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Higher Order Symmetry Operators for the Schödinger Equation

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Abstract

Potential $V(x,t)$ classes, depending on both independent variables x and t, which allow a symmetry of the Schrödinger equation in the differential operators class of the third order are found.

One-dimensional non-relativistic systems can be considered with the help of the Schrödinger equation

$$L\Psi(x,t) = \left[p_0 - \frac{p^2}{2m} - V(x,t)\right]\Psi(x,t) = 0, \tag{1}$$

where $p_0 = i\dfrac{\partial}{\partial t}$, $p = -i\dfrac{\partial}{\partial x}$, $V(x,t)$ is a potential.

For equation (1), its symmetric analysis plays an important role. The Schrödinger equation symmetry is also surveyed in works [1–7].

The operator of a symmetry Q of equation (1) is an operator, which complies with the condition

$$[L,Q] = 0, \tag{2}$$

where

$$[L,Q] = LQ - QL. \tag{3}$$

Let us find classes of potentials $V(x,t)$, which possess symmetry with respect to a differential operator Q_3 of the third order

$$Q_3 = a_3p^3 + a_2p^2 + a_1p + a_0, \tag{4}$$

where a_3, a_2, a_1, a_0 are unknown functions depending on variables x and t. Substituting operator (4) to equation (2) and, after corresponding changes, equating coefficients of the corresponding operators of differentiation, we obtain the system of differential equations

$$\begin{aligned}
&a_3' = 0,\\
&\dot a_3 + \frac{1}{2m}a_2' = 0,\\
&\dot a_2 + \frac{1}{2m}a_1' - 6a_3V' = 0,\\
&\dot a_1 + \frac{1}{2m}a_0' - 4a_2V' = 0,\\
&\dot a_0 - 2a_1V' + 2a_3V''' = 0.
\end{aligned} \tag{5}$$

Integrating system (5), we obtain

$$
\begin{aligned}
a_3 &= a_3(t),\\
a_2 &= -2m\dot{a}_3 x + a_2^0(t),\\
a_1 &= 2m^2\ddot{a}_3 x^2 + 12ma_3V - 2m\dot{a}_2^0 x + a_1^0(t),\\
a_0' &= -4m^3\dddot{a}_3 x^2 - 16m^2\dot{a}_3 xV' - 24m^2\dot{a}_3 V - 24m^2 a_3\dot{V} +\\
&\qquad 8ma_2^0V' + 4m^2\ddot{a}_2^0 x - 2m\dot{a}_1^0,\\
\dot{a}_0 &- 2a_1V' + 2a_3V''' = 0,
\end{aligned}
\tag{6}
$$

where $a_3(t)$, $a_2^0(t)$, $a_1^0(t)$ are unrestricted functions depending on t.

Some classes of potentials $V(x)$ which comply with system (6) are found in [6, 8]:

$$
V(x) = \frac{2c^2}{m\cos^2 cx}, \qquad V(x) = \frac{2}{m}c^2\tan^2 cx, \qquad V(x) = \frac{2c^2}{m}(\tanh^2 cx - 1),
$$
$$
V(x) = \frac{2c^2}{m}(\coth^2 cx - 1), \qquad V(x) = \frac{1}{m}\left(\frac{c^2}{\sinh^2 cx} \pm \frac{c^2\cosh cx}{\sinh^2 cx}\right),
$$

where c is some unrestricted constant.

We succeeded to indentify other kinds of potentiales $V(x,t)$ (depending on variables x and t) with the symmetry under the class of differential operators Q_3.

If we set $a_3 = \mathrm{const}$, $a_2 = a_1^0 = a_2^0 = 0$ in (6), then we obtain the system

$$
\begin{aligned}
a_1 &= 12ma_3V,\\
a_0' &= -24m^2a_3\dot{V},\\
\dot{a}_0 &- 2a_1V' + 2a_3V''' = 0.
\end{aligned}
\tag{7}
$$

After some changes, we obtain the equation in partial derivatives for finding $V(x,t)$:

$$
12m^2\ddot{V} + 12mVV'' + 12m(V')^2 - V'''' = 0. \tag{8}
$$

Equation (8) can be written as

$$
12m^2\ddot{V} = (V'' - 6mV^2)''. \tag{9}
$$

A solution of the given equation is the function

$$
V(x) = \left(-\frac{c_1^2}{2m}t^2 + c_2t + c_3\right)(c_1x + c_4), \tag{10}
$$

where c_1, c_2, c_3, c_4 are unrestricted constants.

If we make the substitution $V(x,t) = \dfrac{U(x,t)}{m}$ in equation (9), then we will obtain the equation

$$
12m^2\ddot{U} = (U'' - 6U^2)''. \tag{11}
$$

We know [9], that a solution of the equation $y'' - 6y^2 = 0$ $(y = y(x))$ is the function $y(x) = \wp(x + c_0)$, where $\wp$ is the Weierstrass function with invariants $g_2 = 0$ and $g_3 = c_1$, and c_1 is an unrestricted constant. Using this fact, we can write a solution of equation (11) as

$$U(x,t) = (\alpha t + \beta)\wp(x + c_0),$$

and a solution of equation (9)

$$V(x,t) = \frac{1}{m}(\alpha t + \beta)\wp(x + c_0), \tag{12}$$

where α, β, c_0 are unrestricted constants.

Thus, the found operators $V(x,t)$ (10), (12) exhibit the symmetry of the Schrödinger equation (1) in the class of differential operators of the third order Q_3.

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