Symmetry in Nonlinear Mathematical Physics 1997, V.2, 254–261.

Quantum Mechanics in Noninertial Reference Frames and Representations of the Euclidean Line Group

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Abstract

The Galilean covariance of nonrelativistic quantum mechanics is generalized to an infinite parameter group of acceleration transformations called the Euclidean line group. Projective representations of the Euclidean line group are constructed and the resulting unitary operators are shown to implement arbitrary accelerations. These unitary operators are used to modify the time-dependent Schrödinger equation and produce the quantum mechanical analog of fictitious forces. The relationship of accelerating systems to gravitational forces is discussed. Solutions of the time-dependent Schrödinger equation for time varying, spatially constant external fields are obtained by transforming to appropriate accelerating reference frames. Generalizations to relativistic quantum mechanics are briefly discussed.

Infinite-dimensional groups and algebras continue to play an important role in quantum physics. The goal of this work, which is dedicated to the memory of Wilhelm Fushchych, is to look at some special representations of a group $\mathcal{E}(3)$, called the Euclidean line group in three dimensions, the group of maps from the real line to the three-dimensional Euclidean group. The motivation for studying such a group arises from a long-standing question in quantum mechanics, namely how to do quantum mechanics in noninertial reference frames. Now to "do" quantum mechanics in noninertial frames means constructing unitary operators that implement acceleration transformations. We will show that representations of $\mathcal{E}(3)$ on an appropriate Hilbert space provide the unitary operators that are needed to implement acceleration transformations.

The natural Hilbert space on which to construct representations of $\mathcal{E}(3)$ is the Hilbert space of a single particle of mass m and spin s, $\mathcal{H}_{m,s}$, which itself is the representation space for the central extension of the Galilei group [1]. Thus, the central extension of $\mathcal{E}(3)$ should contain the central extension of the Galilei group as a subgroup, and moreover, under the restriction of the representation of $\mathcal{E}(3)$ to that of the Galilei group, the Hilbert space should remain irreducible. The representations of $\mathcal{E}(3)$ will be obtained from the generating functions of the corresponding classical mechanics problem. These generating functions carry a representation of the Lie algebra of $\mathcal{E}(3)$ under Poisson bracket operations; using the correspondence between Poisson brackets in classical mechanics and commutators in

[1] Supported in part by DOE.

quantum mechanics gives the desired representation, which is readily exponentiated to give the representation of $\mathcal{E}(3)$. Moreover, this procedure automatically gives a representation of the central extension of $\mathcal{E}(3)$.

Once the unitary operators implementing acceleration transformations on $\mathcal{H}_{m,s}$ are known, it is possible to compute the analog of fictitious forces in quantum mechanics by applying the acceleration operators to the time-dependent Schrödinger equation. The resulting fictitious forces are proportional to the mass of the particle, and for linear accelerations, proportional to the position operator. This means that in a position representation, fictitious forces can simulate constant gravitational forces, which is the principle of equivalence in nonrelativistic physics. As will be shown, it is also possible to explicitly solve the time-dependent Schrödinger equation for such potentials by transforming to an appropriate noninertial frame.

We begin by looking at acceleration transformations that form the Euclidean line group $\mathcal{E}(3)$. Consider the acceleration transformations

$$\begin{aligned} &\vec{x} \to \vec{x}\,' = \vec{x} + \vec{a}(t) \quad \text{(linear acceleration)}, \\ &\vec{x} \to \vec{x}\,' = \vec{R}(t)\vec{x} \quad \text{(rotational acceleration)}, \end{aligned} \tag{1}$$

where $R(t) \in SO(3)$ is a rotation and $\vec{a}(t) \in \mathbb{R}^3$ is a three-dimensional translation. Both of these types of transformations are indexed by the time parameter t, so that the Euclidean line group consists of maps $\mathbb{R} \to E(3)$, from the real line to the Euclidean group in three dimensions. Such an infinite-dimensional group contains all transformations that preserve the distance between two points in the three-dimensional space $(\vec{x} - \vec{y}\,)^2$.

Linear accelerations contain translations and Galilei boosts of the Galilei group,

$$\begin{aligned} &\vec{x}\,' = \vec{x} + \vec{a} \quad \text{(translations)}, \\ &\vec{x}\,' = \vec{x} + \vec{v}t \quad \text{(Galilei boosts)}, \end{aligned} \tag{2}$$

as well as such acceleration transformations as constant accelerations,

$$\vec{x}\,' = \vec{x} + \frac{1}{2}\,\vec{a}t^2 \quad \text{(constant accelerations)}. \tag{3}$$

Similarly, rotational accelerations contain constant rotations of the Galilei group as well as constant angular velocity rotations,

$$R(t) = R(\hat{n}, \omega t), \tag{4}$$

where $\hat{n}$ is the axis of rotation and ωt is the angle of rotation.

Given the group $\mathcal{E}(3)$, we wish to find its unitary projective representations on the Hilbert space for a nonrelativistic particle of mass m and spin s, namely $\mathcal{H}_{m,s} = L^2(\mathbb{R}^3) \times V^s$, where V^s is the $2s+1$ dimensional complex vector space for a particle of spin s [1]. In momentum space, the wave functions $\varphi(\vec{p}, m_s) \in \mathcal{H}_{m,s}$ transform under the Galilei group elements as

$$\begin{aligned} &(U_{\vec{a}}\varphi)(\vec{p}, m_s) = e^{-i\vec{P}\cdot\vec{a}/\hbar}\varphi(\vec{p}, m_s) = e^{-i\vec{p}\cdot\vec{a}/\hbar}\varphi(\vec{p}, m_s), \\ &(U_{\vec{v}}\varphi)(\vec{p}, m_s) = e^{-i\vec{X}\cdot m\vec{v}/\hbar}\varphi(\vec{p}, m_s) = \varphi(\vec{p} + m\vec{v}\,), \\ &(U_R\varphi)(\vec{p}, m_s) = e^{-i\vec{J}\cdot\hat{n}\theta}\varphi(\vec{p}, m_s) = \sum_{m_s'=-s}^{+s} D^s_{m_s m_s'}(R)\varphi(R^{-1}\vec{p}, m_s'), \end{aligned} \tag{5}$$

where $R \in SO(3)$ is a rotation, and $D^s_{m_s m'_s}(R)$ is a Wigner D function. $\vec{P}$, $\vec{X}$, and $\vec{J}$ are, respectively, the momentum, position, and angular momentum operators. The goal is to find $U_{\vec{a}(t)}$ and $U_{R(t)}$ as unitary operators on $\mathcal{H}_{m,s}$, where $\vec{a}(t)$ and $R(t)$ are elements of $\mathcal{E}(3)$.

In Ref. [2], we have shown how to calculate $U_{\vec{a}(t)}$ and $U_{R(t)}$ by looking at the generating functions for a particle in classical mechanics and then changing Poisson brackets to commutators in quantum mechanics. Here we will illustrate the basic idea by considering linear accelerations in only one spatial dimension, namely

$$x' = x + a(t). \tag{6}$$

A generating function for such a transformation is given by

$$\begin{aligned} &F(x,p') = (x + a(t))p' - mx\dot{a}(t), \\ &x' = \frac{\partial F}{\partial p'} = x + a(t), \\ &p = \frac{\partial F}{\partial x} = p' - m\dot{a}(t), \qquad \dot{a}(t) := \frac{da}{dt}, \\ &H'(x',p') = H(x,p) + \frac{\partial F}{\partial t}. \end{aligned} \tag{7}$$

To get the operators that generate various acceleration transformations, we write $a(t)$ as a power series in t,

$$a(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \ , \tag{8}$$

where the expansion coefficients a_n play the role of group parameters for the one-parameter subgroups of linear accelerations. Note that a_0 generates spatial translations, while a_1 generates Galilei boost transformations [see Eq. (2)].

For each one-parameter subgroup specified by a_n, we compute the infinitesimal generating functions $A_n(x,p)$, which, because of the group properties of $\mathcal{E}(3)$, will close under Poisson bracket operations. $A_n(x,p)$ comes from the generating function $F_n(x,p')$ relative to the group element a_n:

$$\begin{aligned} &F_n(x,p') = \left(x + \frac{a_n}{n!}\, t^n\right) p' - mx\, \frac{a_n t^{n-1}}{(n-1)!}, \\ &x' = x + \frac{a_n t^n}{n!}, \\ &p' = p + m\, \frac{a_n}{(n-1)!}\, t^{n-1}, \\ &A_n(x,p) = \frac{t^n}{n!}\, p - \frac{mt^{n-1}}{(n-1)!}\, x\ , \quad n = 1, 2, \dots \\ &A_0(x,p) = p. \end{aligned} \tag{9}$$

Then the Poisson brackets of A_n with $A_{n'}$ close to give

$$\begin{aligned} &\{A_n, A_{n'}\} = \frac{mt^{n+n'-1}}{(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'}\right) \ , \quad n, n' \neq 0, \\ &\{A_n, A_0\} = -\frac{mt^{n-1}}{(n-1)!}. \end{aligned} \tag{10}$$

Since A_0 generates spatial translations, which are related to the momentum operator, we want the canonical commutation relations $\{x, p\} = 1$. But from Eq. (10) it is seen that $\{A_1, A_0\} = -m$. So define

$$\begin{aligned} B_n &:= -\frac{1}{m} A_n , \quad n = 1, 2, \ldots \\ B_0 &:= A_0, \end{aligned} \tag{11}$$

which gives

$$\begin{aligned} \{B_n, B_{n'}\} &= \frac{t^{n+n'-1}}{m(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'} \right) , \quad n, n' \neq 0, \\ \{B_n, B_0\} &= \frac{t^{n-1}}{(n-1)!}. \end{aligned} \tag{12}$$

In particular, $\{B_1, B_0\} = 1$. These equations are the starting point for introducing acceleration operators into quantum mechanics, for they provide a projective representation of the Lie algebra of the one-dimensional linear accelerations, in which commutators replace Poisson brackets. That is

$$\begin{aligned} B_n &\to X_n := \tfrac{t^{n-1}}{(n-1)!} i\hbar \tfrac{\partial}{\partial p} - \tfrac{t^n}{mn!} p , \quad n = 1, 2, \ldots \\ B_0 &\to P = p, \\ [X_n, X_{n'}] &= \frac{i\hbar t^{n+n'-1}}{m(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'} \right) I , \quad n, n' \neq 0, \\ [X_n, P] &= \frac{i\hbar t^{n-1}}{(n-1)!} I, \\ [X_1, P] &= i\hbar I, \end{aligned} \tag{13}$$

where I is the identity operator.

But $X_1 = i\hbar(\partial/\partial p) - (t/m)p$ is not the usual position operator, $X = i\hbar(\partial/\partial p)$. The appendix of Ref. [2] shows that X_1 is unitarily equivalent to X. X_1 is a perfectly good position operator and we continue to use it because the form of $U_{a(t)}$ is particularly simple.

The operators X_n are readily exponentiated; as shown in Ref. [2], the unitary operator implementing the acceleration transformation $a(t)$ is then

$$(U_{a(t)}\varphi)(p) = e^{i(a(t)p/\hbar)} \varphi(p + m\dot{a}(t)) . \tag{14}$$

This can be generalized to the full $\mathcal{E}(3)$ group to give

$$\begin{aligned} (U_{\vec{a}(t)}\varphi)(\vec{p}, m_s) &= e^{i(\vec{a}(t)\cdot\vec{p}/\hbar)} \varphi(\vec{p} + m\dot{\vec{a}}(t)), \\ (U_{R(t)}\varphi)(\vec{p}, m_s) &= \sum_{m'_s} D^s_{m_s m'_s}(R(t)) \varphi(R^{-1}(t)\vec{p}, m'_s), \end{aligned} \tag{15}$$

which are the unitary operators implementing acceleration transformations on $\mathcal{H}_{m,s}$. These operators form a unitary projective representation of $\mathcal{E}(3)$ with multiplier

$$\omega(a_1, a_2) = \frac{m}{\hbar} \dot{\vec{a_1}}(t) \cdot \vec{a}_2(t) .$$

The unitary operators, Eq. (15), can be used to derive the form of fictitious potentials that arise in noninertial reference frames. Let $g(t)$ denote either $\vec{a}(t)$ or $R(t)$ in $\mathcal{E}(3)$ and let $\psi' = U_{g(t)}\psi$ be the wave function in the noninertial frame obtained from the wave function ψ in the inertial frame under the transformation $g(t)$. Applying $U_{g(t)}$ to the time-dependent Schrödinger equation valid in an inertial frame gives

$$\begin{aligned} i\hbar U_{g(t)} \frac{\partial \psi_t}{\partial t} &= U_{g(t)} H \psi_t, \\ i\hbar \left[\frac{\partial}{\partial t}(U_{g(t)}\psi_t) - \frac{\partial U_{g(t)}}{\partial t}\, \psi_t \right] &= U_{g(t)} H U_{g(t)}^{-1} U_{g(t)} \psi_t, \\ i\hbar \frac{\partial}{\partial t}\, \psi'_t &= \left[H' + i\hbar \frac{\partial U_{g(t)}}{\partial t}\, U_{g(t)}^{-1} \right] \psi'_t; \end{aligned} \tag{16}$$

here $H' := U_{g(t)} H U_{g(t)}^{-1}$ is the transformed Hamiltonian in the noninertial reference frame, and

$$\frac{\partial U_{g(t)}}{\partial t} := \lim_{\epsilon \to 0} \frac{U_{g(t+\epsilon)} - U_{g(t)}}{\epsilon} \ . \tag{17}$$

Since $U_{g(t)}$ is known explicitly, the quantum fictitious potential can be computed; the result is that

$$\begin{aligned} i\hbar \frac{\partial U_{\vec{a}(t)}}{\partial t}\, U_{\vec{a}(t)}^{-1} &= m\ddot{\vec{a}}(t) \cdot \vec{X} - \dot{\vec{a}}(t)\vec{P} + m\vec{a}(t) \cdot \ddot{\vec{a}}(t) I, \\ i\hbar \frac{\partial U_{R(t)}}{\partial t}\, U_{R(t)}^{-1} \Big|_{m_s, m'_s} &= -\hbar \vec{\omega}(t) \cdot [\vec{L}\delta_{m_s m'_s} + \vec{S}^s_{m_s m'_s}]; \end{aligned} \tag{18}$$

the angular velocity $\vec{\omega}(t)$ is obtained from the angular momentum matrix $\Omega(t) := \dot{R}(t)$ $R^{-1}(t)$, which is antisymmetric,

$$\omega_i(t) = \frac{1}{2}\, \epsilon_{ijk} \Omega_{jk}(t) \ . \tag{19}$$

$\vec{S}^s_{m_s m'_s}$ are the angular momentum matrices for spin s,

$$\vec{S}^s_{m_s m'_s} := \langle s m_s | \vec{J} | s m'_s \rangle \ . \tag{20}$$

If the Hamiltonian is the free particle Hamiltonian, $H_0 = \vec{P}^2/2m$, then under linear accelerations,

$$\begin{aligned} U_{\vec{a}(t)} H_0 U_{\vec{a}(t)}^{-1} &= \frac{1}{2m} U_{\vec{a}(t)} \vec{P} \cdot \vec{P} U_{\vec{a}(t)}^{-1} = H_0 + \dot{\vec{a}}(t) \cdot \vec{P} + \frac{m}{2}\, \dot{\vec{a}}(t) I, \\ H_{\text{accel}} &:= U_{\vec{a}(t)} H_0 U_{\vec{a}(t)}^{-1} + i\hbar \frac{\partial U_{\vec{a}(t)}}{\partial t} U_{\vec{a}(t)}^{-1} \\ &= H_0 + m \left[\ddot{\vec{a}}(t) \cdot \vec{X} + \tfrac{\dot{\vec{a}}(t) \cdot \dot{\vec{a}}(t)}{2}\, I + \vec{a}(t) \cdot \ddot{\vec{a}}(t) I \right] \end{aligned} \tag{21}$$

and the fictitious potential is proportional to m, as expected.

If it were not known how to couple an external gravitational field to a quantum mechanical particle, the nonrelativistic version of the principle of equivalence could be used to link a constant acceleration with a constant gravitational field. That is, for $\vec{a}(t) = 1/2\vec{a}t^2$, $\vec{a}$ constant, the fictitious potential (in a position representation and neglecting the terms related to I) is

$$V_{\rm fic}(\vec{x}\,) = m\vec{a}\cdot\vec{x} \tag{22}$$

which is the potential for a constant gravitational field.

This reasoning can be turned around. Say, we are given a potential (for simplicity in one dimension) of the form $V_t = f(t)X$, a potential linear in X, but varying arbitrarily in time. An example is a spatially constant external electric field that varies in an arbitrary manner in time. The time-dependent Schrödinger equation for such a system in a momentum representation is

$$i\hbar\frac{\partial\varphi_t}{\partial t} = \left(\frac{p^2}{2m} + f(t)i\hbar\frac{\partial}{\partial p}\right)\varphi_t. \tag{23}$$

If this equation is transformed to a noninertial reference frame, the acceleration $a(t)$ can be chosen so as to cancel off the effect of $f(t)$. That is, if $m\ddot{a}(t) + f(t) = 0$, the $\partial/\partial p$ terms cancel and the Schrodinger equation becomes

$$i\hbar\frac{\partial\varphi'_t}{\partial t} = \left(\frac{p^2}{2m} + \frac{m\dot{a}^2}{2}\right)\varphi'_t,$$

which can readily be solved. Transforming back to the inertial frame gives the solution

$$\begin{aligned}\varphi_t(p) &= U^{-1}_{a(t)}\varphi'_t(p) = \varphi_{t=0}(p - m\dot{a}(t))\\ &\times\exp\left\{-\frac{i}{\hbar}\left[\frac{p^2}{2m}t + (a(t) - t\dot{a}(t))p + \frac{m}{2}\Big(t\dot{a}^2(t) + \int^t(\dot{a}^2 - 2a(t)\ddot{a}(t))\Big)\right]\right\},\end{aligned} \tag{24}$$

with $\varphi_{t=0}(p) \in L^2(\mathbb{R})$ the wave function at $t = 0$.

To conclude this paper, we briefly discuss the question of how to generalize these nonrelativistic results to relativistic quantum mechanics. Relativistic means that the Galilei group is replaced by the Poincaré group, consisting of Lorentz transformations and space-time translations. The first problem that arises is that there are a number of different ways of formulating relativistic quantum mechanics, the most prominent being quantum field theory. These different formulations all carry representations of the Poincaré group in one way or another, but they differ in how interactions are introduced. One way in which interactions can be introduced is through the Poincaré generators. In such a formulation, some generators contain interactions, others not. Dirac [3] classified three such possibilities as instant, front, and point forms. The instant form is the most familiar form, in that the Poincaré generators not containing interactions are the Euclidean subalgebra of rotations and spatial translation generators. More recently, the front form of relativistic quantum mechanics has been of great interest [4].

However, to analyze accelerations in relativistic quantum mechanics, the point form [5], wherein all interactions are put into four-momentum generators, is the natural form to use, for it is the only form which is manifestly covariant under Lorentz transformations (the generators of Lorentz transformations do not contain interactions).

In the point form of relativistic quantum mechanics, the interacting four-momentum operators must satisfy the (Poincaré) conditions

$$[P^\mu, P^\nu] = 0,$$
$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu{}_\nu P^\nu, \tag{25}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ on an appropriate Hilbert space $\mathcal{H}$ (which may be a Fock space or a subspace of a Fock space). Since the four-momentum operators all commute with one another, they can be simultaneously diagonalized, and used to construct the invariant mass operator,

$$M := \sqrt{P^\mu P_\mu}\ . \tag{26}$$

The spectrum of M must be bounded from below; the discrete part of the spectrum of M corresponds to bound states and the continuous part to scattering states.

However, the most important feature of the point form is that the time-dependent Schrödinger equation naturally generalizes to

$$i\hbar \frac{\partial \psi_x}{\partial x_\mu} = P^\mu \psi_x, \tag{27}$$

where x is the space-time point $(ct, \vec{x}\,)$. This relativistic Schrödinger equation simply states that the interacting four-momentum operators act as generators of space-time translations, in a Lorentz covariant manner. But its importance with regard to acceleration transformations is that relativistic fictitious forces will arise in exactly the same way as the nonrelativistic ones arose in Eq. (16).

This leads to a second problem, namely finding the generalization of $\mathcal{E}(3)$ for relativistic acceleration. Since, in relativistic mechanics space and time are on an equal footing, time cannot be an independent parameter, as was the case for $\mathcal{E}(3)$. But since the principle of equivalence links acceleration to general relativity [6], the natural group to consider is the diffeomorphism group on the Minkowski space, the group of invertible differentiable maps from the Minkowski manifold to itself, Diff($\mathcal{M}$). This group has both $\mathcal{E}(3)$ and the Poincaré group as subgroups.

If the transformation

$$x^\mu \to x'^\mu = f^\mu(x^\nu) \tag{28}$$

is an element of Diff($\mathcal{M}$), then what is needed is a unitary representation of Diff($\mathcal{M}$) on $\mathcal{H}$,

$$f \in \mathrm{Diff}(\mathcal{M}) \to U_f \quad \text{on} \quad \mathcal{H}, \tag{29}$$

such that in the limit as $c \to \infty$, the representation of Diff($\mathcal{M}$) contracts to a representation of $\mathcal{E}(3)$. For such U_f, the relativistic fictitious force is given by

$$i\hbar \frac{\partial U_f}{\partial x_\mu} U_f^{-1}, \tag{30}$$

and, from the principle of equivalence, shows how to couple an external gravitational field to a relativistic particle of arbitrary mass and spin. Details of these ideas will be carried out in a future paper.

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