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Symmetry Reduction and Exact Solutions of the $\boldsymbol{SU}(2)$ Yang-Mills Equations

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Abstract

We present a detailed account of symmetry properties of $SU(2)$ Yang-Mills equations. Using a subgroup structure of the conformal group $C(1,3)$, we have constructed $C(1,3)$-inequivalent ansatzes for the Yang-Mills field which are invariant under three-dimensional subgroups of the conformal group. With the aid of these ansatzes, reduction of Yang-Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed.

Classical ideas and methods developed by Sophus Lie provide us with a powerful tool for constructing exact solutions of partial differential equations (see, e.g., [1–3]). In the present paper, we apply the above methods to obtain new explicit solutions of the $SU(2)$ Yang-Mills equations (YME). YME is the following nonlinear system of twelve second-order partial differential equations:

$$\begin{aligned}&\partial_\nu\partial^\nu\mathbf{A}_\mu-\partial^\mu\partial_\nu\mathbf{A}_\nu+e[(\partial_\nu\mathbf{A}_\nu)\times\mathbf{A}_\mu-\\&\quad-2(\partial_\nu\mathbf{A}_\mu)\times\mathbf{A}_\nu+(\partial^\mu\mathbf{A}_\nu)\times\mathbf{A}^\nu]+e^2\mathbf{A}_\nu\times(\mathbf{A}^\nu\times\mathbf{A}_\mu)=0.\end{aligned}\tag{1}$$

Here $\partial_\nu=\dfrac{\partial}{\partial x_\nu}$, $\mu,\nu=0,1,2,3$; $e=\mathrm{const}$, $\mathbf{A}_\mu=\mathbf{A}_\mu(x)=\mathbf{A}_\mu(x_0,x_1,x_2,x_3)$ are three-component vector-potentials of the Yang-Mills field. Hereafter, the summation over the repeated indices μ,ν from 0 to 3 is supposed. Raising and lowering the vector indices are performed with the aid of the metric tensor $g_{\mu\nu}$, i.e., $\partial^\mu=g_{\mu\nu}\partial_\nu$ ($g_{\mu\nu}=1$ if $\mu=\nu=0$, $g_{\mu\nu}=-1$ if $\mu=\nu=1,2,3$ and $g_{\mu\nu}=0$ if $\mu\neq\nu$).

It should be noted that there are several reviews devoted to classical solutions of YME in the Euclidean space R_4. They have been obtained with the help of *ad hoc* substitutions suggested by Wu and Yang, Rosen, 't Hooft, Carrigan and Fairlie, Wilczek, Witten (for more detail, see review [4] and references cited therein). However, symmetry properties of YME were not used explicity. It is known [5] that YME (1) are invariant under the group $C(1,3)\otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group and $SU(2)$ is the infinite-parameter special unitary group. Symmetry properties of YME have been used for obtaining some new exact solutions of equations (1) by W. Fushchych and W. Shtelen in [6].

The present talk is based mainly on the investigations by the author together with W. Fushchych and R. Zhdanov [7–12].

1. Linear form of ansatzes

The symmetry group of YME (1) contains as a subgroup the conformal group $C(1,3)$ having the following generators:

$$\begin{aligned}
&P_\mu = \partial_\mu,\\
&J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu + A^{a\mu}\partial_{A^a_\nu} - A^{a\nu}\partial_{A^a_\mu},\\
&D = x_\mu\partial_\mu - A^a_\mu\partial_{A^a_\mu},\\
&K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^{a\mu}x_\nu\partial_{A^a_\nu} - 2A^a_\nu x^\nu\partial_{A^a_\mu}.
\end{aligned} \tag{2}$$

Here $\partial_{A^a_\mu} = \dfrac{\partial}{\partial A^a_\mu}$, $a = 1,2,3$.

Using the fact that operators (2) realize a linear representation of the conformal algebra, we suggest a direct method for construction of the invariant ansätze enabling us to avoid a cumbersome procedure of finding a basis of functional invariants of subalgebras of the algebra $AC(1,3)$.

Let $L = \langle X_1,\ldots,X_s\rangle$ be a Lie algebra, where

$$X_a = \xi_{a\mu}(x)\partial_\mu + \rho_{amk}(x)u_k\partial_{u_m}. \tag{3}$$

Here $\xi_{a\mu}(x)$, $\rho_{amk}(x)$ are smooth functions in the Minkowski space $R_{1,3}$, $\mu = 0,1,2,3$, $m,k = 1,2,\ldots,n$. Let also $\mathrm{rank} L = 3$, i.e.,

$$\mathrm{rank}||\xi_{a\mu}(x)|| = \mathrm{rank}||\xi_{a\mu}(x),\rho_{amk}(x)|| = 3 \tag{4}$$

at an arbitrary point $x \in R_{1,3}$.

Lemma [3]. *Assume that conditions (3), (4) hold. Then, a set of functionally independent first integrals of the system of partial differential equations*

$$X_a F(x,u) = 0, \qquad u = (u_1,\ldots,u_n)$$

can be chosen as follows

$$\omega = \omega(x), \qquad \omega_i = h_{ik}(x)u_k, \quad i,k = 1,\ldots,n$$

and, in addition,

$$\det||h_{ik}(x)||^{n}_{i=1}{}^{n}_{k=1} \neq 0.$$

Consequently, we can represent L-invariant ansatzes in the form

$$u_i = h_{ik}(x)v_k(\omega)$$

or

$$\mathbf{u} = \Lambda(x)\mathbf{v}(\omega), \tag{5}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1\\ \vdots\\ u_n\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix},$$

$\Lambda(x))$ is a nonsingular matrix in the space $R_{1,3}$.

Let

$$S_{01} = \begin{pmatrix} 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_{02} = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{03} = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \qquad S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \qquad S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where 0 is zero and I is the unit 3×3 matrices, and E is the unit 12×12 matrix. Now we can represent generators (2) in the form

$$\begin{aligned}
&P_\mu = \partial x_\mu, \\
&J_{\mu\nu} = x^\mu \partial x_\nu - x^\nu \partial x_\mu - (S_{\mu\nu} A \cdot \partial_{\mathbf{A}}), \\
&D = x_\mu \partial x_\mu - k(E\ A \cdot \partial_{\mathbf{A}}), \\
&K_0 = 2x_0 D - (x_\nu x^\nu)\partial x_0 - 2x_a (S_{0a}\ \mathbf{A} \cdot \partial_{\mathbf{A}}), \qquad (6)\\
&K_1 = -2x_1 D - (x_\nu x^\nu)\partial x_1 + 2x_0(S_{01} A \cdot \partial_{\mathbf{A}}) - 2x_2(S_{12} A \cdot \partial_{\mathbf{A}}) - 2x_3(S_{13} A \cdot \partial_{\mathbf{A}}), \\
&K_2 = -2x_2 D - (x_\nu x^\nu)\partial x_2 + 2x_0(S_{02} A \cdot \partial_{\mathbf{A}}) + 2x_1(S_{12} A \cdot \partial_{\mathbf{A}}) - 2x_3(S_{23} A \cdot \partial_{\mathbf{A}}), \\
&K_3 = -2x_3 D - (x_\nu x^\nu)\partial x_3 + 2x_0(S_{03} A \cdot \partial_{\mathbf{A}}) + 2x_1(S_{13} A \cdot \partial_{\mathbf{A}}) + 2x_2(S_{23} A \cdot \partial_{\mathbf{A}}).
\end{aligned}$$

Here, the symbol $(* \cdot *)$ denotes a scalar product,

$$A = \begin{pmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \\ \vdots \\ A_3^2 \\ A_3^3 \end{pmatrix}, \qquad \partial_A = (\partial_{A_0^1}, \partial_{A_0^2}, \ldots, \partial_{A_3^3}).$$

Let L be a subalgebra of the conformal algebra $AC(1,3)$ with the basis elements (2) and rank$L = 3$. According to Lemma, it has twelve invariants

$$f_{ma}(x) A_a, \quad a, m = 1, \ldots, 12,$$

which are functionally independent. They can be considered as components of the vector

$$F \cdot A,$$

where $F = ||f_{mn}(x)||$, $m, n = 1, \ldots, 12$. Furthemore, we suppose that the matrix F is non-singular in some domain of $R_{1,3}$. Providing the rank $L = 3$, there is one additional invariant

ω independent of components of A. According to [1], the ansatz $FA = B(\omega)$ reduces system (1) to a system of ordinary differential equations which contains the independent variable ω, dependent variables $B_0^1, B_0^2, \dots, B_3^3$, and their first and second derivatives. This ansatz can be written in the form (5):

$$A = Q(x)B(x), \qquad Q(x) = F^{-1}(x), \tag{7}$$

where a function ω and a matrix F satisfy the equations

$$X_a\omega = 0, \qquad a = 1, 2, 3,$$
$$X_aF = 0, \qquad a = 1, 2, 3,$$

or

$$\begin{aligned} &\xi_{a\mu}(x)\frac{\partial\,\omega}{\partial\,x_\mu} = 0,\\ &\xi_{a\mu}(x)\frac{\partial\,F}{\partial\,x_\mu} + F\Gamma_a(x) = 0, \qquad a = 1, 2, 3, \quad \mu = 0, 1, 2, 3, \end{aligned} \tag{8}$$

where $\Gamma_a(x)$ are certain 12×12 matrices.

It is not difficult to make that matrices Γ_a have the form (6):

$$\begin{aligned} &P_\mu &&: \ \Gamma_\mu = 0;\\ &J_{\mu\nu} &&: \ \Gamma_{\mu\nu} = -S_{\mu\nu};\\ &D &&: \ \Gamma = -E;\\ &K_0 &&: \ \tilde\Gamma_0 = -2x_0E - 2x_aS_{0a} \quad (a = 1, 2, 3);\\ &K_1 &&: \ \tilde\Gamma_1 = 2x_1E + 2x_0S_{01} - 2x_2S_{12} - 2x_3S_{13};\\ &K_2 &&: \ \tilde\Gamma_2 = 2x_2E + 2x_0S_{02} + 2x_1S_{12} - 2x_3S_{23};\\ &K_3 &&: \ \tilde\Gamma_3 = 2x_3E + 2x_0S_{03} + 2x_1S_{13} + 2x_2S_{23}. \end{aligned}$$

It is natural to look for a matrix F in the form

$$\begin{aligned} F(x) = {} & \exp\{(-\ln\theta)E\}\exp(\theta_0S_{03})\exp(-\theta_3S_{12})\exp(-2\theta_1H_1)\times\\ & \exp(-2\theta_2H_2)\exp(-2\theta_4\tilde H_1)\exp(-2\theta_5\tilde H_2), \end{aligned} \tag{9}$$

where $\theta = \theta(x)$, $\theta_0 = \theta_0(x)$, $\theta_m = \theta_m(x)$ $(m = 1, 2, \dots, 5)$ are arbitrary smooth functions, $H_a = S_{0a} - S_{a3}$, $\tilde H_a = S_{0a} + S_{a3}$ $(a = 1, 2)$.

Generators X_a $(a = 1, 2, 3)$ of a subalgera L can be writen in the next general form:

$$X_a = \xi_{a\mu}(x)\partial_{x_\mu} + Q_aA\partial_A,$$

where

$$Q_a = f_aE + f_{0a}S_{03} + f_{1a}H_1 + f_{2a}H_2 + f_{3a}S_{12} + f_{4a}\tilde H_1 + f_{5a}\tilde H_2,$$

and $F_a = f_a(x)$, $f_{0a} = f_{0a}(x)$, $f_{ma} = f_{ma}(x)$ $(m = 1, \ldots, 5)$ are certain functions. Consequently, the determining system for the matrix F (9) reduces to the system for finding functions θ, θ_0, θ_m $(m = 1, \ldots, 5)$:

$$\begin{aligned}
&\xi_{a\mu}\frac{\partial\theta}{\partial x_\mu} = f_a\theta,\\
&\xi_{a\mu}\frac{\partial\theta_0}{\partial x_\mu} = 4(\theta_4 f_{1a} + \theta_5 f_{2a}) - f_{0a},\\
&\xi_{a\mu}\frac{\partial\theta_3}{\partial x_\mu} = 4(\theta_4 f_{2a} - \theta_5 f_{1a}) + f_{3a},\\
&\xi_{a\mu}\frac{\partial\theta_1}{\partial x_\mu} = 4(\theta_1\theta_4 + \theta_2\theta_5) f_{1a} + 4(\theta_1\theta_5 - \theta_2\theta_4) f_{2a} - \theta_1 f_{0a} - \theta_2 f_{3a} + \frac{1}{2} f_{1a},\\
&\xi_{a\mu}\frac{\partial\theta_2}{\partial x_\mu} = 4(\theta_2\theta_4 - \theta_1\theta_5) f_{1a} + 4(\theta_2\theta_5 + \theta_1\theta_4) f_{2a} - \theta_2 f_{0a} + \theta_1 f_{3a} + \frac{1}{2} f_{2a},\\
&\xi_{a\mu}\frac{\partial\theta_4}{\partial x_\mu} = \theta_4 f_{0a} - 2(\theta_4^2 - \theta_5^2) f_{1a} - 4\theta_4\theta_5 f_{2a} - \theta_5 f_{3a} + \frac{1}{2} f_{4a},\\
&\xi_{a\mu}\frac{\partial\theta_5}{\partial x_\mu} = \theta_5 f_{0a} - 4\theta_4\theta_5 f_{1a} + 2(\theta_4^2 - \theta_5^2) f_{2a} + \theta_4 f_{3a} + \frac{1}{2} f_{5a}.
\end{aligned} \tag{10}$$

Here, $\mu = 0, 1, 2, 3$, $a = 1, 2, 3$.

2. Reduction and exact solutions of YME

Substituting (7), (9) into YME we get a system of ordinary differential equations. However, owing to an asymmetric form of the ansatzes, we have to repeat this procedure 22 times (if we consider Poincaré-invariant ansatzes). For the sake of unification of the reduction procedure, we use the solution generation routine by transformations from the Lorentz group (see, for example, [8]). Then ansatz (7), (8) is represented in a unified way for all the subalgebras. In particular, $P(1,3)$-invariant ansatzes have the following form:

$$\mathbf{A}_\mu(x) = a_{\mu\nu}(x)\mathbf{B}^\nu(\omega), \tag{11}$$

where

$$\begin{aligned}
a_{\mu\nu}(x) = {} & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\
& +2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\
& +(\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - \\
& -(c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu).
\end{aligned} \tag{12}$$

Here, $\mu, \nu = 0, 1, 2, 3$; $x = (x_0, \mathbf{x})$, $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary parameters satisfying the equalities

$$\begin{aligned}
& a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
& a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.
\end{aligned}$$

Theorem. *Ansatzes (11), (12) reduce YME (1) to the system*

$$k_{\mu\gamma}\ddot{\mathbf{B}}^\gamma + l_{\mu\gamma}\dot{\mathbf{B}}^\gamma + m_{\mu\gamma}\mathbf{B}^\gamma + e g_{\mu\nu\gamma}\dot{\mathbf{B}}^\nu \times \mathbf{B}^\gamma + e h_{\mu\nu\gamma}\mathbf{B}^\nu \times \mathbf{B}^\gamma + e^2\mathbf{B}_\gamma \times (\mathbf{B}^\gamma \times \mathbf{B}_\mu) = 0. \tag{13}$$

Coefficients of the reduced equations are given by the following formulae:

$$\begin{aligned} &k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \quad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ &m_{\mu\gamma} = R_{\mu\gamma} - G_\mu \dot{H}_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ &h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}, \end{aligned} \tag{14}$$

where F_1, F_2, G_μ, H_μ, $S_{\mu\nu}$, $R_{\mu\nu}$, $T_{\mu\nu\gamma}$ are functions of ω determined by the relations

$$\begin{aligned} &F_1 = \frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu}, \qquad F_2 = \Box\omega, \qquad G_\mu = a_{\gamma\mu}\frac{\partial\omega}{\partial x_\gamma}, \\ &H_\mu = \frac{\partial a_{\gamma\mu}}{\partial x_\gamma}, \qquad S_{\mu\nu} = a^\gamma_\mu \frac{\partial a_{\gamma\nu}}{\partial x_\delta}\frac{\partial\omega}{\partial x^\delta}, \qquad R_{\mu\nu} = a^\gamma_\mu \Box a_{\gamma\nu}, \\ &T_{\mu\nu\gamma} = a^\delta_\mu \frac{\partial a_{\delta\nu}}{\partial x_\sigma}a_{\sigma\gamma} + a^\delta_\nu \frac{\partial a_{\delta\gamma}}{\partial x_\sigma}a_{\sigma\mu} + a^\delta_\gamma \frac{\partial a_{\delta\mu}}{\partial x_\sigma}a_{\sigma\nu}. \end{aligned}$$

A subalgebraic structure of subalgebras of the conformal algebra $AC(1,3)$ is well known (see, for example, [13]). Here we restrict our considerations to the case of the subalgebra $\langle G_a = J_{0a} - J_{03},\ J_{03}, a = 1,2\rangle$ of the algebra $AP(1,3)$. In this case, $\theta = 1$, $\theta_4 = \theta_5 = 0$ and functions f_a, f_{0a}, f_{ma} $(a, m = 1,2,3)$ have following values:

$$\begin{aligned} G_1 &: \ f_1 = f_{01} = f_{21} = f_{31} = 0, \quad f_{11} = -1; \\ G_2 &: \ f_2 = f_{02} = f_{12} = f_{32} = 0, \quad f_{22} = -1; \\ J_{03} &: \ f_3 = f_{13} = f_{23} = f_{33} = 0, \quad f_{03} = -1; \end{aligned}$$

Consequently, system (10) has the form:

$$\begin{aligned} &G_1^{(1)}\theta_0 = G_1^{(1)}\theta_2 = G_1^{(1)}\theta_3 = 0, \qquad G_1^{(1)}\theta_1 = -\frac{1}{2}; \\ &G_2^{(1)}\theta_0 = G_2^{(1)}\theta_1 = G_2^{(1)}\theta_3 = 0, \qquad G_2^{(1)}\theta_2 = -\frac{1}{2}; \\ &J_{03}^{(1)}\theta_0 = 1, \qquad J_{01}^{(1)}\theta_3 = 0, \qquad J_{03}^{(1)}\theta_a = \theta_a \quad (a = 1,2). \end{aligned}$$

Here,

$$\begin{aligned} &G_a^{(1)} = (x_0 - x_3)\partial_{x_a} + x_a(\partial_{x_0} + \partial_{x_3}) \qquad (a = 1,2), \\ &J_{03}^{(1)} = x_0\partial_{x_3} + x_3\partial_{x_0}. \end{aligned}$$

In particular, the system for the function θ_0 reads

$$(x_0 - x_3)\frac{\partial\theta_0}{\partial x_a} + x_a\left(\frac{\partial\theta_0}{\partial x_0} + \frac{\partial\theta_0}{\partial x_3}\right) = 0 \quad (a = 1,2), \qquad x_0\frac{\partial\theta_0}{\partial x_3} + x_3\frac{\partial\theta_0}{\partial x_0} = 1,$$

and the function $\theta_0 = -\ln|x_0 - x_3|$ is its particular solution. In a similar way, we find that $\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}$, $\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)^{-1}$, $\theta_3 = 0$. The function w is a solution of the system

$$G_a^{(1)}w = J_{03}^{(1)}w = 0 \qquad (a = 1,2),$$

and is equal to $x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Finally, we arrive at ansatz (11), (12), where

$$\theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \frac{1}{2}cx(kx)^{-1},$$
$$\theta_3 = 0, \quad w = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = x_\mu x^\mu$, $dx = d_\mu x^\mu$, $kx = k_\mu x^\mu$, $k_\mu = a_\mu + d_\mu$, $\mu = 0, 1, 2, 3$. According to the theorem, the reduced system (13) has the following coefficients (14):

$$\begin{aligned}
k_{\mu\gamma} &= 4wg_{\mu\gamma} - (a_\mu - d_\mu + k_\mu w)(a_\gamma - d_\gamma + k_\gamma w),\\
l_{\mu\gamma} &= 4[2g_{\mu\gamma} - a_\mu a_\gamma + d_\mu d_\gamma - wk_\mu k_\gamma],\\
m_{\mu\gamma} &= -2k_\mu k_\gamma,\\
g_{\mu\nu\gamma} &= \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w),\\
h_{\mu\nu\gamma} &= \frac{3}{2}\epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma),
\end{aligned} \tag{15}$$

where $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$, $\mu, \nu, \gamma = 0, 1, 2, 3$. We did not succeed in finding general solutions of system (13), (15). Nevertheless, we obtain a particular solution of these equations. The idea of our approach to integration of this system is rather simple and quite natural. It is a reduction of this system by the number of components with the aid of an *ad hoc* substitution. Let

$$\mathbf{B}_\mu = b_\mu \mathbf{e}_1 f(w) + c_\mu \mathbf{e}_2 g(w), \tag{16}$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2(0, 1, 0)$, f and g are arbitrary smooth functions. Then the corresponding equations have the form

$$4w\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \qquad 4w\ddot{g} + 8\dot{g} + e^2 f^2 g = 0. \tag{17}$$

System (17) with the substitution $f = g = u(w)$ reduces to

$$w\ddot{u} + 2\dot{u} + \frac{e^2}{4}u^3 = 0. \tag{18}$$

The ordinary differential equation (18) is the Emden-Fowler equation and the function $u = e^{-1}w^{-\frac{1}{2}}$ is its particular solution.

Substituting the result obtained into formula (16) and then into ansatz (11), (12) we get a non-Abelian exact solution of YME (1):

$$\begin{aligned}
\mathbf{A}_\mu = \{&\mathbf{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \mathbf{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}\times\\
&\times e^{-1}\{(ax)^2 - (bx)^2 - (cx)^2 - (dx)^2)\}^{-\frac{1}{2}}.
\end{aligned}$$

Analogously, we consider the rest of subalgebras of the conformal algebra.

For example, for the subalgebra $\langle J_{12}, P_0, P_3\rangle$, we get the following non-Abelian solutions of YME (1):

$$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu Z_0\left[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)\right] + \mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda,$$
$$\begin{aligned}
\mathbf{A}_\mu = \mathbf{e}_1 k_\mu &\left[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}\right]+\\
&+\mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1}.
\end{aligned}$$

Here $Z_0(w)$ is the Bessel function, $\lambda_1, \lambda_2, \lambda_2 = \text{const}$.

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