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On a Mixed Problem for a System of Differential Equations of the Hyperbolic Type with Rotation Points

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Abstract

We propose two approaches to find asymptotic solutions to systems of linear differential equations of the hyperbolic type with slowly varying coefficients in the presence of rotation points.

In [1, 2], an asymptotic solution to a system of linear differential equations of the hyperbolic type with slowly varying coefficients is constructed for the case of constant multiplicity of the spectrum in the whole integration interval. In this paper, we propose two approaches to find asymptotic solutions to systems of such a type in the presence of rotation points.

Let us consider a differential equation of the form

$$\varepsilon^h \frac{\partial^2 u(t;x)}{\partial t^2} = A_1(t;\varepsilon)\frac{\partial^2 u}{\partial x^2} + \varepsilon A_2(t;x;\varepsilon)u(t;x) + \varepsilon A_3(t;x;\varepsilon)\frac{\partial u}{\partial t} \tag{1}$$

with the initial condititons

$$u(0;x) = \varphi_1(x), \qquad \frac{\partial u}{\partial t}(0;x) = \varphi_2(x),$$

and the boundary conditions

$$u(t;0) = u(t;l) = 0,$$

where $0 \le t \le L$, $0 \le x \le l$, $\varepsilon(0 < \varepsilon \le \varepsilon_0)$ is a small real parameter; $u(t;x)$, $\varphi_1(x)$, $\varphi_2(x)$ are n-dimensional vectors; $A_k(t;x;\varepsilon)$, $k = \overline{1,3}$ are matrices of order $n \times n$, $n \in N$.

Let the following conditions be valid:

1) coefficients of system (1) admit expansions in powers of ε

$$A_1(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(t), \qquad A_j(t;x;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_j^{(s)}(t;x), \quad j = 2,3;$$

2) matrices $A_s(t)$, $A_j^{(s)}(t;x)$ are infinitely differentiable with respect to $t \in [0;L]$ and continuous in $x \in [0;l]$ together with derivatives up to the second order inclusively; functions $\varphi_1(x)$, $\varphi_2(x)$ are twice continuously differentiable;

3) in the interval $[0; L]$ for arbitrary $r, s = 0, 1, \ldots$, the series

$$\sum_{k=1}^{\infty} \left\| \frac{d^r A_{mk}^{(s)}(t)}{dt^r} \right\|^2, \quad \sum_{k=1}^{\infty} \left\| \frac{d^r C_{mk}^{(s)}(t)}{dt^r} \right\|^2,$$

converge uniformly, where

$$A_{mk}^{(s)}(t) = \frac{2}{l} \int_0^l A_2^{(s)}(t; x) \sin \omega_k x \sin \omega_m x dx;$$
$$C_{mk}^{(s)}(t) = \frac{2}{l} \int_0^l A_3^{(s)}(t; x) \sin \omega_k x \sin \omega_m x dx.$$

We shall look for a solution of problem (1) in the form

$$u(t; x; \varepsilon) = \sum_{k=1}^{\infty} v_k(x) z_k(t; \varepsilon), \tag{1a}$$

where $v_k(x)$ is the orthonormal system

$$v_k(x) = \sqrt{\frac{2}{l}} \sin \omega_k x, \quad \omega_k = \frac{k\pi}{l}, \quad k = 1, 2 \ldots,$$

and $z_k(t; \varepsilon)$ are n-dimensional vectors that are defined from the denumerable system of differential equations

$$\begin{aligned} \varepsilon^h \frac{d^2 z_k(t; \varepsilon)}{dt^2} &= -\omega_k^2 \sum_{s=0}^{\infty} \varepsilon^s A_1^{(s)}(t) z_k(t; \varepsilon) + \\ &+ \varepsilon \sum_{m=1}^{\infty} \left(\sum_{s=0}^{\infty} \varepsilon^s A_{km}^{(s)}(t) z_m(t; \varepsilon) + \sum_{s=0}^{\infty} \varepsilon^s C_{km}^{(s)}(t) \frac{dz_m(t; \varepsilon)}{dt} \right), \end{aligned} \tag{2}$$

with the initial conditions

$$z_k(0; \varepsilon) = \varphi_1, \qquad \frac{dz_k(0; \varepsilon)}{dt} = \varphi_2,$$

where

$$\varphi_1 = \sqrt{\frac{2}{l}} \int_0^l \varphi_1(x) \sin \omega_k x dx, \quad \varphi_2 = \sqrt{\frac{2}{l}} \int_0^l \varphi_2(x) \sin \omega_k x dx.$$

Putting in (2) $z_k(t; \varepsilon) = q_{1k}(t \ \varepsilon)$, $\dfrac{dz_k(t; \varepsilon)}{dt} = q_{2k}(t; \varepsilon)$, we get the following system of the first order

$$\varepsilon^h \frac{dq_k(t; \varepsilon)}{dt} = H_k(t; \varepsilon) q_k(t; \varepsilon) + \varepsilon \sum_{m=1}^{\infty} H_{km}(t; \varepsilon) q_k(t; \varepsilon), \tag{3}$$

with the initial condition $q_k(0;\varepsilon) = x_{k0}$, where $q_k(t;\varepsilon)$, x_{k0} are $2n$-dimensional vectors, $H_{km}(t;\varepsilon)$, $H_k(t;\varepsilon)$ are square $(2n \times 2n)$-matrices of the form

$$q_k(t;\varepsilon) = \begin{Vmatrix} q_{1k}(t;\varepsilon) \\ q_{2k}(t;\varepsilon) \end{Vmatrix},$$

$$H_{km}(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_{km}^{(s)}(t) = \begin{Vmatrix} 0 & 0 \\ \sum\limits_{s=0}^{\infty} \varepsilon^s A_{km}^{(s)}(t) & \sum\limits_{s=0}^{\infty} \varepsilon^s C_{km}^{(s)}(t) \end{Vmatrix},$$

$$H_k(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_k^{(s)}(t) = \begin{Vmatrix} 0 & E \\ -\sum\limits_{s=0}^{\infty} \omega_k^2 \varepsilon^s A_s(t) & 0 \end{Vmatrix}.$$

Let roots of the characteristic equation

$$\det \left\| H_k^{(0)}(t) - \lambda_k^{(i)}(t)E \right\| = 0, \quad k = 1, 2 \ldots, \quad i = \overline{1, 2n},$$

for the previous system coincide at the point $t = 0$ and be different for $t \in (0; L]$ (i.e., $t = 0$ is a rotation point of system (3)). The following theorem tells us about the form of a formal solution of system (3).

Theorem 1. *If conditions 1)–3) are valid and roots of the equation*

$$det \left\| H_k^{(0)}(t) + \varepsilon H_k^{(1)}(t) - \lambda_k(t;\varepsilon)E \right\| = 0$$

are simple $\forall\, t \in [0; L]$, *then system (3) has the formal matrix solution*

$$Q_k(t;\varepsilon) = U_k(t;\varepsilon) \exp\left(\frac{1}{\varepsilon^h} \int_0^t \Lambda_k(t;\varepsilon) dt \right), \tag{4}$$

where $U_k(t;\varepsilon)$, $\Lambda_k(t;\varepsilon)$ *are square matrices of order* $2n$, *that are presented as formal series*

$$U_k(t;\varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r U_k^{(r)}(t), \qquad \Lambda_k(t;\varepsilon) = \sum_{r=0}^{\infty} \Lambda_k^{(r)}(t). \tag{5}$$

Proof. Hawing substituted (4), (5) into (3), we get the following system of matrix equations

$$H_k^{(1)}(t;\varepsilon) U_k^{(0)}(t;\varepsilon) - U_k^{(0)}(t;\varepsilon) \Lambda_k^{(0)}(t;\varepsilon) = 0, \tag{6}$$

$$H_k^{(1)}(t;\varepsilon) U_k^{(s)}(t;\varepsilon) - U_k^{(s)}(t;\varepsilon) \Lambda_k^{(0)}(t;\varepsilon) = -U_k^{(0)}(t;\varepsilon) \Lambda_k^{(s)}(t;\varepsilon) + B_k^{(s)}(t;\varepsilon), \tag{7}$$

where

$$B_k^{(s)}(t;\varepsilon) = \sum_{r=1}^{s-1} U_k^{(r)}(t;\varepsilon) \Lambda_k^{(s-r)}(t;\varepsilon) + \frac{\partial U_k^{(s-h)}(t;\varepsilon)}{\partial t} - \\ - \sum_{r=2}^{s} H_k^{(r)}(t;\varepsilon) U_k^{(s-r)}(t;\varepsilon) - \sum_{r=1}^{s-1} H_{km}^{(r)}(t;\varepsilon) U_k^{(s-r)}(t;\varepsilon),$$

$$H_k^{(1)}(t;\varepsilon) = H_k^{(0)}(t) + \varepsilon H_k^{(1)}(t).$$

It follows from the conditions of the theorem that there exists a nonsingular matrix $V_k(t;\varepsilon)$ such that

$$H_k^{(1)}(t;\varepsilon)V_k(t;\varepsilon) = V_k(t;\varepsilon)\Lambda_k(t;\varepsilon),$$

where

$$\Lambda_k(t;\varepsilon) = \operatorname{diag}\left\{\lambda_{1k}(t;\varepsilon),\ldots,\lambda_{\nu k}(t;\varepsilon)\right\}, \quad \nu = 2n.$$

Multiply (6) and (7) from the left by the matrix $V_k^{-1}(t;\varepsilon)$ and introduce the notations

$$P_k^{(s)}(t;\varepsilon) = V_k^{-1}(t;\varepsilon)U_k^{(s)}(t;\varepsilon), \quad F_k^{(s)}(t;\varepsilon) = V_k^{-1}(t;\varepsilon)B_k^{(s)}(t;\varepsilon).$$

Finally, we obtain the system

$$\begin{aligned} &\Lambda_k(t;\varepsilon)P_k^{(0)}(t;\varepsilon) - P_k^{(0)}(t;\varepsilon)\Lambda_k^{(0)}(t;\varepsilon) = 0, \\ &\Lambda_k(t;\varepsilon)P_k^{(s)}(t;\varepsilon) - P_k^{(s)}(t;\varepsilon)\Lambda_k^{(0)}(t;\varepsilon) = P_k^{(0)}(t;\varepsilon)\Lambda_k^{(s)}(t;\varepsilon) + F_k^{(s)}(t;\varepsilon). \end{aligned} \tag{8}$$

Let us put here $P_k^{(0)}(t;\varepsilon) = E$. Then $\Lambda_k^{(0)}(t;\varepsilon) = \Lambda_k(t;\varepsilon)$. From (8), we determine $\Lambda_k^{(s)}(t;\varepsilon) = -F_k^{(s0)}(t;\varepsilon)$, where $F_k^{(s0)}(t;\varepsilon)$ is a diagonal matrix that consists of diagonal elements of the matrix $F_k^{(s)}(t;\varepsilon)$. Elements $P_k^{(s)}(t;\varepsilon)$ that are not situated on the main diagonal, are determined by formulas

$$\left\{P_k^{(s)}(t;\varepsilon)\right\}_{ij} = \frac{\{F_k^{(s)}(t;\varepsilon)\}_{ij}}{\lambda_{ik}(t;\varepsilon) - \lambda_{jk}(t;\varepsilon)}, \quad i \neq j, \ i,j = \overline{1,2n},$$

and diagonal elements $\left\{P_k^{(s)}(t;\varepsilon)\right\}_{ii} = 0$. The theorem is proved.

In investigating formal solutions, it has been shown that the following asymptotic equalities ave valid for $t \in [0; L\varepsilon]$:

$$P_k^{(s)}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^{\alpha_1}}\right), \quad \Lambda_k^{(s)}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^{\alpha_2}}\right),$$

where α_1, α_2 are positive numbers, and $t \in (L\varepsilon; L]$, then $P_k^{(s)}(t;\varepsilon)$ and $\Lambda_k^{(s)}(t;\varepsilon)$ are bounded for $\varepsilon \to 0$. Let us consider the character of formal solutions in the sense [3]. Let us write down the p-th approximation

$$q_{kp}(t;\varepsilon) = Q_{kp}(t;\varepsilon)a_k(\varepsilon),$$

where

$$Q_{kp}(t;\varepsilon) = U_{kp}(t;\varepsilon)\exp\left(\frac{1}{\varepsilon^h}\int_0^t \Lambda_{kp}(t;\varepsilon)dt\right),$$

$$U_{kp}(t;\varepsilon) = \sum_{m=1}^{p} \varepsilon^m U_k^{(m)}(t;\varepsilon), \quad \Lambda_{kp}(t;\varepsilon) = \sum_{m=1}^{p} \varepsilon^r \Lambda_k^{(m)}(t;\varepsilon),$$

$a_k(\varepsilon)$ is an arbitrary constant vector. Having substituted the p-th approximation of solutions (4) in the differential operator

$$Mq_k \equiv \varepsilon^h \frac{dq_k}{dt} - H_k(t;\varepsilon)q_k(t;\varepsilon) + \varepsilon \sum_{m=1}^{\infty} H_{km}(t;\varepsilon)q_k(t;\varepsilon),$$

and taking into account (6), (7), we obtain

$$Mq_{kp}(t;\varepsilon) = O\left(\varepsilon^{p+1}\right) Y_{kp}(t;\varepsilon) \exp\left(\frac{1}{\varepsilon^h}\int_0^t \Lambda_{kp}(t;\varepsilon)dt\right),$$

where

$$Y_{kp}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^\beta}\right), \quad \beta > 0 \quad \text{for } t \in [0;L\varepsilon],$$
$$Y_{kp}(t;\varepsilon) = O(1), \quad \text{for } t \in [L\varepsilon;L],$$

Let, in addition, the following condititons are valid:

4) $\operatorname{Re}\lambda_{ik}(t;\varepsilon) < 0\ \forall\, t \in [0;L]$;

5) $\operatorname{Re}\Lambda_k^{(p)}(t;\varepsilon) < 0$; for $h = 1$, $\forall\, t \in [0;L\varepsilon]$.

Then there exists ε_{k1} $(0 < \varepsilon_{k1} < \varepsilon_0)$ such that, for all $t \in [0;L]$, the asymptotic equality $Mq_{kp}(t;\varepsilon) = O(\varepsilon^p)$ is fulfilled. The following theorem is true.

Theorem 2. *If the conditions of Theorem 1 and conditions 4), 5) are fulfilled, and for* $t = 0, q_{kp}(0;\varepsilon) = q_k(0;\varepsilon)$, *where* $q_k(t;\varepsilon)$ *are exact solutions of system (3), then, for every* $L_k > 0$, *there exist constants* $C_k > 0$ *not depending on* ε *and such that, for all* $t \in [0;L]$ *and* $\varepsilon \in (0;\varepsilon_{k1}]$, *the following inequalities*

$$\|q_{kp} - q_k\| < C_k \varepsilon^{p+1-h-\frac{1}{2n}}$$

are fulfilled.

Proof. Vector functions $y_k(t;\varepsilon) = q_k(t;\varepsilon) - q_{kp}(t;\varepsilon)$ are solutions to the equations

$$\varepsilon^h \frac{dy_k}{dt} = H_k(t;\varepsilon)y_k + O\left(\varepsilon^{p+1}\right) + \sum_{m=1}^{\infty} H_{km}y_k.$$

With the help of the transformation

$$y_k(t;\varepsilon) = V_k(t;\varepsilon)z_k(t;\varepsilon),$$

we reduce the latter system to the form

$$\varepsilon^h \frac{dz_k}{dt} = \left(\Lambda_k(t;\varepsilon) + \varepsilon B_{1k}(t;\varepsilon)\right) z(t;\varepsilon) + O\left(\varepsilon^{p+1-\frac{1}{2n}}\right). \tag{9}$$

Let us replace system (9) by the equivalent system of integral equations

$$z_k(t;\varepsilon) = \int_0^t \exp\left(\frac{1}{\varepsilon^h}\int_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\left(B_k(t;\varepsilon)z_k(t;\varepsilon) + O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right).$$

Let us bound $\|z_k(t;\varepsilon)\|$:

$$\|z_k(t;\varepsilon)\| \le \int\limits_0^t \left\|\exp\left(\frac{1}{\varepsilon^h}\int\limits_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\right\| \left(\|B_k(t;\varepsilon)\|\,\|z_k(t;\varepsilon)\| + \left\|O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right\|\right). \quad (10)$$

Since

$$\left\|\exp\left(\frac{1}{\varepsilon^h}\int\limits_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\right\| \le 1, \quad \forall\, t \in [0;L],$$

$$\|B_k(t;\varepsilon)\| \le \frac{C_{1k}}{\varepsilon^\alpha}, \quad \alpha > 0,\ \forall\, t \in [0;L\varepsilon];$$

$$\left\|O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right\| \le C_{2k}\varepsilon^{p+1-h-\frac{1}{2n}},$$

we have

$$\|z_k(t;\varepsilon)\| \le C_{3k}\int\limits_0^t \|z_k(t_1;\varepsilon)\|dt_1 + C_{2k}L\varepsilon^{p+1-h-\frac{1}{2n}},$$

where $C_{3k} = C_{1k}/\varepsilon^\alpha$.

Using the Gronwall-Bellman lemma, we get the inequality

$$\|z_k(t;\varepsilon)\| \le C_k\varepsilon^{p+1-h-\frac{1}{2n}}.$$

Then

$$\|y_k(t;\varepsilon)\| = \|q_k - q_{kp}\| \le \|V_k(t;\varepsilon)\|\cdot\|z_k\| \le C_k\varepsilon^{p+1-h-\frac{1}{2n}}.$$

The theorem is proved.

The other approach to constructing an asymptotic solution is based on the "joining" of solutions in a neighbourhood of a rotation point with solutions that are constructed outside this neighbourhood. For this purpose, we suppose that the following conditions are fulfilled:

4) the equation $\det\|H_k^{(0)}(0) - \lambda_k(t)E\| = 0$ has a multiple root with an elementary divisor;

5) an matrix element

$$\left\{T_k^{-1}\left(\frac{dH_k^{(0)}(t)}{dt}\right)_{t=0}\frac{t}{\varepsilon} + P_k^{(1)}(0)T_k\right\}_{n1}$$

differs from zero for all $t \in [0;L\varepsilon]$, where T_k is a transformation matrix of the matrix $H_k^{(0)}(0)$;

6) matrices $H_k^{(r)}(t)$ and $H_{km}^{(r)}(t)$ are expandable in the interval $t \in [0;L\varepsilon]$ into convergent Taylor series

$$\begin{aligned} H_k^{(r)}(t) &= \sum_{s=0}^{\infty}\frac{1}{s!}\frac{d^sH_k^{(r)}(t)}{dt^s}\Big|_{t=0}t^s, \quad r = 0,1,\ldots \\ H_{km}^{(r)}(t) &= \sum_{s=0}^{\infty}\frac{1}{s!}\frac{d^sH_{km}^{(r)}(t)}{dt^s}t^s. \end{aligned} \quad (11)$$

To construct an expansion of a solution in the interval $[0; L\varepsilon]$, let us introduce a new variable $t_1 = \dfrac{t}{\varepsilon}$. Let us pass to this variable in system (3). Having grouped together coefficients of the same powers of ε on the right-hand side, we get

$$\varepsilon^{h-1}\frac{dq_k}{dt_1} = F_k(t_1;\varepsilon)q_k(t;\varepsilon) + \varepsilon\sum_{m=1}^{\infty} F_{km}(t_1;\varepsilon)q_k(t;\varepsilon), \tag{12}$$

where

$$F_k(t_1;\varepsilon) = \sum_{r=0}^{\infty} F_k(t_1)\varepsilon^r, \quad F_{km}(t_1;\varepsilon) = \sum_{r=0}^{\infty} F_{km}(t_1)\varepsilon^r,$$
$$F_k(t_1) = \sum_{s=0}^{\infty}\frac{1}{s!}\frac{d^s H_k^{(0;r-s)}(t)}{dt^s}t_1^s, \quad F_{km}(t_1) = \sum_{s=0}^{r}\frac{1}{s!}\frac{d^s H_{km}^{(0;r-s)}}{dt^s}t_1^s.$$

Roots of the characteristic equation for system (12) satisfy condition 4), therefore according to [2], we can look for a solution of equation (12) for $t \in [0; L\varepsilon]$ in the form

$$x_k^{(i)}(t;\varepsilon) = u_k^{(i)}\left(\frac{t}{\varepsilon};\mu\right)\exp\left(\frac{1}{\varepsilon^{h-1}}\int_0^{\frac{t}{\varepsilon}}\lambda_k^{(i)}(t;\mu)dt\right), \tag{13}$$

where a $2n$-dimensional vector $u_k^{(i)}(t_1;\mu)$ and the function $\lambda_k^{(i)}(t_1;\mu)$ admit expansions

$$u_k^{(i)}(t_1;\mu) = \sum_{r=0}^{\infty}\mu^r u_{kr}^{(i)}(t_1), \quad \lambda_k^{(i)}(t_1;\mu) = \sum_{r=0}^{\infty}\mu^r\lambda_{kr}^{(i)}(t_1), \quad \mu = \sqrt[2n]{\varepsilon}.$$

In the interval $[L\varepsilon; L]$, roots $\lambda_{ik}(t)$, $i = \overline{1, 2n}$, of the characteristic equation for system (3) are simple. Then, in this interval $2n$, independent formal solutions to system (3) are constructed in the form

$$y_k^{(i)}(t;\varepsilon) = v_k^{(i)}(t;\varepsilon)\exp\left(\frac{1}{\varepsilon^h}\int_0^t \xi_k^{(i)}(t;\varepsilon)dt\right), \tag{14}$$

where $v_k^{(i)}(t;\varepsilon)$ is an n-dimensional vector and $\xi_k^{(i)}(t;\varepsilon)$ is a scalar function which admit the expansions

$$v_k^{(i)}(t;\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r v_{kr}^{(i)}(t), \quad \xi_k^{(i)}(t;\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r\xi_{kr}^{(i)}(t).$$

The functions $u_{kr}^{(i)}(t_1)$, $\lambda_{kr}^{(i)}(t_1)$, $v_{kr}^{(i)}(t)$, $\xi_{kr}^{(i)}(t)$ are determined by the method from [2].

Denote by $x_{kp}^{(i)}(t;\varepsilon)$, $y_{kp}^{(i)}(t;\varepsilon)$ p-th approximations of solutions (13), (14), that are formed by cutting off the corresponding expansions at the p-th place. The p-th approximation of a general solution for $t \in [0; L\varepsilon]$ is of the form

$$\overline{x}_{kp}(t;\varepsilon) = \sum_{i=1}^{2n} x_{kp}^{(i)}(t;\varepsilon)a_{ki}(\varepsilon),$$

and, in the interval $[L\varepsilon; L]$:

$$\overline{y}_{kp}(t;\varepsilon) = \sum_{i=1}^{2n} y_{kp}^{(i)}(t;\varepsilon) b_{ki}(\varepsilon),$$

where $a_{ki}(\varepsilon)$, $b_{ki}(\varepsilon)$ are arbitrary numbers. We choose numbers $a_{ki}(\varepsilon)$ from the initial condition for system (3), that is equivalent to the relation $\overline{x}_{kp}(0;\varepsilon) = x_{0k}$. Let us "join" the constructed p-th approximations $\overline{x}_{kp}(t;\varepsilon)$, $\overline{y}_{kp}(t;\varepsilon)$ at the point $t = L\varepsilon$. We can do this by choosing numbers $b_{ki}(\varepsilon)$ in $\overline{y}_{kp}(L\varepsilon;\varepsilon)$ so that the equality

$$\overline{x}_{kp}(L\varepsilon;\varepsilon) = \overline{y}_{kp}(L\varepsilon;\varepsilon). \tag{15}$$

is fulfilled. Therefore, Theorem 2 is proved.

Theorem 3. *If conditions 1)–5) and relation (15) are fulfilled, then the Cauchy problem for system (3) has the p-th approximation of a solution of the form*

$$q_{kp}(t;\varepsilon) = \begin{cases} \overline{x}_{kp}(t;\varepsilon) & \text{for } 0 \le t \le L\varepsilon; \\ \overline{y}_{kp}(t;\varepsilon) & \text{for } L\varepsilon \le t \le L. \end{cases}$$

So, the theorem on the asymptotic character of formal solutions is proved.

Theorem 4. *If the conditions of Theorem 3 are fulfilled, then the following asymptotic bounds are valid:*

$$\|q_{kp}(t;\varepsilon) - q_k(t;\varepsilon)\| \le C \cdot \mu^{p+3-2n-h} \sup_{t\in[0;L\varepsilon]} \exp\left(\varepsilon^{1-h} \int_0^{\frac{t}{\varepsilon}} \sum_{m=0}^{2n(h-1)} \mu^k \, Re\, \lambda_{km}^{(i)}(t) dt\right)$$

for $t \in [0; L\varepsilon]$,

$$\|q_{kp}(t;\varepsilon) - q_k(t;\varepsilon)\| \le C \cdot \varepsilon^{p+1-h} \sup_{t\in[L\varepsilon;L]} \exp\left(\varepsilon^{-h} \int_{L\varepsilon}^{t} \sum_{m=0}^{h-1} \varepsilon^k \xi_{km}^{(i)}(t) dt\right)$$

for $t \in [L\varepsilon; L]$.

So, we get asymptotics of equation (1) for the case, when the rotation point is some inner point $t = L$ of the interval $[0; L]$ and also for two rotation points.

Now let us consider an inhomogeneous system of the hyperbolic type

$$\varepsilon^h \frac{\partial^2 u(t;x)}{\partial t^2} = A_1(t;\varepsilon) \frac{\partial^2 u}{\partial x^2} + g(t;x,\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right) \tag{16}$$

where

$$g(t;x;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s g_s(t;x).$$

With the help of transformation (1a), system (16) takes the form

$$\varepsilon^h \frac{dq_k}{dt} = H_k(t;\varepsilon) q_k(t;\varepsilon) + p_k(t;\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{17}$$

where

$$p_k(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s p_k^{(s)}(t) = \left\| \begin{matrix} 0 \\ \sum\limits_{s=0}^{\infty} \varepsilon^s f_k^{(s)}(t) \end{matrix} \right\|,$$

$$f_k^{(s)}(t) = \frac{2}{l} \int_0^l g_s(t;x) \sin \omega_k x dx.$$

Let one of the following cases hold: 1) "nonresonance", when the function $ik(t)$ $\left(k(t) = \dfrac{d\theta(t)}{dt}\right)$ is not equal to any root of the characteristic equation for all $t \in [0;L]$; 2) "resonance" when the function $ik(t)$ is equal identically to one of roots of the characteristic equation, for example, $ik(t) = \lambda_k^{(1)}(t)$. Then, in the "nonresonance" case, the following theorem is valid.

Theorem 5. *If the conditions of Theorem 1 are fulfilled, then, in the "nonresonance" case, system (17) has a partial formal solution of the form*

$$q_k(t;\varepsilon) = \sum_{m=0}^{\infty} \overline{q}_k(t;\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{18}$$

where $\overline{q}_k(t;\varepsilon)$ is an n-dimensional vector that admits the expansion

$$\overline{q}_k(t;\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m q_k^{(m)}(t). \tag{19}$$

Proof. Having substituted (19), (18) in (17) and equated coefficients of the same powers of ε, we get

$$\begin{aligned} &\left(H_k^{(0)}(t) - ik(t)\right) q_k^{(0)}(t) = -p_k^{(0)}(t), \\ &\left(H_k^{(0)}(t) - ik(t)\right) q_k^{(s)}(t) = \frac{dq_k^{(s-h)}}{dt} - p_k^{(s)}(t) - \sum_{m=1}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t),\ s = 1, 2, \ldots, \end{aligned} \tag{20}$$

Let us prove that system (20) has a solution. Since $\forall\, t \in [0;L]$ $ik(t) \neq \lambda_k^{(j)}(t)$, we have

$$\det \|H_k^{(0)}(t) - ik(t)E\| \neq 0, \quad j = \overline{1, 2n}.$$

For this reason,

$$\begin{aligned} &q_k^{(0)}(t) = -\left(H_k^{(0)}(t) - ik(t)E\right)^{-1} P_k^{(0)}(t), \\ &q_k^{(s)}(T) = \left(H_k^{(0)}(t) - ik(t)E\right)^{-1} \left(\frac{dq_k^{(s-h)}}{dt} - p_k^{(s)}(t) - \sum_{m=1}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t)\right). \end{aligned}$$

Theorem 5 is proved.

So, in the case of "nonresonance", the presence of a rotation point doesn't influence the form of a formal solution. In the "resonance" case, the following theoren is true.

Theorem 6. *If the conditions of Theorem 1 are fulfilled, then, in the case of "resonance", system (16) has a partial formal solution of the form*

$$q_k(t;\varepsilon) = \overline{q}_k(t;\varepsilon)\exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{21}$$

where $\overline{q}_k(t;\varepsilon)$ *are* $2n$*-dimensional vectors presented by the formal series*

$$\overline{q}_k(t;\varepsilon) = \sum_{m=0}^{\infty}\varepsilon^m q_k^{(m)}(t;\varepsilon). \tag{22}$$

Proof. Substitute (21), (22) in (17) and determine vectors $q_k^{(m)}(t;\varepsilon), m = 0, 1\ldots$, from the identity obtained with the help of equalities

$$\begin{aligned} &\left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right) q_k^{(0)}(t;\varepsilon) = -P_k^{(0)}(t), \\ &\left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right) q_k^{(s)}(t;\varepsilon) = h_k^{(s)}(t;\varepsilon), \end{aligned} \tag{23}$$

where

$$h_k^{(s)}(t;\varepsilon) = -p_k^{(s)}(t) + \frac{\partial p_k^{(s-h)}(t)}{\partial t} - \sum_{m=2}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t;\varepsilon).$$

Prove that the system of equations (23) has a solution. Since $ik(t)$ coincides with a root $\lambda_k^{(1)}(t)$, but $\lambda_{ik}(t;\varepsilon) \neq ik(t)$, $i = \overline{1, 2n}$, we get $\det \|H_k^{(1)}(t;\varepsilon) - ik(t)E\| \neq 0$. Therefore, from (23) we obtain

$$\begin{aligned} &q_k^{(0)}(t;\varepsilon) = -\left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right)^{-1} p_k^{(0)}(t), \\ &q_k^{(s)}(t) = \left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right)^{-1} h_k^{(s)}(t;\varepsilon), \quad s = 1, 2, \ldots . \end{aligned}$$

Theorem 6 is proved.

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