of finite values of λ, corresponding to the terms such as vector fields, differential forms, and Lie derivatives. The answer is "yes" [13]. In order to show that, we first define a difference operator by

$$\boldsymbol{\nabla}_{\boldsymbol{a}_x,\boldsymbol{a}_p} := \frac{1}{\lambda}\sinh\left[\lambda(\boldsymbol{a}\boldsymbol{\partial})\right], \qquad (\boldsymbol{a}\boldsymbol{\partial}) := \boldsymbol{a}_x\boldsymbol{\partial}_x+\boldsymbol{a}_p\boldsymbol{\partial}_p \tag{4}$$

For a given $\boldsymbol{C}^\infty$ function $f(\boldsymbol{x},\boldsymbol{p})$, the Hamiltonian vector field is defined as [13]

$$\boldsymbol{X}_f^D := \left(\frac{\lambda}{2\pi}\right)^{2N} \int d\boldsymbol{a}_x d\boldsymbol{a}_p \int d\boldsymbol{b}_x d\boldsymbol{b}_p \, e^{-i\lambda(\boldsymbol{a_x b}_p - \boldsymbol{a}_p \boldsymbol{b}_x)} f(\boldsymbol{x}+\lambda\boldsymbol{b}_x, \boldsymbol{p}+\lambda\boldsymbol{b}_p) \boldsymbol{\nabla}_{\boldsymbol{a}_x,\boldsymbol{a}_p}. \quad (5)$$

The operation of the Hamiltonian vector field to a function g on the phase space yields the Moyal bracket:

$$\boldsymbol{X}_f^D g(\boldsymbol{x},\boldsymbol{p}) = i\{f,g\}_M. \quad (6)$$

The above definition of the Hamiltonian vector fields enables us to extend the concept of Lie derivative in the differential geometry to a discrete phase space [13]. In fact, we can show directly the following relation between two vector fields:

$$\left[\boldsymbol{X}_f^D, \boldsymbol{X}_g^D\right] = \boldsymbol{X}^D_{-i\{f,g\}_M}. \quad (7)$$

This exhibits a very large symmetry generated by Hamiltonian vector fields, which should be shared commonly by the physical systems formulated in our prescription. This algebra itself was derived and discussed [14, 15, 16] in various contexts including some geometrical arguments.

For the vector field $\boldsymbol{X}_f^D$ to be associated with a quantum operator, we must introduce a difference one form whose pairing with the vector field yields the expectation value of f. It is shown in [13] that such a pairing can be defined properly by considering a quantum state characterized by the Wigner distribution function [17].

4 Miwa transformation

The Miwa transformation [3] of variables enables us to interpret the shift operators in HBDE (2) as the Hamiltonian vector field discussed above. It is defined by

$$t_n := \frac{1}{n}\sum_j k_j z_j^n, \qquad n = 1,2,3,\ldots. \quad (8)$$

In this expression, k_j ($j \in \boldsymbol{N}$) are integers among which $k_1,\ k_2,\ k_3$ belong to (1). t_n's are new variables which describe soliton coordinates of the KP hierarchy. For instance, $t_1 = t$ and $t_3 = x$ are the time and space variables, respectively, of the KdV equation. z_j's are complex parameters which are defined on the Riemann surface.

In the language of string models, the soliton variables t_n's correspond to the oscillation parts of open strings [7]. The center of momenta x_0 and the total momentum p_0 should be also included as dynamical variables, which are related to k_j's by

$$p_0 = \sum_j k_j, \qquad x_0 = i\sum_j k_j \ln \bar{z}_j. \quad (9)$$

In addition to (8), we also define

$$\bar{t}_n := \frac{1}{n}\sum_j k_j \bar{z}_j^{-n}, \qquad n = 1,2,3,\ldots, \quad (10)$$

so that the physical space of oscillations is doubled. The phase space of oscillations are described by t_n's and $\bar{t}_n$'s as

$$\begin{aligned} x_n &= \sqrt{\frac{n}{2}}(t_n + \bar{t}_n) = \sqrt{\frac{1}{2n}} \sum_j k_j \left(z_j^n + \bar{z}_j^{-n} \right), \qquad n = 1, 2, 3, \ldots, \\ p_n &= \frac{1}{i}\sqrt{\frac{n}{2}}(t_n - \bar{t}_n) = \frac{1}{i}\sqrt{\frac{1}{2n}} \sum_j k_j \left(z_j^n - \bar{z}_j^{-n} \right), \qquad n = 1, 2, 3, \ldots. \end{aligned} \tag{11}$$

In terms of these new variables, the shift operators appearing in HBDE (2) become

$$\exp \partial_{k_j} = \exp \left[\lambda (\boldsymbol{a}_j \boldsymbol{\partial}) \right]. \tag{12}$$

Here $(\boldsymbol{a}_j \boldsymbol{\partial})$ is the sum of an infinite number of components in the notation of (4). The values of the components of $\boldsymbol{a}_{j,x}$ and $\boldsymbol{a}_{j,p}$ are specialized to

$$\begin{aligned} a_{j,x_0} &= \frac{i}{\lambda} \ln \bar{z}_j, \qquad a_{j,p_0} = \frac{1}{\lambda}, \\ a_{j,x_n} &= \frac{z_j^n + \bar{z}_j^{-n}}{\lambda\sqrt{2n}}, \quad a_{j,p_n} = \frac{z_j^n - \bar{z}_j^{-n}}{i\lambda\sqrt{2n}}, \qquad n = 1, 2, 3, \ldots. \end{aligned} \tag{13}$$

5 Gauge covariant shift operator

We now gauge the shift operator to obtain a gauge covariant shift operator:

$$e^{\partial_{k_j}} \quad \to \quad \boldsymbol{X}^S_{u_j} := U(\boldsymbol{k}) e^{\partial_{k_j}} U^{-1}(\boldsymbol{k}). \tag{14}$$

Here $U(\boldsymbol{k})$ is a function of $\boldsymbol{k} = \{k_1, k_2, \cdots\}$. Using the Miwa transformation, we can write it in terms of the soliton variables as

$$\boldsymbol{X}^S_{u_j} = u_j(\boldsymbol{x}, \boldsymbol{p}) e^{\lambda (\boldsymbol{a}_j \boldsymbol{\partial})}, \qquad u_j(\boldsymbol{x}, \boldsymbol{p}) := U(\boldsymbol{k}) U^{-1}(\boldsymbol{k}^{(j)}), \tag{15}$$

where $\boldsymbol{k}^{(j)}$ denotes the set of k's but k_j is replaced by $k_j + 1$.

The covariant shift operators also form a closed algebra as follows:

$$\left[\boldsymbol{X}^S_{u_j},\ \boldsymbol{X}^S_{v_l} \right] = \boldsymbol{X}^S_{\{u_j, v_l\}_S} \tag{16}$$

where

$$\{u_j, v_l\}_S := \left(e^{\lambda (\boldsymbol{a}_l \boldsymbol{\partial}')} - e^{\lambda (\boldsymbol{a}_j \boldsymbol{\partial})} \right) u_j(\boldsymbol{x}, \boldsymbol{p}) v_l(\boldsymbol{x}', \boldsymbol{p}') \Big|_{\boldsymbol{x}' = \boldsymbol{x}, \boldsymbol{p}' = \boldsymbol{p}}. \tag{17}$$

In this formula, $\boldsymbol{\partial}$ and $\boldsymbol{\partial}'$ act on u_j and v_l selectively.

We now want to establish the correspondence of gauge covariant shift operators to the Hamiltonian vector field of the Moyal quantum algebra discussed before. For this purpose, we denote by $\boldsymbol{X}^D_{u_j}$ the antisymmetric part of $-\boldsymbol{X}^S_{u_j}$,

$$\boldsymbol{X}^D_{u_j} = \frac{1}{2\lambda} \left(\overline{\boldsymbol{X}}^S_{u_j} - \boldsymbol{X}^S_{u_j} \right), \tag{18}$$

Here $\overline{\boldsymbol{X}}_{u_j}^S$ is obtained from $\boldsymbol{X}_{u_j}^S$ by reversing the direction of shift in (15). Then it is not difficult to convince ourselves that $\boldsymbol{X}_{u_j}^D$ is a Hamiltonian vector field of (5) with f being specialized to u_j, and u_j itself is given by $u_j(\boldsymbol{x},\boldsymbol{p}) = \exp\left[i\lambda\left(\boldsymbol{a}_{j,p}\boldsymbol{x} - \boldsymbol{a}_{j,x}\boldsymbol{p}\right)\right]$. We thus obtain an expression for the shift operator

$$\boldsymbol{X}_{u_j}^S = \exp\left[i\lambda\left(\boldsymbol{a}_{j,p}\boldsymbol{x} - \boldsymbol{a}_{j,x}\boldsymbol{p}\right)\right]\exp\left[\lambda\left(\boldsymbol{a}_{j,x}\boldsymbol{\partial}_x + \boldsymbol{a}_{j,p}\boldsymbol{\partial}_p\right)\right], \tag{19}$$

from which we can form a Hamiltonian vector field of the Moyal algebra. We notice that this operator $\boldsymbol{X}_{u_j}$ is nothing but the vertex operator for the interaction of closed strings [7].

Does there exist U associated to this $u_j(\boldsymbol{x},\boldsymbol{p})$? We can see that

$$U(\boldsymbol{k}) := \prod_{j,l}\left(\frac{\bar{z}_j - z_l}{\bar{z}_l - z_j}\right)^{-k_l k_j} \tag{20}$$

satisfies this requirement if we substitute it to (15) and use the Miwa transformations.

6 Symmetry of HBDE

One of the symmetries which characterize the solution space of HBDE is generated by the Bäcklund transformation [4, 5]. The generators of this symmetry are given by

$$\boldsymbol{B}(z_j, z_l) := 4\pi V(z_j)\bar{V}(z_l) \tag{21}$$

where $V(z_j)$ is obtained from $\boldsymbol{X}_{u_j}^S$ of (19) simply by ignoring the $\bar{t}_n$ and x_0 dependence. In terms of the soliton variables, we can express it in the form

$$\begin{aligned} V(z_j) &:= \exp\left[p_0 \ln \bar{z}_j - \sum_{n=1}^{\infty} \bar{z}_j^{-n} t_n\right]\exp\left[\partial_{p_0} + \sum_{n=1}^{\infty}\frac{1}{n} z_j^n \partial_{t_n}\right], \\ \bar{V}(z_j) &:= \exp\left[-p_0 \ln \bar{z}_j + \sum_{n=1}^{\infty} \bar{z}_j^{-n} t_n\right]\exp\left[-\partial_{p_0} - \sum_{n=1}^{\infty}\frac{1}{n} z_j^n \partial_{t_n}\right]. \end{aligned} \tag{22}$$

These vertex operators must be thought being local field operators which might behave singularly when their coordinates z get close with each other. For instance, after simple calculation we find

$$\begin{aligned} V(z_j)\bar{V}(z_l) &= \frac{1}{\bar{z}_l - z_j}\exp\left[p_0 \ln\left(\frac{\bar{z}_j}{\bar{z}_l}\right) - \sum_{n=1}^{\infty}\left(\bar{z}_j^{-n} - \bar{z}_l^{-n}\right)t_n\right]\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(z_j^n - z_l^{-n}\right)\partial_{t_n}\right], \\ \bar{V}(z_l)V(z_j) &= \frac{1}{\bar{z}_j - z_l}\exp\left[p_0 \ln\left(\frac{\bar{z}_j}{\bar{z}_l}\right) - \sum_{n=1}^{\infty}\left(\bar{z}_j^{-n} - \bar{z}_l^{-n}\right)t_n\right]\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(z_j^n - z_l^{-n}\right)\partial_{t_n}\right]. \end{aligned} \tag{23}$$

If we are interested in the behaviour of these quantities on the real axis, the summation of $V(z_j)\bar{V}(z_l)$ and $\bar{V}(z_l)V(z_j)$ is zero. If we are interested in the behaviour, not on the real axis but near the real axis, we must be more careful. Let us write $\bar{z}_m - z_m = -2i\epsilon,\ {}^\forall m$, and take the limit of $\epsilon \to 0$. We find

$$V(z_j)\bar{V}(z_l) + \bar{V}(z_l)V(z_j) = 4\pi i\delta(z_l - z_j). \tag{24}$$

This is the result known as bosonization. Using this result, we can show that $\boldsymbol{B}$'s form the algebra $gl(\infty)$, which characterizes the symmetry of the universal Grassmannian [4, 5],

$$[\boldsymbol{B}(z_j, z_k),\ \boldsymbol{B}(z_l, z_m)] = i\boldsymbol{B}(z_j, z_m)\delta(z_k - z_l) - i\boldsymbol{B}(z_k, z_l)\delta(z_j - z_m). \tag{25}$$

This symmetry includes the conformal symmetry of the theory [18]. In fact, the Virasoro generators are included in (21) in the limit of $z_j \to z_l$. $W_{1+\infty}$ symmetry of the KP-hierarchy [19, 20, 21] can be also described in our scheme.

7 Quantum deformation

We now look at the vertex operator $V(z_j)$ from other view point. It must be also represented as a gauge covariant shift operator as $\boldsymbol{X}^S_{u_j}$ in (14). Indeed, we can rewrite it in the form of (14), i.e., $U \exp \partial_{k_j} U^{-1}$, by choosing $U(\boldsymbol{k})$ as

$$U(\boldsymbol{k}) = \prod_{j\neq l} (\bar{z}_l - z_j)^{k_j k_l}. \tag{26}$$

It then follows, from their construction, that $V(z_j)$ and $\bar{V}(z_l)$ must commute with each other; $[V(z_j),\ \bar{V}(z_l)] = 0$, $\forall j, l$. By the same reason, we should have $\left[\boldsymbol{X}^S_{u_j},\ \boldsymbol{X}^S_{u_l}\right] = 0$, $\forall j, l$, as long as $\boldsymbol{X}^S_{u_j}$ and $\boldsymbol{X}^S_{u_l}$ are induced by the same function $U(\boldsymbol{k})$. Apparently, this is not compatible with the previous result, e.g., (24). How can we get rid of this contradiction?

Instead of trying to avoid this problem, we like to interpret it as a transition from a classical view to a quantum view. To do this, we recall that the shift operators $\exp \partial_{k_j}$ define HBDE, a classical soliton equation. The change of variables, via Miwa transformations, introduces infinitely many new variables and the shift operator is represented in terms of quantized local fields. If we expand $\exp \partial_{k_j}$ in (1) into powers of ∂_{t_n}'s, we obtain infinitely many classical soliton equations which belong to the KP hierarchy [4, 5]. But their infinite collection may not be classical anymore.

This transition is, however, not sufficient to claim that $V(z_j)$, expressed in the form of (22), is an operator of quantum mechanics. A quantum mechanical operator must be described in terms of a Moyal Hamiltonian vector field of (5). From this point of view, the operator $\boldsymbol{X}^D_{u_j}$ of (18) could be a candidate of a quantum mechanical object.

Before closing this note, let us see the algebra corresponding to (24), but $V(z_j)$ is replaced by $\boldsymbol{X}^S_{u_j}$ of (19). It is more convenient to introduce the notations $\boldsymbol{V}(z_j) := \boldsymbol{X}^S_{u_j}$ and $\bar{\boldsymbol{V}}(z_j)$ being defined by substituting $-\lambda$ in the place of λ on the right hand side of (19). After some calculations similar to (23), we find

$$\boldsymbol{V}(z_j)\bar{\boldsymbol{V}}(z_l) = \frac{\bar{z}_j - z_l}{\bar{z}_l - z_j} \exp\left[i\lambda\left\{(\boldsymbol{a}_{j,p} - \boldsymbol{a}_{l,p})\boldsymbol{x} - (\boldsymbol{a}_{j,x} - \boldsymbol{a}_{l,x})\boldsymbol{p}\right\}\right] \exp\left[\lambda\left((\boldsymbol{a}_j - \boldsymbol{a}_l)\boldsymbol{\partial}\right)\right],$$

$$\bar{\boldsymbol{V}}(z_l)\boldsymbol{V}(z_j) = \frac{\bar{z}_l - z_j}{\bar{z}_j - z_l} \exp\left[i\lambda\left\{(\boldsymbol{a}_{j,p} - \boldsymbol{a}_{l,p})\boldsymbol{x} - (\boldsymbol{a}_{j,x} - \boldsymbol{a}_{l,x})\boldsymbol{p}\right\}\right] \exp\left[\lambda\left((\boldsymbol{a}_j - \boldsymbol{a}_l)\boldsymbol{\partial}\right)\right],$$

from which we obtain, in the limit of small $\bar{z}_m - z_m = -2i\epsilon$,

$$\boldsymbol{V}(z_j)\bar{\boldsymbol{V}}(z_l) - \bar{\boldsymbol{V}}(z_l)\boldsymbol{V}(z_j) = \lim_{\epsilon\to 0} \frac{8i\epsilon(z_j - z_l)}{(z_j - z_l)^2 + 4\epsilon^2}. \tag{27}$$

This does not vanish in the limit of $\epsilon \to 0$ *iff* $z_j - z_l$ is the same order of ϵ.

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