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On Reduction and Some Exact Solutions of the Euler-Lagrange-Born-Infeld Equation

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Abstract

The ansatzes which reduce the Euler-Lagrange-Born-Infeld equation to differential equations with a smaller number of independent variables are constructed using the subgroup structure of the Poincaré group $P(1,3)$. The corresponding symmetry reduction is made. Some classes of exact solutions of the investigated equation are found.

In [1, 2], the symmetry properties of the Euler-Lagrange-Born-Infeld equation were studied and multiparametric families of exact solutions of the equation have been found using special ansatzes.

Consider the equation

$$\Box u\,(1-u_\nu u^\nu)+u_{\mu\nu}u^\mu u^\nu=0, \tag{1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2)\in\mathbf{R}_3$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $u_\nu\equiv\dfrac{\partial u}{\partial x_\nu}$, $\mu,\nu=0,1,2$, $\Box$ is the d'Alembertian.

The symmetry group [1, 2] of equation (1) includes the Poincaré group $P(1,3)$ as a subgroup. We construct ansatzes which reduce equation (1) to differential equations with a smaller number of independent variables, using the invariants [3, 4] of the subgroups of the group $P(1,3)$. The corresponding symmetry reduction is performed. Using the solutions of the reduced equations, we have found some classes of exact solutions of the Euler-Lagrange-Born-Infeld equation.

1. Below we write ansatzes which reduce equation (1) to ordinary differential equations (ODEs), and we list the ODEs obtained as well as some exact solutions of the Euler-Lagrange-Born-Infeld equation.

1. $u=\varphi(\omega)$, $\omega=\left(x_1^2+x_2^2\right)^{1/2}$, $\omega\varphi''+\varphi'^3+\varphi'=0$;

$$u=c\ln\left((x_1^2+x_2^2)^2+\sqrt{x_1^2+x_2^2-c^2}\right);$$

2. $u=-\varphi(\omega)+x_0$, $\omega=\left(x_1^2+x_2^2\right)^{1/2}$, $\varphi'=0$; $u=x_0-c$;

3. $u^2=-\varphi^2(\omega)+x_0^2$, $\omega=x_1$, $\varphi''\varphi-\varphi'^2+1=0$;

$$u^2=-\frac{1}{c_1^2}\sin^2(c_1x_1+c_2)+x_0^2;\quad u^2=-(\varepsilon x_1+c)^2+x_0^2,\quad \varepsilon=\pm1;$$

4. $u = -\varphi(\omega) + x_0, \quad \omega = \left(x_0^2 - x_1^2 - u^2\right)^{1/2},$

$\omega\varphi^2\varphi'' - 2\omega^2\varphi'^3 + 4\omega\varphi\varphi'^2 - \varphi^2\varphi' = 0;$

$u = \frac{1}{2c}\left(1 + \varepsilon\sqrt{1 - 4c(x_0 - c(x_0^2 - x_1^2))}\right); \quad u = x_0 - c(x_0^2 - x_1^2 - u^2)^{1/4};$

5. $u^2 = -\varphi^2(\omega) + x_0^2, \quad \omega = \left(x_1^2 + x_2^2\right)^{1/2},$

$\omega\varphi''\varphi - \varphi'^3\varphi - \omega\varphi'^2 + \varphi'\varphi + \omega = 0;$

$u^2 = x_0^2 - \left(\varepsilon(x_1^2 + x_2^2)^{1/2} + c\right)^2, \quad \varepsilon = \pm 1;$

6. $u^2 = -\varphi^2(\omega) + x_0^2 - x_1^2 - x_2^2, \quad \omega = x_0 - u,$

$\omega^2\varphi''\varphi + \omega^2\varphi'^2 - 6\omega\varphi\varphi' + 3\varphi^2 = 0;$

$u^2 = x_0^2 - x_1^2 - x_2^2 - c^2(x_0 - u)^6; \; u = \frac{1}{2}\left(c + \varepsilon\sqrt{c^2 - 4(cx_0 - x_0^2 + x_1^2 + x_2^2)}\right),$

$\varepsilon = \pm 1;$

7. $u = \varphi(\omega), \quad \omega = (x_0^2 - x_1^2 - x_2^2)^{1/2}, \quad \omega\varphi'' - 2\varphi'^3 + 2\varphi' = 0;$

$u = \varepsilon(x_0^2 - x_1^2 - x_2^2)^{1/2} + c, \quad \varepsilon = \pm 1;$

8. $u^2 = \varphi^2(\omega) - x_1^2 - x_2^2, \quad \omega = x_0, \quad \varphi''\varphi + 2\varphi'^2 - 2 = 0;$

$u^2 = (\varepsilon x_0 + c)^2 - x_1^2 - x_2^2, \quad \varepsilon = \pm 1;$

9. $u = \varphi(\omega) - \alpha \arctan\frac{x_2}{x_1}, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$(\alpha^2 + 1)\omega^3\varphi'' + \omega^2\varphi'^3 + (2\alpha^2 + \omega^2)\varphi' = 0;$

10. $u = -\varphi(\omega) + x_0 + \arctan\frac{x_2}{x_1}, \quad \omega = (x_1^2 + x_2^2)^{1/2}, \quad \omega^3\varphi'' + \omega^2\varphi'^3 + 2\varphi' = 0.$

Ansatzes (1)–(10) can be written in the following form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x),$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

11. $\omega x_2^2 - (1 - \omega)\left((2x_0 + \omega)\omega + x_1^2\right) = \varphi(\omega), \quad \omega = u - x_0,$

$\omega^2(1 - \omega)^2\varphi'' - 4\omega(1 - \omega)(1 - 2\omega)\varphi' + 2(7\omega^2 - 7\omega + 2)\varphi = 0;$

$(u - x_0)x_2^2 + (x_0 - u + 1)(x_0^2 - x_1^2 - u^2) = \widetilde{c}_1(u - x_0)^4 \times$

$(6(u - x_0)^3 - 21(u - x_0)^2 + 25(u - x_0) + 10) + c_2(u - x_0)(u - x_0 - 1).$

Ansatz (11) can be written in the following form:

$$h(\omega, x) = f(x) \cdot \varphi(\omega) + g(x),$$

where $h(\omega, x), f(x), g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

12. $\alpha \ln(x_0 - u) = \varphi(\omega) - x_1, \quad \omega = (x_0^2 - u^2)^{1/2},$

$\omega(\alpha^2+\omega^2)\varphi''-\omega^2\varphi'^3+\alpha(2+\alpha)\omega\varphi'^2+(\omega^2-\alpha^2)\varphi'=0;$

13. $\alpha\ln(x_0-u)=\varphi(\omega)-x_1,\quad \omega=x_2,\quad \varphi''=0;$

$u=x_0-\exp\left((c_1x_2-x_1+c_2)/\alpha\right);$

14. $x_0+u-x_1(x_0-u)+\frac{1}{6}(x_0-u)^3=\varphi(\omega),\quad \omega=\frac{1}{4}(x_0-u)^2-x_1,$

$2\omega\varphi''-\varphi'=0;$

$x_0+u-x_1(x_0-u)+\frac{1}{6}(x_0-u)^3=\frac{2c_1}{3}\left(\frac{1}{4}(x_0-u)^2-x_1\right)^{3/2}+c_2;$

15. $(x_0-u)^2=4\varphi(\omega)+4x_1,\quad \omega=x_2,\quad \varphi''=0;$

$u=x_0+2\varepsilon\sqrt{x_1+c_1x_2+c_2},\quad \varepsilon=\pm1;$

16. $\alpha\ln(x_0-u)=\varphi(\omega)-x_2,\quad \omega=(x_0^2-x_1^2-u^2)^{1/2},$

$\omega(\alpha^2+\omega^2)\varphi''-2\omega^2\varphi'^3+5\alpha\omega\varphi'^2+(2\omega^2-\alpha^2)\varphi'=0.$

Ansatzes (12)–(16) can be written in the following form:

$$h(u,x)=f(x)\cdot\varphi(\omega)+g(x),$$

where $h(u,x),f(x),g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega=\omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

2. Next we consider the reduction of the investigated equation to two-dimensional partial differential equations (PDEs). The PDEs obtained can be written in the form:

$$A\left(\varphi_{11}\varphi_2^2+\varphi_{22}\varphi_1^2-2\varphi_{12}\varphi_1\varphi_2\right)+B_1\varphi_{11}+B_2\varphi_{22}+2B_3\varphi_{12}+V=0,$$

$$\varphi_i\equiv\frac{\partial\varphi}{\partial\omega_i},\quad \varphi_{ij}\equiv\frac{\partial^2\varphi}{\partial\omega_i\partial\omega_j},\quad i=1,2.$$

Below, we present the ansatzes, which reduce equation (1) to two-dimensional PDEs, and the corresponding coefficients $A,\ B_1,\ B_2,\ B_3,\ V$ of the reduced equation.

1. $u=\varphi(\omega_1,\omega_2),\quad \omega_1=x_1,\quad \omega_2=x_2;$

$A=B_1=B_2=1,\quad B_3=V=0.$

2. $u=-\varphi(\omega_1,\omega_2)+x_0,\quad \omega_1=x_1,\quad \omega_2=x_2;$

$A=1,\quad B_1=B_2=B_3=V=0.$

3. $u=\varphi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$A=B_1=-B_2=\omega_2,\quad B_3=0,\quad V=\varphi_2\left(\varphi_1^2-\varphi_2^2-1\right).$

4. $u^2=-\varphi^2(\omega_1,\omega_2)+x_0^2,\quad \omega_1=x_1,\quad \omega_2=x_2;$

$A=-B_1=-B_2=\varphi^3,\quad B_3=0,\quad V=\varphi^2\left(\varphi_1^2+\varphi_2^2-1\right).$

5. $u^2=-\varphi^2(\omega_1,\omega_2)+x_0^2-x_1^2,\quad \omega_1=x_0-u,\quad \omega_2=x_2;$

$A=0,\quad B_1=\omega_1^2\varphi,\quad B_2=\varphi^2\left(\varphi-2\omega_1\varphi_1\right),\quad B_3=\omega_1\varphi^2\varphi_2,$

$V=\omega_1\varphi_1\left(\omega_1\varphi_1-4\varphi\right)-2\varphi^2\left(\varphi_2^2-1\right).$

6. $u = \varphi(\omega_1, \omega_2), \quad \omega_1 = \alpha \arctan \frac{x_2}{x_1} + x_0, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2};$

$A = B_1 = \omega_2 \left(\omega_2^2 - \alpha^2\right), \quad B_2 = -\omega_2^3, \quad B_3 = 0,$

$V = \omega_2^2 \varphi_2 \left(\varphi_1^2 - \varphi_2^2 - 1\right) - 2\alpha^2 \varphi_1^2 \varphi_2.$

7. $u = \varphi(\omega_1, \omega_2) - \alpha \arctan \frac{x_2}{x_1}, \quad \omega_1 = x_0, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2};$

$A = \omega_2^3, \quad B_1 = -B_2 = \omega_2 \left(\omega_2^2 + \alpha^2\right), \quad B_3 = 0,$

$V = \omega_2^2 \varphi_2 \left(\varphi_1^2 - \varphi_2^2 - 1\right) - 2\alpha^2 \varphi_2.$

8. $u^2 = -\varphi^2(\omega_1, \omega_2) + x_0^2 - x_1^2, \quad \omega_1 = x_0 - u, \quad \omega_2 = x_1 - x_2 (x_0 - u);$

$A = 0, \quad B_1 = \omega_1^2 \varphi, \quad B_2 = \varphi \left(\omega_2^2 + \varphi \left(\omega_1^2 + 1\right)(\varphi - 2\omega_1 \varphi_1)\right),$

$B_3 = \omega_1 \varphi \left(\omega_2 + \left(\omega_1^2 + 1\right) \varphi_2 \varphi\right),$

$V = (\omega_1 \varphi_1 + \omega_2 \varphi_2)^2 - \varphi^2 \varphi_2^2 - 4\varphi (\omega_1 \varphi_1 + \omega_2 \varphi_2) + 2\varphi^2.$

9. $\alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2) - \arctan \frac{x_2}{x_1}, \quad \omega_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad \omega_2 = \left(x_0^2 - u^2\right)^{1/2};$

$A = \omega_1^3 \omega_2^2, \quad B_1 = -\omega_1^3 \omega_2^2 (\omega_2 + 2\alpha \varphi_2), \quad B_2 = \omega_1 \omega_2 \left(\alpha^2 \omega_1^2 + \omega_2^2\right),$

$B_3 = \alpha \omega_1^3 \omega_2^2 \varphi_1, \quad V = \omega_1^2 \omega_2^2 \left(\varphi_1^2 - \varphi_2^2\right)(\omega_1 \varphi_2 - \omega_2 \varphi_1) + 3\alpha \omega_1^3 \omega_2 \varphi_2^2 -$

$-2\alpha \omega_1^2 \omega_2^2 \varphi_1 \varphi_2 + \omega_1 \left(\omega_2^2 - \alpha^2 \omega_1^2\right) \varphi_2 - 2\omega_2^3 \varphi_1.$

10. $\left(x_1^2 + x_2^2\right)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = \arctan \frac{x_2}{x_1} + x_0 - u;$

$A = 4\varphi^3, \quad B_1 = 0, \quad B_2 = -\varphi, \quad B_3 = 2\varphi^3, \quad V = 2\varphi_2^2 - \varphi^2 (4\varphi_1 \varphi_2 - 1).$

11. $\alpha \ln(x_0 - u) = \varphi(\omega_1, \omega_2) - x_1, \quad \omega_1 = \left(x_0^2 - u^2\right)^{1/2}, \quad \omega_2 = x_2;$

$A = \omega_1^3, \quad B_1 = \omega_1 \left(\alpha^2 + \omega_1^2\right), \quad B_2 = -2\alpha \omega_1^2 \varphi_1, \quad B_3 = \alpha \omega_1^2 \varphi_2,$

$V = -\omega_1^2 \varphi_1 \left(\varphi_1^2 - \varphi_2^2\right) + 3\alpha \omega_1 \varphi_1^2 + \left(\omega_1^2 - \alpha^2\right) \varphi_1.$

12. $u + x_0 - x_1(x_0 - u) + \frac{1}{6}(x_0 - u)^3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = \frac{1}{4}(x_0 - u)^3 - x_1, \quad \omega_2 = x_2;$

$A = 1, \quad B_1 = B_2 = -4\omega_1, \quad B_3 = 0, \quad V = 2\varphi_1.$

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