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Symmetry Reduction of Nonlinear Equations of Classical Electrodynamics

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Abstract

Symmetry reduction of generalized Maxwell equations is carried out on the three-dimensional subgroup of the extended Poincaré group. Some their exact solutions are constructed.

1. Electromagnetic field is described by the familiar Maxwell equations that, with the help of a real covariant vector of electromagnetic potential $A = (A_0, A_1, A_2, A_3)$, can be presented in the form (see, e.g., [1])

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{1}$$

We use the notation:

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}, \qquad \partial_\mu = \frac{\partial}{\partial x_\mu},$$

and we sum over repeated indices (from 0 to 3). The raising and lowering of indices is performed with the help of the metric tensor $g = g_{\alpha\beta}$, where $g_{\alpha\beta} = \operatorname{diag}[1, -1, -1, -1]$.

The equation

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = F(A_\nu A^\nu)A_\mu \tag{2}$$

is a natural generalization of system (1) [2].

If $F = \lambda(A_\nu A^\nu)$, then equation (2) is invariant with respect to the algebra $AC(1,3)$ [2], [3] with generators

$$\begin{aligned} &P_\mu = \partial_\mu, \quad J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu + A^\mu\frac{\partial}{\partial A_\nu} - A^\nu\frac{\partial}{\partial A_\mu}, \quad D = x_\mu\partial_\mu - A_\mu\frac{\partial}{\partial A_\mu}, \\ &K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^\mu x_\nu\frac{\partial}{\partial A_\nu} - 2A_\nu x^\nu\frac{\partial}{\partial A_\mu}. \end{aligned} \tag{3}$$

If $F = -m^2 + \lambda(A_\nu A^\nu)$ and $m \neq 0$, then the maximal invariance algebra of equation (2) is the Poincaré algebra $AP(1,3) = \langle P_\mu, J_{\mu\nu} \mid \mu, \nu = 0, 1, 2, 3\rangle$. Note, that system (1) is also invariant with respect to the algebra $AC(1,3)$ with basis (3).

Yehorchenko [4] considered the problem of symmetry reduction of equations (1), (2) by subalgebras of the Poincaré algebra $AP(1,3)$. In this paper, we consider the problem of symmetry reduction of the system

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = \lambda(A_\nu A^\nu)A_\mu, \quad (\mu, \nu = 0, 1, 2, 3) \tag{4}$$

on subalgebras of the algebra $A\tilde{P}(1,3)$ to a system of ordinary differential equations.

2. The symmetry reduction of equation (4) to ordinary differential equations is carried out by subalgebras of the algebra $A\tilde{P}(1,3)$ of rank 3. The list of such subalgebras is known [7]:

$L_1 = \langle D, P_0, P_3 >$, $L_2 = \langle J_{12} + \alpha D, P_0, P_3 >$, $L_3 = \langle J_{12}, D, P_0\rangle$,
$L_4 = \langle J_{12}, D, P_3\rangle$, $L_5 = \langle J_{03} + \alpha D, P_0, P_3\rangle$, $L_6 = \langle J_{03} + \alpha D, P_1, P_2\rangle$,
$L_7 = \langle J_{03} + \alpha D, M, P_1\rangle$ $(\alpha \neq 0)$, $L_8 = \langle J_{03} + D + 2T, P_1, P_2\rangle$,
$L_9 = \langle J_{03} + D + 2T, M, P_1\rangle$, $L_{10} = \langle J_{03}, D, P_1\rangle$, $L_{11} = \langle J_{03}, D, M\rangle$,
$L_{12} = \langle J_{12} + \alpha J_{03} + \beta D, P_0, P_3\rangle$, $L_{13} = \langle J_{12} + \alpha J_{03} + \beta D, P_1, P_2\rangle$,
$L_{14} = \langle J_{12} + \alpha(J_{03} + D + 2T), P_1, P_2\rangle$, $L_{15} = \langle J_{12} + \alpha J_{03}, D, M\rangle$,
$L_{16} = \langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle (0 \leq |\alpha| \leq 1,\ \beta \geq 0, |\alpha| + |\beta| \neq 0)$,
$L_{17} = \langle J_{03} + D + 2T, J_{12} + \alpha T, M\rangle$ $(\alpha \geq 0)$, $L_{18} = \langle J_{03} + D, J_{12} + 2T, M\rangle$,
$L_{19} = \langle J_{03}, J_{12}, D\rangle$, $L_{20} = \langle G_1, J_{03} + \alpha D, P_2\rangle$ $(0 < |\alpha| \leq 1)$,
$L_{21} = \langle J_{03} + D, G_1 + P_2, M\rangle$, $L_{22} = \langle J_{03} - D + M, G_1, P_2\rangle$,
$L_{23} = \langle J_{03} + 2D, G_1 + 2T, M\rangle$, $L_{24} = \langle J_{03} + 2D, G_1 + 2T, P_2\rangle$,

where $M = P_0 + P_3$, $G_1 = J_{01} - J_{13}$, $T = \dfrac{1}{2}(P_0 - P_3)$, unless otherwise stated, $\alpha, \beta > 0$.

The structure of generators of the algebra $AP(1,3)$ (3) allows one to construct linear invariant ansatzes that correspond to subalgebras of the algebra $AP(1,3)$, ([5], [6])

$$A = \Lambda(x)B(\omega), \tag{5}$$

where $\Lambda(x)$ is a known nondegenerate square matrix of order 4, and $B(\omega)$ is a new unknown vector function for invariants of the subalgebra $\omega = \omega(x)$, $x = (x_0, x_1, x_2, x_3)$.

Using the approach suggested by Fushchych, Zhdanov, and Lahno [8], [9], ansatzes (5) for subalgebras of the extended Poincaré algebra $AP(1,3)$ can be represented in the form

$$A_\mu(x) = \theta(x)a_{\mu\nu}(x)B^\nu(\omega), \tag{6}$$

where $B^\nu = B^\nu(\omega)$ are new unknown functions of ω,

$$\begin{aligned} a_{\mu\nu} = {} & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\ & 2(a_\mu + d_\mu)[\theta_2 \cos\theta_1 b_\nu - \theta_2 \sin\theta_1 c_\nu + \theta_2^2 \exp(-\theta_0)(a_\nu + d_\nu)] + \\ & (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_1 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_1 - 2\exp(-\theta_0)\theta_2 b_\mu(a_\nu + d_\nu). \end{aligned} \tag{7}$$

Here $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary constants satisfying the following equalities:

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \quad a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0,$$

$\mu, \nu = 0, 1, 2, 3$. The form of the (non-zero) functions θ, θ_i $(i = 0, 1, 2)$, ω is determined by subalgebras L_j, $(j = \overline{1, 24})$ of the algebra $A\tilde{P}(1,3)$, and we given them below for each of these subalgebras.

$L_1 : \theta = |bx|^{-1}$, $\omega = cx(bx)^{-1}$;
$L_2 : \theta = \Psi_1^{-\frac{1}{2}}$, $\theta_1 = \Phi$, $\omega = \ln\Psi_1 + 2\Phi$;
$L_3 : \theta = |dx|^{-1}$, $\theta_1 = \Phi$, $\omega = \Psi_1(dx)^{-2}$;
$L_4 : \theta = |ax|^{-1}$, $\theta_1 = \Phi$, $\omega = \Psi_1(ax)^{-2}$;
$L_5 : \theta = |bx|^{-1}$, $\theta_0 = \alpha^{-1}\ln|bx|$, $\omega = cx(bx)^{-1}$;

$L_6: \theta = |\Psi_2|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|(ax-dx)(kx)^{-1}|, \quad \omega = |ax-dx|^{1-\alpha}|kx|^{1+\alpha};$
$L_7: \theta = |cx|^{-1},\ \theta_0 = \alpha^{-1}\ln|cx|, \quad \omega = |kx|^{\alpha}|cx|^{1-\alpha};$
$L_8: \theta = |ax-dx|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|ax-dx|, \quad \omega = kx - \ln|ax-dx|;$
$L_9: \theta = |cx|^{-1},\ \theta_0 = \ln|cx|, \quad \omega = kx - 2\ln|cx|;$
$L_{10}: \theta = |cx|^{-1},\ \theta_0 = \ln|(ax-dx)(cx)^{-1}|, \quad \omega = \Psi_2(cx)^{-2};$
$L_{11}: \theta = |cx|^{-1},\ \theta_0 = -\ln|kx(cx)^{-1}|, \quad \omega = cx(bx)^{-1};$
$L_{12}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\alpha\Phi,\ \theta_1 = \Phi,\ \omega = \ln\Psi_1 + 2\beta\Phi;$
$L_{13}: \theta = |\Psi_2|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|(ax-dx)(kx)^{-1}|,$
$\quad \theta_1 = -\frac{1}{2\alpha}\ln|(ax-dx)(kx)^{-1}|, \quad \omega = |ax-dx|^{\alpha-\beta}|kx|^{\alpha+\beta};$
$L_{14}: \theta = |ax-dx|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|ax-dx|,\ \theta_1 = -\frac{1}{2}\ln|ax-dx|,\ \omega = kx - \ln|ax-dx|;$
$L_{15}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\alpha\Phi, \quad \theta_1 = \Phi, \quad \omega = \ln[\Psi_1(kx)^{-2}] + 2\alpha\Phi;$
$L_{16}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln[\Psi_1(kx)^{-2}], \quad \theta_1 = \Phi, \quad \omega = \ln[\Psi_1^{1-\alpha}(kx)^{2\alpha}] + 2\beta\Phi;$
$L_{17}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln\Psi_1, \quad \theta_1 = \Phi, \quad \omega = kx - \ln\Psi_1 + 2\alpha\Phi;$
$L_{18}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln\Psi_1, \quad \theta_1 = \Phi,\ \omega = kx + 2\Phi;$
$L_{19}: \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\frac{1}{2}\ln|kx(ax-dx)^{-1}|, \quad \theta_1 = \Phi, \quad \omega = \Psi_1|\Psi_2|^{-1};$
$L_{20}: \theta = |\Psi_3|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2\alpha}\ln|\Psi_3|,\ \theta_2 = \frac{1}{2}bx(kx)^{-1},\ \omega = |kx|^{2\alpha}|\Psi_3|^{1-\alpha};$
$L_{21}: \theta = |cx\,kx - bx|^{-1},\ \theta_0 = \ln|cx\,kx - bx|, \quad \theta_2 = \frac{1}{2}cx,\ \omega = kx;$
$L_{22}: \theta = |kx|^{-\frac{1}{2}}, \quad \theta_0 = -\frac{1}{2}\ln|kx|, \quad \theta_2 = \frac{1}{2}bx(kx)^{-1},$
$\quad \omega = ax - dx + \ln|kx| - (bx)^2(kx)^{-1};$
$L_{23}: \theta = |cx|^{-1},\ \theta_0 = \frac{1}{2}\ln|cx|, \quad \theta_2 = -\frac{1}{4}kx,\ \omega = [4bx + (kx)^2](cx)^{-1};$
$L_{24}: \theta = |4bx + (kx)^2|^{-1},\ \theta_0 = \frac{1}{2}\ln|4bx + (kx)^2|,$
$\quad \theta_2 = -\frac{1}{4}kx,\ \omega = [ax - dx + bxkx + \frac{1}{6}(kx)^3]^2[4bx + (kx)^2]^{-3}.$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $kx = ax + dx$, $\Phi = \arctan\dfrac{cx}{bx}$, $\Psi_1 = (bx)^2 + (cx)^2$, $\Psi_2 = (ax)^2 - (dx)^2$, $\Psi_3 = (ax)^2 - (bx)^2 - (dx)^2$.

3. The covariant form of ansatz (6), (7) which we have obtained enables us to perform the $\tilde{P}(1,3)$-invariant reduction of equation (4) in general form.

Theorem. *Ansatz (6), (7) reduces equation (4) to the system of ODEs*

$$k_{\mu\gamma}\ddot{B}^\gamma + l_{\mu\gamma}\dot{B}^\gamma + m_{\mu\gamma}B^\gamma = \lambda(B^\gamma B_\gamma)B_\mu, \tag{8}$$

where

$$k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \qquad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu\dot{G}_\gamma,$$
$$m_{\mu\gamma} = R_{\mu\gamma}, \qquad \dot{B}^\gamma = \frac{dB^\gamma}{d\omega}, \qquad \ddot{B}^\gamma = \frac{d^2B^\gamma}{d\omega^2}, \qquad \dot{G}_\gamma = \frac{dG_\gamma}{d\omega}. \tag{9}$$

In (9), $F_1 = F_1(\omega)$, $F_2 = F_2(\omega)$, $G_\mu = G_\mu(\omega)$, $Q_{\mu\gamma} = Q_{\mu\gamma}(\omega)$, $H_\gamma = H_\gamma(\omega)$, $R_{\mu\gamma} = R_{\mu\gamma}(\omega)$ are smooth functions of ω and are determined from the relations

$$\frac{\partial\omega}{\partial x_\mu}\cdot\frac{\partial\omega}{\partial x^\mu} = F_1(\omega)\theta^2; \quad \theta\,\Box\omega + 2\frac{\partial\theta}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu} = F_2(\omega)\theta^3; \quad a_{\nu\mu}\frac{\partial\omega}{\partial x_\nu} = G_\mu(\omega)\theta;$$

$$\theta\frac{\partial a_{\nu\mu}}{\partial x_\nu}+3a_{\nu\mu}\frac{\partial\theta}{\partial x_\nu}=H_\mu(\omega)\theta^2;\quad a_\mu^\gamma\frac{\partial a_{\gamma\nu}}{\partial x^\delta}\frac{\partial\omega}{\partial x_\delta}+G_\mu(\omega)a_{\delta\nu}\frac{\partial\theta}{\partial x_\delta}-G_\nu(\omega)a_{\delta\mu}\frac{\partial\theta}{\partial x_\delta}=Q_{\mu\nu}(\omega)\theta^2;$$

$$g_{\mu\nu}\Box\theta+2a_\mu^\gamma\frac{\partial a_{\gamma\nu}}{\partial x^\delta}\frac{\partial\theta}{\partial x_\delta}-a_\mu^\gamma a_{\delta\nu}\frac{\partial^2\theta}{\partial x^\gamma\partial x_\delta}-a_\mu^\gamma\frac{\partial a_{\delta\nu}}{\partial x_\delta}\frac{\partial\theta}{\partial x^\gamma}-a_\mu^\gamma\frac{\partial a_{\delta\nu}}{\partial x^\gamma}\frac{\partial\theta}{\partial x_\delta}$$

$$+\theta(a_\mu^\gamma\Box a_{\gamma\nu}-a_\mu^\gamma\frac{\partial^2 a_{\delta\nu}}{\partial x^\gamma\partial x_\delta})=R_{\mu\nu}(\omega)\theta^3;$$

where $\mu,\nu,\gamma,\delta=0,1,2,3;$ $\Box$ *is the d'Alembertian.*

Using the results of the theorem for each subalgebra L_j ($j=\overline{1,24}$), we obtain the corresponding systems of ordinary differential equations which for the case of equation (4) are, in general, nonlinear. Integration of the reduced equations we obtain and substitution of their solutions into ansatz (6), (7) lead to solutions of the original system. We give below some typical solutions of both the linear Maxwell system (1) and the nonlinear equations (4).

Solutions of equations (1)

1) $$A_\mu=\frac{a_\mu}{2}\left\{G_1+\frac{E}{(bx)^2+(cx)^2}(AF_1+BF_2)\right\}+\frac{d_\mu}{2}\left\{G_1-\frac{E}{(bx)^2+(cx)^2}(AF_1+BF_2)\right\}+$$

$$\frac{c_\mu}{(bx)^2+(cx)^2}\left\{\frac{E}{2(1+\alpha^2)}[(\zeta B+\chi A)F_1+(\chi B-\zeta A)F_2]+\chi(G_1+C_1\omega_1+C_2)+\zeta C_3\right\}+$$

$$\frac{b_\mu}{(bx)^2+(cx)^2}\left\{\frac{E}{2(1+\alpha^2)}[(\zeta A-\chi B)F_1+(\zeta B+\chi A)F_2]+\zeta(G_1+C_1\omega_1+C_2)-\chi C_3\right\}.$$

Here

$$E=\exp\left(-\frac{\omega_1}{1+\alpha^2}\right);\quad F_1=\sin\left(\frac{\alpha\omega_1}{1+\alpha^2}\right);\quad F_2=\cos\left(\frac{\alpha\omega_1}{1+\alpha^2}\right).$$

2) $$A_\mu=C(a_\mu-d_\mu)\frac{(kx)^2}{\Psi}+k_\mu\left\{2C\varepsilon(kx)^{-2}+C_1\frac{bx}{kx}|\Psi|^{-\frac12}+C(bx)^2|\Psi|^{-1}\right\}+c_\mu C_2|\Psi|^{-\frac12}-$$

$$b_\mu\left\{2C\ bx\ kx\ |\Psi|^{-1}+C_1|\Psi|^{-\frac12}\right\},$$

3)

$$A_\mu=a_\mu\left\{\frac{\epsilon}{kx}(G_2+C_1\omega_2+C_2)-\epsilon\left[1+\frac{(bx)^2}{(kx)^2}\right](C_1\omega_2+C_2-G_2)-\frac{bx}{|kx|^{\frac32}}(C_3\omega_2+C_4)\right\}+$$

$$d_\mu\left\{\frac{\epsilon}{kx}(G_2+C_1\omega_2+C_2)+\epsilon\left[1-\frac{(bx)^2}{(kx)^2}\right](C_1\omega_2+C_2-G_2)-\frac{bx}{|kx|^{\frac32}}(C_3\omega_2+C_4)\right\}+$$

$$b_\mu\left\{2\epsilon\frac{bx}{kx}(C_1\omega_2+C_2-G_2)+|kx|^{-\frac12}(C_3\omega_2+C_4)\right\}+c_\mu|kx|^{-\frac12}(C_5\omega_2+C_6).$$

Solutions of equations (4)

$$A_\mu(x)=2\delta\ A_\mu^*(x);$$

where

$$\delta=\sqrt{-\frac{2}{3\lambda}}\quad\text{for }\epsilon=1\qquad\text{and}\quad\delta=\sqrt{\frac{2}{\lambda}}\quad\text{for }\epsilon=-1,$$

$$A^*_\mu = -\frac{\epsilon}{2}(a_\mu - d_\mu)\frac{1}{\omega_2 + C_0} + k_\mu \left\{ \frac{1}{2} \cdot \frac{1}{\omega_2 + C_0} \left(|kx|^{-1} - \epsilon \frac{(bx)^2}{(kx)^2} \right) - \epsilon |kx|^{-\frac{3}{2}} \cdot \frac{1}{\omega_2 + C_1} \right\} +$$
$$c_\mu \cdot \frac{|kx|^{-\frac{1}{2}}}{\omega_2 + C_2} + b_\mu \left\{ \frac{bx}{kx} \cdot \frac{\epsilon}{\omega_2 + C_0} + \frac{|kx|^{-\frac{1}{2}}}{\omega_2 + C_1} \right\}$$

We use the following notation:
$\zeta = \alpha cx + bx$; $\chi = \alpha bx - cx$; $\Psi = (ax)^2 - (bx)^2 - (dx)^2$; $kx = ax + dx$; $\varepsilon = 1$ for $\Psi > 0$ and $\varepsilon = -1$ for $\Psi < 0$; $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$;
$\omega_1 = kx - \ln[(bx)^2 + (cx)^2] + 2\alpha \arctan \frac{cx}{bx}$, $\omega_2 = ax - dx + \ln|kx| - (bx)^2(kx)^{-1}$.

$G_1 = G_1(\omega_1)$; $G_2 = G_2(\omega_2)$ are arbitrary smooth functions of ω_1 and ω_2, respectively. A, B, C, C_i $(i = \overline{0,6})$ are constants of integration.

References

[1] Bogolyubov N.N. and Shirokov D.V., Introduction to Theory of Quantized Fields, Moscow, Nauka, 1976 (in Russian).

[2] Fushchych W.I. and Tsyfra I.M., On the symmetry of nonlinear electrodynamics equations, *Theor. Math. Phys.*, 1985. V.64, N 1, 41–50

[3] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solution of Equations of Nonlinear Mathematical Physics, Kluwer Academic Publishers, Dordrecht, 1993.

[4] Yehorchenko I.A., Symmetry property of nonlinear equations for complex vector field, Preprint, Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1989 (in Russian).

[5] Fushchych W.I., Symmetry in mathematical physics problems, in: Algebraic-Theoretical Studies in Mathematical Physics, Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1981, 6–28.

[6] Fushchych W.I., On symmetry and exact solutions of multidimensional nonlinear wave equations, *Ukrain. Math. J.*, 1987. V.39, N 1, 116–123.

[7] Fushchych W.I., Barannyk L.F. and Barannyk A.F., Subgroup Analysis of the Galilei and Poincaré Groups and Reduction of Nonlinear Equations, Kyiv, Naukova Dumka, 1991 (in Russian).

[8] Lahno V., Zhdanov R. and Fushchych W., Symmetry Reduction and Exact Solutions of the Yang–Mills Equations, *J. Nonlin. Math. Phys.*, 1995, V.2, N 1, 51–72.

[9] Zhdanov R.Z., Lahno V.I. and Fushchych W.I., Reduction of the self-dual Yang–Mills equations. I.The Poincaré group, *Ukrain. Math. J.*, 1995, V.47, N 4, 456–462.