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Conformal Invariance of the Maxwell-Minkowski Equations

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Abstract

We study the symmetry of Maxwell's equations for external moving media together with the additional Minkowski constitutive equations (or Maxwell-Minkowski equations). We have established the sufficient condition for a solution found with the help of conditional symmetry operators to be an invariant solution of the considered equation in the classical Lie sense.

In the present paper, we study the symmetry properties of Maxwell's equations in an external moving medium

$$\begin{aligned} &\frac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H} - \vec{j}, \qquad &&\frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\vec{E}, \\ &\mathrm{div}\vec{D} = \rho, &&\mathrm{div}\vec{B} = 0, \end{aligned} \tag{1}$$

together with the additional Minkowski constitutive equations

$$\begin{aligned} &\vec{D} + \vec{u}\times\vec{H} = \varepsilon(\vec{E} + \vec{u}\times\vec{B}), \\ &\vec{B} + \vec{E}\times\vec{u} = \mu(\vec{H} + \vec{D}\times\vec{u}), \end{aligned} \tag{2}$$

where $\vec{u}$ is the velocity of the medium, ε is the permittivity and μ is the permeance of the stationary medium, ρ and $\vec{j}$ are the charge and current densities.

In [1], we have established the infinite symmetry of (1) for $\rho = 0$, $\vec{j} = 0$. In this case, the Maxwell equations admit an infinite-dimensional algebra whose elements have the form

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta_{E_a}\frac{\partial}{\partial E_a} + \eta_{B_a}\frac{\partial}{\partial B_a} + \eta_{D_a}\frac{\partial}{\partial D_a} + \eta_{H_a}\frac{\partial}{\partial H_a}, \tag{3}$$

where

$$\begin{aligned}
&\eta^{E_1} = \xi^3_0 B_2 - \xi^2_0 B_3 - (\xi^1_1 + \xi^0_0)E_1 - \xi^2_1 E_2 - \xi^3_1 E_3, \\
&\eta^{E_2} = -\xi^3_0 B_1 + \xi^1_0 B_3 - (\xi^2_2 + \xi^0_0)E_2 - \xi^1_2 E_1 - \xi^3_2 E_3, \\
&\eta^{E_3} = \xi^2_0 B_1 - \xi^1_0 B_2 - (\xi^3_3 + \xi^0_0)E_3 - \xi^2_3 E_2 - \xi^1_3 E_1, \\
&\eta^{B_1} = \xi^1_2 B_2 - \xi^1_3 B_3 - (\xi^2_2 + \xi^3_3)B_1 - \xi^0_3 E_2 + \xi^0_2 E_3, \\
&\eta^{B_2} = \xi^0_3 E_1 - \xi^0_1 E_3 - (\xi^1_1 + \xi^3_3)B_2 - \xi^2_1 B_1 + \xi^2_3 E_3, \\
&\eta^{B_3} = \xi^3_1 B_1 + \xi^3_2 B_2 - (\xi^1_1 + \xi^2_2)B_3 - \xi^0_2 E_1 + \xi^0_1 E_2, \\
&\eta^{D_1} = \xi^1_2 D_2 + \xi^1_3 D_3 - (\xi^1_1 + \xi^0_0)D_1 + \xi^0_3 H_2 - \xi^0_2 H_3, \\
&\eta^{D_2} = -\xi^0_3 H_1 + \xi^0_1 H_3 - (\xi^2_2 + \xi^0_0)D_2 - \xi^2_1 D_1 + \xi^2_3 D_3,
\end{aligned}$$

$$\eta^{D_3} = \xi^3_1 D_1 + \xi^3_2 D_2 - (\xi^3_3 + \xi^0_0) D_3 - \xi^0_2 H_1 - \xi^0_1 H_2,$$
$$\eta^{H_1} = -\xi^3_0 D_2 + \xi^2_0 D_3 - (\xi^1_1 + \xi^0_0) H_1 - \xi^2_1 H_2 - \xi^3_1 H_3,$$
$$\eta^{H_2} = \xi^3_0 D_1 - \xi^1_0 D_3 - (\xi^2_2 + \xi^0_0) H_2 - \xi^1_2 H_1 - \xi^3_2 H_3,$$
$$\eta^{H_3} = -\xi^2_0 D_1 + \xi^1_0 D_2 - (\xi^3_3 + \xi^0_0) H_3 - \xi^3_2 H_2 - \xi^1_3 H_1,$$

and $\xi^\mu(x)$ are arbitrary smooth functions of $x = (x_0, x_1, x_2, x_3)$, $\xi^\mu_\nu \equiv \dfrac{\partial \xi^\mu}{\partial x_\nu}$, $\mu, \nu = \overline{0,3}$, $a = \overline{1,3}$. We prove that the Maxwell equations (1) with charges and currents ($\rho \neq 0$, $\vec{j} \neq 0$) are invariant with respect to an infinite-parameter group provided that ρ and $\vec{j}$ are transformed in appropriate way.

Theorem 1. *The system of equations (1) is invariant with respect to an infinite-dimensional Lie algebra whose elements are given by*

$$Q = X + \eta^{j^a} \frac{\partial}{\partial j^a} + \eta^\rho \frac{\partial}{\partial \rho}, \tag{4}$$

where

$$\eta^{j^a} = -d j^a + \xi^a_b j^b + \xi^a_0 \rho, \quad \eta^\rho = -d\rho + \xi^0_0 \rho + \xi^0_b j^b, \quad d = -(\xi^0_0 + \xi^1_1 + \xi^2_2 + \xi^3_3) \tag{5}$$

and the summation from 1 *to* 3 *is understood over the index* b.

Proof. The proof of Theorem 1 requires long cumbersome calculations which are omitted here. We use in principle the standard Lie scheme which is reduced to realization of the following algorithm:
Step 1. The prolongating operator (4) is constructed by using the Lie formulae (see, e.g., [4]).
Step 2. Using the invariance condition

$$\underset{1}{Q} L\Psi \Big|_{L\Psi=0} = 0, \tag{6}$$

where $\underset{1}{Q}$ is the first prolongation of operator (4) and $L\Psi = 0$ is the system of equations (1), we obtain the determining equations for the functions ξ^μ, η^{j^a}, and η^ρ.
Step 3. Solving the corresponding determining equations, we obtain the condition of Theorem 1.

From the invariance condition (6) for the equation $\dfrac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H} - \vec{j}$, we obtain $\eta^{j^a} = -dj^a + \xi^a_b j^b + \xi^a_0 \rho$. By applying criterium (6) to the equation $\mathrm{div}\,\vec{D} = \rho$, we get $\eta^\rho = -d\rho + \xi^0_0 \rho + \xi^0_b j^b$. As follows from [1] the invariance condition for the equations $\dfrac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\,\vec{E}$ and $\mathrm{div}\,\vec{B} = 0$ gives no restriction on η^{j^a} and η^ρ. Theorem 1 is proved.

The invariance algebra (4), (5) of the Maxwell equations contains the Galilei algebra $AG(1,3)$, Poincaré algebra $AP(1,3)$, and conformal algebra $AC(1,3)$ as subalgebras.

It is well known that the Maxwell equations in vacuum are invariant with respect to the conformal group [2]. In [1], we showed that there exists the class of conformally invariant constitutive equations of the type

$$\vec{D} = M(I)\vec{E} + N(I)\vec{B}, \quad \vec{H} = M(I)\vec{B} - N(I)\vec{E}, \quad I = \frac{\vec{E}^2 - \vec{B}^2}{\vec{B}\vec{E}}, \tag{7}$$

where M, N are smooth functions of I. System (1), (7) admits the conformal algebra $AC(1,3)$.

It is surprisingly but true that the symmetry of equations for electromagnetic fields in a moving medium was not investigated at all. The following statement gives information on a local symmetry of system (1), (2) which may be naturally called Maxwell's equations with the supplementary Minkowski conditions (or Maxwell-Minkowski equations).

Theorem 2. *System (1), (2) is invariant with respect to the conformal algebra AC(1,3) whose basis operators have the form*

$$
\begin{aligned}
&P_0 = \partial_0 = \frac{\partial}{\partial t}, \qquad P_a = -\partial_a, \qquad \partial_a = \frac{\partial}{\partial_{x_a}}, \quad a, b = \overline{1,3} \\
&J_{ab} = x_a\partial_b - x_b\partial_a + \tilde{S}_{ab} + V_{ab} + R_{ab}, \\
&J_{0a} = x_0\partial_a + x_a\partial_0 + \tilde{S}_{0a} + V_{0a} + R_{0a}, \\
&D = t\partial_t + x_k\partial_{x_k} - 2\left(E_k\partial_{E_k} + B_k\partial_{B_k} + D_k\partial_{D_k} + H_k\partial_{H_k}\right) - 3\left(j_k\partial_{j_k} + \rho\partial_\rho\right), \\
&K_\mu = 2x_\mu D - x^2 P_\mu + 2x^\nu(\tilde{S}_{\mu\nu} + V_{\mu\nu} + R_{\mu\nu}), \quad \mu, \nu = \overline{0,3},
\end{aligned}
\tag{8}
$$

where $\tilde{S}_{ab}, \tilde{S}_{0a}$ *are given by (6), and* $V_{ab}, V_{0a}, R_{ab}, R_{0a}$ *have the form*

$$
\begin{aligned}
&V_{ab} = u_a\partial_{u_b} - u_b\partial_{u_a}, \quad V_{0a} = \partial_{u_a} - u_a\left(u_b\partial_{u_b}\right), \\
&R_{ab} = j_a\partial_{j_b} - j_b\partial_{j_a}, \quad R_{0a} = j_a\partial_\rho + \rho\partial_{j_a}.
\end{aligned}
\tag{9}
$$

To prove the theorem, we use in principle the standard Lie scheme and therefore it is given without proof.

As follows from the theorem, vectors $\vec{D}, \vec{B}, \vec{E}, \vec{H}$ are transformed according to a standard linear representation of the Lorentz group, and the velocity of a moving medium and the density of charge are nonlinearity transformed

$$
u_a \to u'_a = \frac{u_a + \theta_a}{1 + \vec{u}\vec{\theta}}, \qquad \rho \to \rho' = \frac{\rho(1 - \vec{\theta}\vec{u})}{\sqrt{1 - \vec{\theta}^2}}.
$$

Components of the velocity vector $\vec{u}$ are transformed in the following way:

$$
u_k \to u'_k = \frac{u_k\sigma - 2b_0x_k - 2b_0^2x_k(x_0 - \vec{x}\vec{u})}{1 + 2b_0(x_0 - \vec{x}\vec{u}) + b_0^2(x_0^2 + \vec{x}^2 - 2x_0\vec{x}\vec{u})}, \tag{10}
$$

where $\sigma = 1 + 2b_0x_0 + b_0^2x^2$, $x^2 = x_0^2 - \vec{x}^2$, b_0 is a group parameter under the transformations generated by K_μ.

Operators K_a generate the following transformations for the velocity vector:

$$
u_a \to u'_a = \frac{u_a\sigma + 2(x_0 - \vec{x}\vec{u})(b_a - b_a^2x_a) - 2b_au_a(x_a + b_ax^2)}{\sigma + 2b_a^2x_0(x_0 - \vec{x}\vec{u}) - 2b_au_ax_0}, \tag{11}
$$

$$
u_c \to u'_c = \frac{u_c\sigma + 2(x_0 - \vec{x}\vec{u})b_a^2x_c - 2b_au_ax_c}{\sigma + 2b_a^2x_0(x_0 - \vec{x}\vec{u}) - 2b_au_ax_0}, \qquad c \neq a, \tag{12}
$$

where $\sigma = 1 - 2b_ax_a - b_a^2x^2$, b_a are group parameters and there is no summation over a.

If the permittivity ε and permeance μ are functions of the ratio of invariants of electromagnetic field, i.e.,

$$\varepsilon = \varepsilon\left(\frac{\vec{B}^2 - \vec{E}^2}{\vec{B}\vec{E}}\right), \qquad \mu = \mu\left(\frac{\vec{B}^2 - \vec{E}^2}{\vec{B}\vec{E}}\right),$$

then system (1), (2) is invariant with respect to the conformal algebra AC(1,3).

Thus, the system of Maxwell's equations (1), (2) in a moving external medium is invariant with respect to the conformal group C(1,3). Here, the velocity is changed nonlinearly under the transformations generated by K_μ according to formulae (10), (11), (12).

In the all above-given equations, the fields $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ are transformed in a linear way.

Here, we give one more system of nonlinear equations for which a nonlinear representation of the Poincaré algebra AP(1,3) is realized on the set of its solutions. The system has the form

$$\frac{\partial \Sigma_k}{\partial x_0} + \Sigma_l \frac{\partial \Sigma_k}{\partial x_l} = 0, \qquad k, l = 1, 2, 3, \tag{13}$$

where $\Sigma_k = E_k + iH_k$. The complex system (12) is equivalent to the real system of equations for $\vec{E}$ and $\vec{H}$:

$$\begin{aligned} &\frac{\partial E_k}{\partial x_0} + E_l \frac{\partial E_k}{\partial x_l} - H_l \frac{\partial H_k}{\partial x_l} = 0, \\ &\frac{\partial H_k}{\partial x_0} + H_l \frac{\partial E_k}{\partial x_l} + E_l \frac{\partial H_k}{\partial x_l} = 0. \end{aligned} \tag{14}$$

Having used the Lie algorithm [4], we have proved the theorem.

Theorem 3. *The system of equations (14) is invariant with respect to the 24-dimensional Lie algebra with basis operators*

$$\begin{aligned} &P_\mu = \frac{\partial}{\partial_{x_\mu}} = \partial_\mu, \qquad \mu = \overline{0,3} \\ &J^{(1)}_{kl} = x_k\partial_l - x_l\partial_k + E_k\partial_{E_l} - E_l\partial_{E_k} + H_k\partial_{H_l} - H_l\partial_{H_k}, \\ &J^{(2)}_{kl} = x_k\partial_l + x_l\partial_k + E_k\partial_{E_l} + E_l\partial_{E_k} + H_k\partial_{H_l} + H_l\partial_{H_k}, \\ &G^{(1)}_a = x_0\partial_a + \partial_{E_a}, \\ &G^{(2)}_a = x_a\partial_0 - (E_aE_k - H_aH_k)\partial_{E_a} - (E_aH_k + H_aE_k)\partial_{H_k}, \\ &D_0 = x_0\partial_0 - E_k\partial_{E_k} - H_k\partial_{H_k}, \\ &D_a = x_a\partial_a + E_a\partial_{E_a} + H_a\partial_{H_a} \qquad \textit{(there is no summation by k)}, \\ &K_0 = x_0^2\partial_0 + x_0x_k\partial_k + (x_k - x_0E_k)\partial_{E_k} - x_0H_k\partial_{H_k}, \\ &K_a = x_0x_a\partial_0 + x_ax_k\partial_k + [x_kE_a - x_0(E_aE_k - H_aH_k)]\partial_{E_k} + \\ &\qquad [x_kH_a - x_0(H_aE_k + E_aH_k)]\partial_{H_k}. \end{aligned} \tag{15}$$

The invariance algebra of (14) given by (15) contains the Poincaré algebra AP(1,3), the conformal algebra AC(1,3), and the Galilei algebra AG(1,3) as subalgebras.

The operators $J_{0k} = G_k^{(1)} + G_k^{(2)}$ generate the standard transformations for x:

$$
\begin{aligned}
&x_0 \to x'_0 = x_0 \text{ ch } \theta_k + x_0 \text{ sh } \theta_k, \\
&x_k \to x'_k = x_k \text{ ch } \theta_k + x_0 \text{ sh } \theta_k, \\
&x_l \to x'_l = x_l, \qquad \text{if } l \neq k,
\end{aligned}
\tag{16}
$$

and nonlinear transformations for $\vec{E}, \vec{H}$:

$$
\begin{aligned}
&E_k + iH_k \to E'_k + iH'_k = \frac{(E_k + iH_k) \text{ ch } \theta_k + \text{ sh } \theta_k}{(E_k + iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \\
&E_k - iH_k \to E'_k - iH'_k = \frac{(E_k - iH_k) \text{ ch } \theta_k + \text{ sh } \theta_k}{(E_k - iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&E_l + iH_l \to E'_l + iH'_l = \frac{E_l + iH_l}{(E_k + iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \quad l \neq k, \\
&E_l - iH_l \to E'_l - iH'_l = \frac{E_l - iH_l}{(E_k - iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \quad l \neq k.
\end{aligned}
\tag{18}
$$

There is no summation over k in formulas (16), (17), (18).

Conformal invariance can be used to construct exact solutions of Maxwell's equations. In conclusion, we give the theorem determining the relationship betwen invariant and conditionally invariant solutions of differential equations.

Let consider a nonlinear partial differential equation

$$
Lu = 0. \tag{19}
$$

Suppose that (19) is Q-conditionally-invariant under the k-dimensional algebra AQ_k [4, 5, 6] with basis elements $\langle Q_1, Q_2, \ldots, Q_k \rangle$, where

$$
Q_i = \xi_i^a \partial_{x_a} + \eta_i \partial_u,
$$

and the ansatz corresponding to this algebra reduces equation (19) to an ordinary differential equation. A general solution of the reduced equation is called the general conditionally invariant solution of (19) with respect to AQ_k. Then the following theorem has been proved.

Theorem 4. *Let (19) is invariant (in the Lie sense) with respect to the m-dimensional Lie algebra AG_m and Q-conditionally invariant under the k-dimensional Lie algebra AQ_k. Suppose that a general conditionally invariant solution of (19) depends on t constants c_1, c_2, …, c_t.*

If the system

$$
\xi_i^a \frac{\partial u}{\partial x_a} = \eta_i(x, u), \quad i = \overline{1, t}, \tag{20}
$$

is invariant with respect to a p-dimensional subalgebra of AG_m and $p \geq t + 1$, then the conditionally invariant solution of (19) with respect to AQ_k is an invariant solution of this equation in the classical Lie sense.

Thus, we obtain the sufficient condition for the solution found with the help of conditional symmetry operators to be an invariant solution in the classical sense. It is obvious that this theorem can be generalized and applicable to construction of exact solutions of partial differential equations by using the method of differential constraints [7], Lie-Bäcklund symmetry method [8], and the approach suggested in [9].

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