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The Conditional Symmetry and Connection Between the Equations of Mathematical Physics

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Abstract

With help of the conditional symmetry method, the connections between the linear heat equation and nonlinear heat and Burgers ones, between the generalized Harry-Deam equation and Korteweg-de Vries one are obtained. The nonlocal general formulae for solutions of the generalized Harry-Deam equation are constructed.

More than ten years ago, we proposed an idea of condition symmetry. This concept-was worked out under the leadership and with the direct participation of Wilhelm Illich Fushchych (see [1]). It is worth noting that at papers of western scientists a notion of conditional symmetry is persistently identified with nonclassical symmetry. A definition of this symmetry is given at [2]. But, in fact, these notions do not agree. The notion of conditional symmetry is more wider than one of nonclassical symmetry. A great number of our papers illustrate this fact convincingly. The conditional symmetry includes the Lie's symmetry, widening it essentially. After introducting conditional symmetry, there is a necessity to revise symmetry properties of many basic equations of theoretical and mathematical physics from this point of view.

Many articles written by W.I. Fushchych and his pupils are devoted to this problem last ten years. Conditional symmetry is used usually to build exact (conditional invariant) solutions of investigated equations. But it isn't a unique use of conditional symmetry. In this paper, we'll show a way of use conditional symmetry to give connection formulae between differential equations.

Investigating the conditional symmetry of the nonlinear heat equation

$$\frac{\partial U}{\partial t} + \vec{\nabla}(f(U)\vec{\nabla}U) = g(U) \tag{1}$$

or the equivalent equation

$$H(u)\frac{\partial u}{\partial x_0} + \triangle u = F(u), \tag{2}$$

where $u = u(x), x = (x_0, \vec{x}) \in R_{1+n}$, we receive the following result in the case $n = 1$.

Theorem 1 *The equation*

$$H(u)u_0 + u_{11} = F(u), \tag{3}$$

where $u_0 = \dfrac{\partial u}{\partial x_0}$, $u_{11} = \dfrac{\partial^2 u}{\partial x_1^2}$, $u = u(x_0, x_1)$, $H(u)$, $F(u)$ *are arbitrary smooth functions, is Q-conditionally invariant with respect to the operator*

$$Q = A(x_0, x_1, u)\partial_0 + B(x_0, x_1, u)\partial_1 + C(x_0, x_1, u)\partial_u, \tag{4}$$

if the functions A, B, C satisfy the system of differential equations in one from the following cases.
1) $A \neq 0$
(we can assume without restricting the generality that $A = 1$)

$$\begin{aligned} &B_{uu} = 0, \qquad C_{uu} = 2(B_{1u} + HBB_u), \\ &3B_uF = 2(C_{1u} + HB_uC) - (HB_0 + B_{11} + 2HBB_1 + \dot{H}BC), \\ &C\dot{F} - (C_u - 2B_1)\, F = HC_0 + C_{11} + 2HCB_1 + \dot{H}C^2; \end{aligned} \tag{5}$$

2) $A = 0,\ B = 1.$

$$C\dot{F} - C_uF = HC_0 + C_{11} + 2CC_{1u} + C^2C_{uu} + \frac{C\dot{H}}{H}\,(F - CC_u + C_1)\,. \tag{6}$$

A subscript means differentiation with respect to the corresponding argument.

If in formula (6), as the particular case , we take

$$H(u) \equiv 1, \qquad F(u) \equiv 0, \qquad C(x_0, x_1, u) = w(x_0, x_1)u, \tag{7}$$

then operator (4) has the form

$$Q = \partial_1 + w(x_0, x_1)u\partial_u, \tag{8}$$

and the function w is a solution of the Burgers equation

$$w_0 + 2ww_1 + w_{11} = 0. \tag{9}$$

With our assumptions (8), equation (3) become the linear heat equation

$$u_0 + u_{11} = 0. \tag{10}$$

We find the connection between equations (9) and (10) from the condition

$$Qu = 0. \tag{11}$$

In this case, equation (11) has the form

$$u_1 - wu = 0, \tag{12}$$

or

$$w = \partial_1(\ln u). \tag{13}$$

Thus, we receive the well-known Cole-Hopf substitution, which reduces the Burgers equation to the linear heat equation.

Let us consider other example. G. Rosen in 1969 and Blumen in 1970 showed that the nonlinear heat equation

$$w_t + \partial_x(w^{-2}w_x) = 0, \qquad w = w(t, x), \tag{14}$$

is reduced to the linear equation (10) by the substitutions

$$\begin{aligned} &1.\quad t = t, \quad x = x, \quad w = v_x; \\ &2.\quad t = x_0, \quad x = u, \quad v = x_1. \end{aligned} \tag{15}$$

Connection (15) between equations (14) and (10) is obtained from condition (11) if we take in (6)

$$H(u) \equiv 1, \quad F(u) \equiv 0, \quad C(x_0, x_1, u) = \frac{1}{w(x_0, u)}. \tag{16}$$

Equation (6) will have the form (14), and (11) will be written in the form

$$u_1 = \frac{1}{w(x_0, u)}, \tag{17}$$

that equivalent to (15).

Provided that

$$H(u) = 1, \quad F(u) = \lambda u \ln u, \quad \lambda = \text{const}, \tag{18}$$

formula (13) connects the equations

$$u_0 + u_{11} = \lambda u \ln u \quad \text{and} \quad w_0 + 2ww_1 + w_{11} = \lambda w. \tag{19}$$

When

$$H(u) = \frac{1}{f(u)}, \quad F(u) \equiv 0, \quad C(x_0, x_1, u) = \frac{1}{w(x_0, u)}, \tag{20}$$

then substitutions (15) set the connection between the equations

$$u_0 + f(u)u_{11} = 0 \quad \text{and} \quad w_t + \partial_x[f(x)w^{-2}w_x)] = 0. \tag{21}$$

It should be observed that equations (19) and (21) are widely used to decribe real physical processes.

If we assume

$$C \equiv 0, \quad B = -\frac{f_1}{f}, \quad H(u) \equiv -1, \quad F(u) \equiv 0, \tag{22}$$

in formulae (5), where $f = f(x)$ is an arbitrary solution of the equation

$$f_0 = f_{11}, \tag{23}$$

then we take the operator

$$Q = \partial_0 - \frac{f_1}{f}\partial_1.$$

From condition (11), we receive the connection between two solutions of the linear equation (23)

$$fu_0 - f_1u_1 = 0. \tag{24}$$

The characteristic equation

$$f_1dx_0 + fdx_1 = 0 \tag{25}$$

corresponds to equation (24).

Theorem 2 *If a function f is a solution of equation (23), and a function $u(x)$ is a common integral of the ordinary differential equation (25), then $u(x)$ is a solution of equation (23).*

Theorem 2 sets a generation algorithm for solutions of equation (23). Thus starting from the "old" solution $u \equiv 1$, we obtain a whole chain of solutions of equation (23):

$$1 \longrightarrow x_1 \longrightarrow x_0 + \frac{x_1^2}{2!} \longrightarrow x_0 x_1 + \frac{x_1^3}{3!} \longrightarrow \cdots$$

Prolongating this process, we find next solutions of equation (23):

$$\frac{x_1^{2m}}{(2m)!} + \frac{x_0}{1!}\frac{x_1^{2m-2}}{(2m-2)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{2!} + \frac{x_0^m}{m!},$$
$$\frac{x_1^{2m+1}}{(2m+1)!} + \frac{x_0}{1!}\frac{x_1^{2m-1}}{(2m-1)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^3}{3!} + \frac{x_0^m}{m!}\frac{x_1}{1!},$$

where $m = 0, 1, 2, \cdots$

In conclusion, we give still one result, which sets the connection between the generalized Harry-Dym and Korteweg-de Vries equations.

Let us generate the Korteweg-de Vries equation

$$u_0 + \lambda u u_1 + u_{111} = 0, \qquad u = u(x_0, x_1), \tag{26}$$

and the Harry-Dym equation

$$w_t + \partial_{xx}(w^{-3/2} w_x) = 0, \qquad w = w(t, x), \tag{27}$$

by following equations

$$u_0 + f(u) u_1 + u_{111} = 0, \tag{28}$$

and

$$w_t + \partial_{xx}(F(w) w_x) = G(x, w). \tag{29}$$

Theorem 3 *The generalized Korteweg-de Vries equation (28) is Q-conditionally invariant with respect to the operator*

$$Q = \partial_1 + C(x_0, x_1, u)\partial_u, \tag{30}$$

if the function $C(x_0, x_1, u)$ is a solution of the equation

$$C_0 + C_{111} + 3CC_{11u} + 3(C_1 + CC_u)(C_{1u} + CC_{uu}) + 3C^2 C_{1uu} + C^3 C_{uuu} + C_1 f + C^2 \dot{f} = 0. \tag{31}$$

If we assume in (31) that

$$C = \frac{1}{w(x_0, u)},$$

then we obtain

$$w_0 + \partial_{uu}(w^{-3} w_u) = \dot{f}(u). \tag{32}$$

Thus, formula (17) sets the connection between the generalized Korteweg-de Vries (28) equation and Harry-Dym equation

$$w_t + \partial_{xx}(w^{-3}w_x) = \dot{f}(x). \tag{33}$$

In particular at $f(u) = \lambda u$, we obtain the connection between the following equations:

$$u_0 + \lambda u u_1 + u_{111} = 0 \quad \text{and} \quad w_t + \partial_{xx}(w^3 w_x) = \lambda, \tag{34}$$

where λ is an arbitrary constant.

So, we have shown that it is possible to obtain nonlocal connection formulae between some differential equations, using the operators of conditional symmetry. It is, for example, the well-known connection between the Burgers and linear heat equations, that is realized by the Cole-Hopf substitution. We have obtained also that there is the nonlocal connection between the generalized Harry-Dym and the Korteweg-de Vries equations.

References

[1] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

[2] Bluman G.W. and Cole I.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, N 1, 1035–1047.