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On Parasupersymmetries in a Relativistic Coulomb Problem for the Modified Stueckelberg Equation

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Abstract

We consider a Coulomb problem for the modified Stueckelberg equation. For some specific values of parameters, we establish the presence of parasupersymmetry for spin-1 states in this problem and give the explicit form of corresponding parasupercharges.

Introduction

In spite of the striking progress of quantum field theory (QFT) during last three decades, the problem of description of bound states in QFT context isn't yet completely solved. This state of things stimulates the use of various approximate methods in order to obtain physically important results.

In this paper, we work within the frame of the so-called one-particle approximation (OPA) which consists in neglecting creation and annihilation of particles and the quantum nature of external fields which the only particle we consider interacts with. This approximation is valid for the case where the energy of the interaction between the particle and the field is much less than the mass of the particle. In OPA, the operator of the quantized field corresponding to our particle is replaced by the "classical" (i.e., non-secondly quantized) quantity which is in essence nothing but the matrix element of that operator between vacuum and one-particle state (cf. [1] for the case of spin 1/2 particles). This quantity may be interpreted as a wave function of the particle and satisfies a *linear* equation, in which external field also appears as a classical quantity.

But for the case of massive charged spin-1 particles interacting with external electromagnetic field, even such relatively simple equations lead to numerous difficulties and inconsistencies [2, 3]. Only in 1995, Beckers, Debergh and Nikitin [2] overcame most of them and suggested a new equation describing spin-1 particles. They exactly solved it and pointed out its parasupersymmetric properties for the case of external constant homogeneous magnetic field [2]. (To find out more about parasupersymmetry, refer to [4] and references therein.)

The evident next step is to study this model for another physically interesting case of external Coulomb field (and in particular to check the possibility of existence of the parasupersymmetry in this case too).

In order to overcome some difficulties arising in the process of such a study, we consider in [3] and here a "toy" model corresponding to a particle with two possible spin states:

spin-1 and spin-0. Beckers, Debergh and Nikitin in [2] suggested a new equation for this case too, but our one is slightly more general (refer to Section 1).

The plan of the paper is as follows. In Section 1, we write down the equation describing our model (we call it modified Stueckelberg equation). In Section 2, we briefly recall the results from [3] concerning the exact solution of this model for the case of the Coulomb field of attraction. Finally, in Section 3, we point out the existence of the parasupersymmetry in the Coulomb field for spin-1 states at particular values of some parameters of our model.

1 Modified Stueckelberg equation

We consider the so-called modified Stueckelberg equation written in the second-order formalism [3]:

$$(D_\mu D^\mu + m_{eff}^2)B^\nu + iegF^\mu_\rho B^\rho = 0, \tag{1}$$

where

$$m_{eff}^2 = M^2 + k_2 \mid e^2 F_{\mu\nu}F^{\mu\nu} \mid^{1/2}, \quad D_\mu = \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

We use here the $\hbar = c = 1$ units system and following notations: small Greek letters denote indices which refer to the Minkowski 4-space and run from 0 to 3 (unless otherwise stated); we use the following metrics of the Minkowski space: $g_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$; the four-vector is written as $N^\mu = (N^0, \mathbf{N})$, where the bold letter denotes its three-vector part; coordinates and derivatives are $x^\mu = (t, \mathbf{r})$, $\partial_\mu = \partial/\partial x^\mu$; A^μ are potentials of external electromagnetic field; e, g and M are, respectively, charge, gyromagnetic ratio and mass of the particle described by (1). Its wave function is given by the four-vector B^μ. In the free ($e = 0$) case [2] this particle has two possible spin states: spin-0 and spin-1 ones with the same mass M.

Equation (1) generalizes the modification of the Stueckelberg equation from [2] for the case of an arbitrary gyromagnetic ratio g (authors of [2] consider only the $g = 2$ case). We also put the module sign at expression (2) for m_{eff}^2 (in spite of [2]) in order to avoid complex energy eigenvalues in the Coulomb problem (else μ_i (9) and, hence, energy eigenvalues of the discrete spectrum E^{inj} will be complex).

2 Coulomb field

The 4-potential, corresponding to the Coulomb field of attraction, is:

$$\mathbf{A} = 0, \quad A^0 = -Ze/r, \quad Z > 0. \tag{3}$$

Since it is static and spherically symmetric, energy E and total momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ ($\mathbf{L}$ is an angular momentum and $\mathbf{S}$ is a spin) are integrals of motion. Let us decompose the wavefunctions of stationary states with fixed energy E in the basis of common eigenfunctions of $\mathbf{J}^2$ and $\mathbf{J}_z$ with eigenvalues $j(j+1)$ and m, respectively, $j = 0, 1, 2, \ldots$, for a given j, $m = -j, -j+1, \ldots, j$. The corresponding eigenmodes are:

$$\begin{aligned} B^0_{Ejm} &= iF_{Ej}(r)Y_{jm}\exp(-iEt), \\ \mathbf{B}_{Ejm} &= \exp(-iEt)\sum_{\sigma=-1,0,1} B^{(\sigma)}_{Ej}(r)\mathbf{Y}^{(\sigma)}_{jm}, \end{aligned} \tag{4}$$

where $\mathbf{Y}^{(\sigma)}_{jm}$ are spherical vectors (see [5] for their explicit form) and Y_{jm} are usual spherical functions. It may be shown that F_{Ej} and B^{σ}_{Ej}, ($\sigma = -1, 0, 1$) in fact don't depend on m [5]. For the sake of brevity from now on, we often suppress indices Ej at F_{Ej} and $B^{(\sigma)}_{Ej}$ ($\sigma = -1, 0, 1$).

The substitution of (3) and (4) into (1) yields the following equations for F and $B^{(\sigma)}$:

$$\mathrm{T}V = (2/r^2)\mathrm{P}V, \qquad \mathrm{T}B^{(0)} = 0, \tag{5}$$

where

$$V = (FB^{(-1)}B^{(1)})^{\dagger}, \qquad \mathrm{P} = \begin{pmatrix} 0 & -b & 0 \\ b & 1 & -a \\ 0 & -a & 0 \end{pmatrix}$$

($\dagger$ denotes matrix transposition, $a = \sqrt{j(j+1)}$, $\beta = Ze^2$, $b = \beta g/2$);

$$\mathrm{T} = (E + \beta/r)^2 + d^2/dr^2 + (2/r)d/dr - j(j+1)/r^2 - M^2 - k_2\beta/r^2. \tag{6}$$

Notice that for $j = 0$ $B^{(0)}_{Ej} = B^{(1)}_{Ej} \equiv 0$ [5].

By the appropriate replacement of the basis (introducing new unknown functions $K^{(i)}$, $i = 1, 2, 3$, being linear combinations of $B^{(\sigma)}$, $\sigma = -1, 1$ and $F(r)$), equations (5) for radial functions may be reduced to the following form [3] (for convenience, we denote $K^{(0)} \equiv B^{(0)}$):

$$\mathrm{T}K^{(i)} = (2\lambda_i/r^2)K^{(i)} \qquad i = 0, 1, 2, 3, \tag{7}$$

where

$$\lambda_l = 1/2 + (-1)^{l-1}\sqrt{(j+1/2)^2 - (\beta g/2)^2}, \qquad l = 1, 2, \quad \lambda_0 = \lambda_3 = 0. \tag{8}$$

Energy eigenvalues of the discrete spectrum for (7) are [3]:

$$E^{inj} = M/\sqrt{1 + \beta^2/(n + \mu_i + 1)^2}, \tag{9}$$

where $n = 0, 1, 2, \ldots$; $j = 0, 1, 2\ldots$; $i = 0, 1, 2, 3$ and

$$\mu_i = -1/2 + \sqrt{(j+1/2)^2 - \beta^2 + 2\lambda_i + k_2\beta} \tag{10}$$

(index i corresponds to the following eigenmode: $K^{(i)} \neq 0$, other functions $K^{(l)} = 0$; since for $j = 0$ $K^{(0)} = K^{(3)} \equiv 0$, in this case, we have only two branches corresponding to $i = 1, 2$). Branches of the spectrum for $i = 0$ and $i = 3$ are completely identical, i.e., we meet here a twofold degeneracy.

The discrete spectrum eigenfunctions are [3]

$$K^{(i)nj} = c^{inj}x^{\mu_i}\exp(-x/2)L^{\mu_i}_n(x), \tag{11}$$

where c^{inj} are normalization constants, $x = 2r\sqrt{M^2 - E^2}$, L^{α}_n are Laguerre polynomials, $n = 0, 1, 2, \ldots$; $j = 0, 1, 2\ldots$; $i = 0, 1, 2, 3$.

3 Parasupersymmetry

We consider the case $k_2 = 0, g = 2, j > 0$ where [3]

$$\mu_1 = \lambda_1, \quad \mu_0 = \lambda_1 - 1, \quad \mu_2 = \lambda_1 - 2$$

and hence the above energy eigenvalues (9) possess the threefold extra degeneracy:

$$E^{1,n+1,j} = E^{0,n,j} = E^{3,n,j} = E^{2,n-1,j}, \quad n > 1. \tag{12}$$

Moreover, we restrict ourselves by considering only spin-1 states, setting the condition (compatible with (1) for the case of Coulomb field if $k_2 = 0$, $g = 2$ for the states with $j > 0$ [3])

$$\begin{aligned} D_\mu B^\mu_{Ejm} = \Big\{ E K^{(3)}_{Ej} + \sum_{i=1}^{2} \Big[dK^{(i)}_{Ej}/dr + \\ +(1+\lambda_i) K^{(i)}_{Ej}/r - E\beta K^{(i)}_{Ej}/\lambda_i \Big] \Big\} \exp(-iEt) Y_{jm} = 0. \end{aligned} \tag{13}$$

We use here the term "spin-1 states" by the analogy with the free ($e = 0$) case (cf. [1]), i.e., we call spin-1 states the states satisfying (13). In virtue of (13) the component $K^{(3)}$ is expressed via the remaining ones and isn't more independent [3]. This implies that we must deal with only three branches of the discrete spectrum E^{inj}, $i = 0, 1, 2$.

Equations (7) which remain after the exclusion of $K^{(3)}$ may be rewritten in the following form:

$$\mathrm{H}\psi = \varepsilon\psi, \tag{14}$$

where

$$\begin{aligned} &\psi = (K^{(1)} K^{(0)} K^{(2)})^\dagger, \quad \mathrm{H} = 2\mathrm{diag}\,[\mathrm{H}_1, \mathrm{H}_0, \mathrm{H}_2]; \\ &\mathrm{H}_i = -d^2/dr^2 - (2/r)d/dr - \lambda_i(\lambda_i+1)/r^2 - 2\beta E/r + \\ &\qquad +(1/2)\beta E(1/\lambda_1 + 1/\lambda_2) \qquad \text{for} \quad i = 1, 2, \\ &\mathrm{H}_0 = -d^2/dr^2 - (2/r)d/dr - \lambda_1(\lambda_1 - 1)/r^2 - 2\beta E/r + (1/2)\beta E(1/\lambda_1 + 1/\lambda_2), \\ &\varepsilon = \beta E(1/\lambda_1 + 1/\lambda_2) - 2(M^2 - E^2). \end{aligned} \tag{15}$$

Let us introduce parasupercharges Q^+ and Q^- of the form:

$$\mathrm{Q}^+ = \begin{pmatrix} 0 & \mathrm{S}_1 & 0 \\ 0 & 0 & \mathrm{R}_2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{Q}^- = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{R}_1 & 0 & 0 \\ 0 & \mathrm{S}_2 & 0 \end{pmatrix},$$

where

$$\mathrm{R}_i = d/dr + (1+\lambda_i)/r - E\beta/\lambda_i, \qquad \mathrm{S}_i = -d/dr + (\lambda_i - 1)/r - E\beta/\lambda_i. \tag{16}$$

Now it is straightforward to check that Q^+, Q^-, $\mathrm{Q}_1 = (\mathrm{Q}^+ + \mathrm{Q}^-)/2$, $\mathrm{Q}_2 = (i/2)(\mathrm{Q}^+ - \mathrm{Q}^-)$ and H satisfy the commutation relations of the so-called $p = 2$ parasupersymmetric quantum mechanics of Rubakov and Spiridonov (see, e.g., [2, 4]):

$$\begin{aligned} &(\mathrm{Q}^\pm)^3 = 0, \qquad \mathrm{Q}_i^3 = \mathrm{H}\mathrm{Q}_i, \qquad [\mathrm{Q}^\pm, \mathrm{H}] = 0, \\ &\{\mathrm{Q}_i^2, \mathrm{Q}_{3-i}\} + \mathrm{Q}_i \mathrm{Q}_{3-i} \mathrm{Q}_i = \mathrm{H}\mathrm{Q}_{3-i}, \qquad i = 1, 2, \end{aligned} \tag{17}$$

where $[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - \mathrm{BA}$, $\{\mathrm{A}, \mathrm{B}\} = \mathrm{AB} + \mathrm{BA}$.

We see that the parasupercharges Q^+, Q^- commute with H and hence are (non-Lie) symmetries of our Coulomb problem. Their existence is just the reason of the above-mentioned threefold extra degeneracy.

4 Conclusions and discussion

Thus, in this paper, we the explained the threefold extra degeneracy of discrete spectrum levels in a Coulomb problem for the modified Stueckelberg equation for $k_2 = 0$, $g = 2$, $j > 0$, $n > 1$ by the presence of parasupersymmetry. Our results present a natural generalization of supersymmetry in a Coulomb problem for the Dirac equation [4].

It is also worth noticing that, in the case of Coulomb field for $k_2 = 0$, $g = 2$, the equations of our model for the spin-1 states with $j > 1$ coincide with the equations for the same states of the Corben-Schwinger model [6] for $g = 2$. Hence, their model also is parasupersymmetric and possesses non-Lie symmetries – parasupercharges Q^+, Q^-.

Finally, we would like to stress that, to the best of author's knowledge, our model give one of the first examples of parasupersymmetry in a relativistic system with non-oscillator-like interaction.

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