Symmetry in Nonlinear Mathematical Physics 1997, V.1, 241–244.

Reduction and Some Exact Solutions of the Eikonal Equation

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Abstract

Using the nonsplitting subgroups of the generalized Poincaré group $P(1,4)$, ansatzes which reduce the eikonal equation to differential equations with a lesser number of independent variables are constructed. The corresponding symmetry reduction is made. Some classes of exact solutions of the investigated equation are presented.

The relativistic eikonal (the relativistic Hamilton-Jacobi) equation is fundamental in theoretical and mathematical physics. We consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{1}$$

In [1], it was shown that the maximal local (in the sense of Lie) invariance group of equation (1) is the conformal group $C(1,4)$ of the five-dimensional Poincaré-Minkowski space. Using special ansatzes, the multiparametric families of exact solutions of the eikonal equation were constructed [1–4].

The conformal group $C(1,4)$ contains the generalized Poincaré group $P(1,4)$ as a subgroup. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Poincaré–Minkowski space. For the investigation of equation (1), we have used the nonsplitting subgroups [5–7] of the group $P(1,4)$. We have constructed ansatzes which reduce equation (1) to differential equations with a smaller number of independent variables using invariants [8] of the nonsplitting subgroups of the group $P(1,4)$. The corresponding symmetry reduction is performed. Using solutions of the reduced equations, we have found some classes of exact solutions of the eikonal equation.

Below we write ansatzes which reduce equation (1) to ordinary differential equations (ODEs), and we list the ODEs obtained as well as some solutions of the eikonal equation.

1. $\frac{c}{\alpha}x_3 + \ln(x_0+u) = \varphi(\omega), \quad \omega = (x_1^2+x_2^2+u^2-x_0^2)^{1/2},$

   $(\varphi')^2 - \frac{2}{\omega}\varphi' + \frac{c^2}{\alpha^2} = 0;$

2. $(x_0^2-x_1^2-x_2^2-u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 + \alpha\ln(x_0+u),$

   $(\varphi')^2 - \frac{2\alpha}{\varphi}\varphi' - 1 = 0;$

3. $(u^2+x_3^2-x_0^2)^{1/2} = \varphi(\omega), \quad \omega = x_2 + \alpha\ln(x_0+u),$

$$(\varphi')^2 - \frac{2\alpha}{\varphi}\varphi' - 1 = 0;$$

4. $u = \exp\left(\varphi(\omega) + \frac{x_2}{\widetilde{a}_2}\right) - x_0, \quad \omega = x_3,$

$$(\varphi')^2 = -\frac{1}{\widetilde{a}_2^2};$$

$$u = \exp\left(\frac{\varepsilon i x_3 + x_2}{\widetilde{a}_2} + c\right) - x_0;$$

5. $u = -\exp\left(\varphi(\omega) - \frac{x_2}{\widetilde{a}_2}\right) + x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i x_3 - x_2}{\widetilde{a}_2} + c\right) + x_0;$$

6. $u = \exp\left(\varphi(\omega) - \frac{x_1}{a}\right) - x_0, \quad \omega = x_2,$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = \exp\left(\frac{\varepsilon i x_2}{a^2} + \frac{x_1}{a} + c\right) - x_0;$$

7. $u = -\exp\left(\varphi(\omega) - \frac{x_3}{a}\right) + x_0, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i}{a}(x_1^2 + x_2^2)^{1/2} - \frac{x_3}{a} + c\right) + x_0;$$

8. $u = \exp\left(\frac{x_3}{a} - \varphi(\omega)\right) - x_0, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = \exp\left(\frac{x_3}{a} - \frac{\varepsilon i}{a}(x_1^2 + x_2^2)^{1/2} - c\right) - x_0;$$

9. $u = -\exp\left(\varphi(\omega) - \frac{x_2}{\widetilde{a}_2}\right) + x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i}{\widetilde{a}_2}x_3 - \frac{x_2}{\widetilde{a}_2} + c\right) + x_0;$$

10. $u = \exp\left(\frac{x_2}{\widetilde{a}_2} - \varphi(\omega)\right) - x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = \exp\left(\frac{x_2}{\widetilde{a}_2} - \frac{\varepsilon i}{\widetilde{a}_2}x_3 + c\right) - x_0.$$

Ansatzes (1)–(10) can be written in the following form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x), \tag{2}$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x)$ is a one-dimensional invariant of the nonsplitting subgroups of the group $P(1,4)$.

11. $\arcsin \frac{x_3}{\omega} = \varphi(\omega) - \frac{x_0}{\alpha}, \quad \omega = (x_3^2 + u^2)^{1/2},$

$$(\varphi')^2 + \frac{1}{\omega^2} - \frac{1}{\alpha^2} = 0;$$

12. $\arcsin \frac{x_3}{\omega} = \varphi(\omega) - \frac{c}{\alpha} x_0, \quad \omega = (x_3^2 + u^2)^{1/2} \ \ (0 < c < 1,\ \alpha > 0),$

$$(\varphi')^2 + \frac{1}{\omega^2} - \frac{c^2}{\alpha^2} = 0;$$

13. $\operatorname{arch} \frac{x_0}{\omega} = \varphi(\omega) - \frac{c}{\alpha} x_3, \quad \omega = (x_0^2 - u^2)^{1/2} \ \ (\alpha > 0),$

$$(\varphi')^2 - \left(\frac{1}{\omega^2} + \frac{c^2}{\alpha^2}\right) = 0;$$

$$\frac{c}{\alpha} x_3 + \operatorname{arch} \frac{x_0}{\sqrt{x_0^2 - u^2}} = (x_0^2 - u^2)^{1/2} \left(\frac{c^2}{\alpha^2} + \frac{1}{x_0^2 - u^2}\right)^{1/2} -$$

$$- \ln \left[\frac{1}{(x_0^2 - u^2)^{1/2}} + \left(\frac{c^2}{\alpha^2} + \frac{1}{x_0^2 - u^2}\right)^{1/2}\right], \quad (c > 0);$$

14. $\frac{\varepsilon}{3} \left(2(\omega - x_3)\right)^{3/2} + \varepsilon x_3 \left(2(\omega - x_3)\right)^{1/2} = \varphi(\omega) - x_0,$

$$\omega = x_3 + \frac{(x_0 + u)^2}{2},$$

$$(\varphi')^2 - 2\omega - 1 = 0;$$

15. $x_0 + \frac{x_3}{\alpha}(x_0 + u) + \frac{(x_0 + u)^3}{3\alpha^2} = \varphi(\omega), \quad \omega = \alpha x_3 + \frac{(x_0 + u)^2}{2},$

$$(\varphi')^2 - \frac{1}{\alpha^2} - \frac{2\omega}{\alpha^4} = 0;$$

$$x_0 + \frac{x_0 + u}{\alpha} x_3 + \frac{(x_0 + u)^3}{3\alpha^2} = \frac{2\sqrt{2}}{3\alpha^2} \left[\alpha x_3 + \frac{(x_0 + u)^2}{2} + \frac{\alpha^2}{2}\right]^{3/2};$$

16. $\frac{(x_0 + u)^3}{3} + \varepsilon x_3 (x_0 + u) + \frac{1}{2}(x_0 - u) = \varphi(\omega),$

$$\omega = \frac{(x_0 + u)^2}{2} + \varepsilon x_3 \ \ (\varepsilon = \pm 1), \qquad (\varphi')^2 - 2\omega = 0;$$

$$\frac{(x_0 + u)^3}{3} + \varepsilon x_3 (x_0 + u) + \frac{1}{2}(x_0 - u) = \frac{2\sqrt{2}}{3} \left[\frac{(x_0 + u)^2}{2} + \varepsilon x_3\right]^{3/2} + c.$$

Ansatzes (11)–(16) can be written in the following form:

$$h(\omega, x) = f(x) \cdot \varphi(\omega) + g(x), \tag{3}$$

where $h(\omega, x)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the nonsplitting subgroups of the group $P(1,4)$.

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