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On Lie Reduction of the MHD Equations to Ordinary Differential Equations

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Abstract

The MHD equations describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity are reduced to ordinary differential equations by means of Lie symmetries.

The MHD equations (the MHDEs) describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity have the following form:

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p + \vec{H}\times \mathrm{rot}\,\vec{H} = \vec{0}, \\ &\vec{H}_t - \mathrm{rot}\,(\vec{u}\times\vec{H}) - \nu_m\triangle\vec{H} = \vec{0}, \qquad \mathrm{div}\,\vec{u} = 0, \qquad \mathrm{div}\,\vec{H} = 0. \end{aligned} \tag{1}$$

System (1) is very complicated and the construction of new exact solutions is a difficult problem. Following [1], in this paper we reduce the MHDEs (1) to ordinary differential equations by means of three-dimensional subalgebras of the maximal Lie invariance algebra of the MHDEs.

In (1) and below, $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{H} = \{H^a(t,\vec{x})\}$ denotes the magnetic intensity, ν_m is the coefficient of magnetic viscosity, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The kinematic coefficient of viscosity and fluid density are set equal to unity, permeability is done $(4\pi)^{-1}$. Subscript of a function denotes differentiation with respect to the corresponding variables. The maximal Lie invariance algebra of the MHDEs (1) is an infinite-dimensional algebra $A(\mathrm{MHD})$ with the basis elements (see [2])

$$\begin{aligned} &\partial_t, \quad D = t\partial_t + \tfrac{1}{2}x_a\partial_a - \tfrac{1}{2}u^a\partial_{u^a} - \tfrac{1}{2}H^a\partial_{H^a} - p\partial_p, \\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} + H^a\partial_{H^b} - H^b\partial_{H^a}, \ a<b, \\ &R(\vec{m}) = R(\vec{m}(t)) = m^a\partial_a + m^a_t\partial_{u^a} - m^a_{tt}x_a\partial_p, \quad Z(\eta) = Z(\eta(t)) = \eta\partial_p, \end{aligned} \tag{2}$$

where $m^a = m^a(t)$ and $\eta = \eta(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbf{R})$). We sum over repeated indices. The indices a, b take values in $\{1,2,3\}$ and the indices i, j in $\{1,2\}$. The algebra $A(\mathrm{MHD})$ is isomorphic to the maximal Lie invariance algebra $A(\mathrm{NS})$ of the Navier-Stokes equations [3, 4, 5].

Besides continuous transformations generated by operators (2), the MHDEs admit discrete transformations I_b of the form

$$\begin{aligned} &\tilde{t} = t, \qquad \tilde{x}_b = -x_b, \qquad \tilde{x}_a = x_a, \\ &\tilde{p} = p, \quad \tilde{u}^b = -u^b, \quad \tilde{H}^b = -H^b, \quad \tilde{u}^a = u^a, \quad \tilde{H}^a = H^a, \ a \neq b, \end{aligned}$$

where b is fixed.

We construct a complete set of A(MHD)-inequivalent three-dimensional subalgebras of A(MHD) and choose those algebras from this set which can be used to construct ansatzes for the MHD field. The list of the classes of these algebras is given below.

1. $A^3_1 = \langle D,\ \partial_t,\ J_{12}\rangle$.

2. $A^3_2 = \langle D + \frac{1}{2}\kappa J_{12},\ \partial_t,\ R(0,0,1)\rangle$,

where $\kappa \geq 0$. Here and below, κ, σ_{ij}, ε_i, μ, and ν are real constants.

3. $A^3_3 = \langle D,\ J_{12} + R(0,0,\nu|t|^{1/2}\ln|t|) + Z(\nu\varepsilon_2|t|^{-1}\ln|t| + \varepsilon_1|t|^{-1}),$
$R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon_2|t|^{\sigma-1})\rangle$,

where $\sigma\nu = 0$, $\sigma\varepsilon_2 = 0$, $\varepsilon_1 \geq 0$, and $\nu \geq 0$.

4. $A^3_4 = \langle \partial_t,\ J_{12} + R(0,0,\nu t) + Z(\nu\varepsilon_2 t + \varepsilon_1),\ R(0,0,e^{\sigma t}) + Z(\varepsilon_2 e^{\sigma t})\rangle$,

where $\sigma\nu = 0$, $\sigma\varepsilon_2 = 0$, $\sigma \in \{-1;0;1\}$, and, if $\sigma = 0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0;1\}.$$

5. $A^3_5 = \langle D + \kappa J_{12},\ R(|t|^{1/2} f^{ij}(t)\hat{A}(t)\vec{e}_j) + Z(|t|^{-1} f^{ij}(t)\varepsilon_j), i = 1,2\rangle$,

6. $A^3_6 = \langle \partial_t + \kappa J_{12},\ R(f^{ij}(t)\check{A}(t)\vec{e}_j) + Z(f^{ij}(t)\varepsilon_j), i = 1,2\rangle$,

Here, $(f^{1j}(t), f^{2j}(t)), j = 1,2$, are solutions of the Cauchy problems

$$f^{ij}_\tau = \sigma_{ik} f^{kj}, \quad f^{ij}(0) = \delta_{ij},$$

where $\tau = \ln|t|$ in the case of the algebra A^3_5 and $\tau = t$ in the case of the algebra A^3_6. $\vec{e}^i = \mathrm{const} : \vec{e}^i \cdot \vec{e}^j = \delta_{ij}$, δ_{ij} is the Kronecker delta,

$$\sigma_{ii}(\sigma_{12} - \sigma_{21} - 2\kappa K\vec{e}_1 \cdot \vec{e}_2) = 0, \quad \varepsilon_i(\sigma_{ik}\sigma_{kj}\vec{e}_j - 2\kappa\sigma_{ij}K\vec{e}_j) = \vec{0}, \quad \sigma_{12} + \sigma_{21} = 0.$$

$$A(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta & 0 \\ \sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} \hat{A}(t) = A(\kappa\ln|t|), \\ \check{A}(t) = A(\kappa t). \end{array}$$

$\kappa\varepsilon_l K\vec{e}_k \cdot \vec{e}_l = 0$ if $\sigma_{kj} = 0$ and $\varepsilon_k(2\kappa K\vec{e}_k \cdot \vec{e}_l - \sigma_{kl}) = 0$ if $\sigma_{kj} = \sigma_{ll} = 0$, where k and l take fixed values from $\{1;2\}$, $k \neq l$. To simplify parameters in A^3_5 and A^3_6, one can also use the adjoint actions generated by I_b, D, J_{12}, and, if $\kappa = 0$, J_{23} and J_{31}.

7. $A^3_7 = \langle J_{12} + R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)\rangle$,

where η^a are smooth functions of t, $\eta^i\eta^i \neq 0$, $\eta^3 \neq 0$, $\eta^1_{tt}\eta^2 - \eta^1\eta^2_{tt} = 0$. The algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde{\eta}^1,\tilde{\eta}^2,\tilde{\eta}^3)$ are equivalent if $\exists E_i \in \mathbf{R}\backslash\{0\}$, $\exists\delta \in \mathbf{R}$, $\exists(b_{ij}) \in O(2)$:

$$\tilde{\eta}^i(\tilde{t}) = E_2 b_{ij}\eta^i(t), \quad \tilde{\eta}^3(\tilde{t}) = E_1\eta^3(t),$$

where $t = E_1^2\tilde{t} + \delta$.

8. $A^3_8 = \langle R(\vec{m}^a),\ a = \overline{1,3}\rangle$,

where $\vec{m}^a$ are smooth functions of t, rank $(\vec{m}^1,\vec{m}^2,\vec{m}^3) = 3$, $\vec{m}^a_{tt} \cdot \vec{m}^b - \vec{m}^a \cdot \vec{m}^b_{tt} = 0$. The algebras $A^3_8(\vec{m}^1,\vec{m}^2,\vec{m}^3)$ and $A^3_8(\vec{\tilde{m}}^1,\vec{\tilde{m}}^2,\vec{\tilde{m}}^3)$ are equivalent if $\exists E_1 \in \mathbf{R}\backslash\{0\}$, $\exists\delta \in \mathbf{R}$, $\exists B \in O(3)$, and $\exists(d_{ab}) : \det(d_{ab}) \neq 0$ such that

$$\vec{\tilde{m}}^a(\tilde{t}) = d_{ab}B\vec{m}^b(t),$$

where $t = E_1^2\tilde{t} + \delta$.

The way of obtaining (and the form of writing down) the algebras $A^3_1 - A^3_8$ differs slightly from the one used in [1].

By means of subalgebras $A^3_1 - A^3_8$, one can construct the following ansatzes that reduce the MHDEs to ODEs:

1. $u^1 = x_1R^{-2}\varphi^1 - x_2(Rr)^{-1}\varphi^2 + x_1x_3r^{-1}R^{-2}\varphi^3,$
   $u^2 = x_2R^{-2}\varphi^1 + x_1(Rr)^{-1}\varphi^2 + x_2x_3r^{-1}R^{-2}\varphi^3,$
   $u^3 = x_3R^{-2}\varphi^1 - rR^{-2}\varphi^3,$
   $p\ = R^{-2}h,$

where $\omega = \arctan r/x_3$, the expressions for H^a are obtained by means of the substitution of ψ^a for φ^a in the expressions for u^a.

Here and below, $\varphi^a = \varphi^a(\omega)$, $\psi^a = \psi^a(\omega)$, $h = h(\omega)$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $r = (x_1^2 + x_2^2)^{1/2}$. The numeration of ansatzes and reduced systems corresponds to that of the algebras above. All the parameters satisfy the equations given for these algebras.

2. $u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2), \quad u^3 = r^{-1}\varphi^3, \quad p = r^{-2}h,$

where $\omega = \arctan x_2/x_1 - \kappa \ln r$, the expressions for H^a are obtained by means of substituting ψ^a for φ^a in the expressions for u^a.

3. $u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2) + \frac{1}{2}x_1t^{-1},$
   $u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2) + \frac{1}{2}x_2t^{-1},$
   $u^3 = |t|^{-1/2}\varphi^3 + (\sigma + \frac{1}{2})x_3t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1,$
   $H^1 = r^{-2}(x_1\psi^1 - x_2\psi^2), \quad H^2 = r^{-2}(x_2\psi^1 + x_1\psi^2), \quad H^3 = |t|^{-1/2}\psi^3,$
   $p\ = |t|^{-1}h + \frac{1}{8}x_ax_at^{-2} - \frac{1}{2}\sigma^2x_3^2t^{-2} + \varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2x_3|t|^{-3/2},$

where $\omega = |t|^{-1/2}r$.

4. $u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2),$
   $u^3 = \varphi^3 + \sigma x_3 + \nu\arctan x_2/x_1,$
   $H^1 = r^{-2}(x_1\psi^1 - x_2\psi^2), \quad H^2 = r^{-2}(x_2\psi^1 + x_1\psi^2), \quad H^3 = \psi^3,$
   $p\ = h - \frac{1}{2}\sigma^2x_3^2 + \varepsilon_1\arctan x_2/x_1 + \varepsilon_2x_3,$

where $\omega = r$.

5. $$\vec{u} = |t|^{1/2}t^{-1}\hat{A}(t)\vec{v} + \tfrac{1}{2}\vec{x}t^{-1} - \kappa K\vec{x}t^{-1}, \quad \vec{H} = |t|^{1/2}t^{-1}\hat{A}(t)\vec{G},$$
   $$p = |t|^{-1}q + \tfrac{1}{8}x_ax_at^{-2} + \tfrac{1}{2}\kappa^2x_ix_it^{-2}, \qquad (3)$$

where $\vec{y} = |t|^{-1/2}\hat{A}(t)^T\vec{x}$.

6. $$\vec{u} = \check{A}(t)\vec{v} - \kappa K\vec{x}, \quad \vec{H} = \check{A}(t)\vec{G}, \quad p = q + \tfrac{1}{2}\kappa^2x_ix_i, \qquad (4)$$

where $\vec{y} = \check{A}(t)^T\vec{x}$.

In (3) and (4) $\vec{v}$, $\vec{G}$, q, and ω are defined by means of the following formulas:

$$\vec{v} = \vec{\varphi}(\omega) + \sigma_{ij}(\vec{e}_i\cdot\vec{y})\vec{e}_j, \quad \vec{G} = \vec{\psi}(\omega),$$
$$q = h(\omega) + \varepsilon_i(\vec{e}_i\cdot\vec{y}) - \tfrac{1}{2}(\vec{d}_i\cdot\vec{y})(\vec{e}_i\cdot\vec{y}) - \tfrac{1}{2}(\vec{d}_i\cdot\vec{e}_3)\omega(\vec{e}_i\cdot\vec{y}),$$

where $\omega = (\vec{e}_3\cdot\vec{y})$, $\vec{e}_3 = \vec{e}_1\times\vec{e}_2$, $\vec{d}_i = \sigma_{ik}\sigma_{kj}\vec{e}_j - 2\kappa\sigma_{ij}K\vec{e}_j$.

7. $$u^1 = \varphi^1\cos z - \varphi^2\sin z + x_1\theta^1 + x_2\theta^2,$$
$$u^2 = \varphi^1\sin z + \varphi^2\cos z - x_1\theta^2 + x_2\theta^1,$$
$$u^3 = \varphi^3 + \eta^3_t(\eta^3)^{-1}x_3,$$
$$H^1 = \psi^1\cos z - \psi^2\sin z, \quad H^2 = \psi^1\sin z + \psi^2\cos z, \quad H^3 = \psi^3,$$
$$p = h - \tfrac{1}{2}\eta^3_{tt}(\eta^3)^{-1}x_3^2 - \tfrac{1}{2}\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2,$$

where $\omega = t$, $\theta^1 = \eta^i_t\eta^i(\eta^j\eta^j)^{-1}$, $\theta^2 = (\eta^1_t\eta^2 - \eta^1\eta^2_t)(\eta^j\eta^j)^{-1}$, $z = x_3/\eta^3$.

8. $$\vec{u} = \vec{\varphi} + \lambda^{-1}(\vec{n}^a\cdot\vec{x})\vec{m}^a_t, \quad \vec{H} = \vec{\psi},$$
$$p = h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{x})(\vec{n}^a\cdot\vec{x}) + \tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{x})(\vec{n}^b\cdot\vec{x}),$$

where $\omega = t$, $\vec{m}^a_{tt}\cdot\vec{m}^b - \vec{m}^a\cdot\vec{m}^b_{tt} = 0$, $\lambda = \vec{m}^1\vec{m}^2\vec{m}^3 \neq 0$,

$$\vec{n}^1 = \vec{m}^2\times\vec{m}^3, \quad \vec{n}^2 = \vec{m}^3\times\vec{m}^1, \quad \vec{n}^3 = \vec{m}^1\times\vec{m}^2.$$

Substituting the ansatzes 1–8 into the MHDEs, we obtain the following systems of ODE in the functions φ^a, ψ^a, and h:

1. $$\varphi^3\varphi^1_\omega - \psi^3\psi^1_\omega - \varphi^1_{\omega\omega} - \varphi^1_\omega\cot\omega - \varphi^a\varphi^a - 2h = 0,$$
$$\varphi^3\varphi^2_\omega - \psi^3\psi^2_\omega - \varphi^2_{\omega\omega} - \varphi^2_\omega\cot\omega + \varphi^2\sin^{-2}\omega + (\varphi^3\varphi^2 - \psi^3\psi^2)\cot\omega = 0,$$
$$\varphi^3\varphi^3_\omega - \psi^3\psi^3_\omega - \varphi^3_{\omega\omega} - \varphi^3_\omega\cot\omega + \varphi^3\sin^{-2}\omega - 2\varphi^1_\omega + h_\omega + \psi^a_\omega\psi^a - ((\varphi^2)^2 - (\psi^2)^2)\cot\omega = 0,$$
$$\varphi^3\psi^1_\omega - \psi^3\varphi^1_\omega - \nu_m(\psi^1_{\omega\omega} + \psi^1_\omega\cot\omega) = 0,$$
$$\varphi^3\psi^2_\omega - \psi^3\varphi^2_\omega - \nu_m(\psi^2_{\omega\omega} + \psi^2_\omega\cot\omega - \psi^2\sin^{-2}\omega) + 2(\psi^1\varphi^2 - \psi^2\varphi^1) + (\psi^3\varphi^2 - \psi^2\varphi^3)\cot\omega = 0,$$
$$\varphi^3\psi^3_\omega - \psi^3\varphi^3_\omega - \nu_m(\psi^3_{\omega\omega} + \psi^3_\omega\cot\omega - \psi^3\sin^{-2}\omega + 2\psi^1_\omega) + 2(\psi^1\varphi^3 - \psi^3\varphi^1) = 0,$$
$$\varphi^3_\omega + \varphi^3\cot\omega + \varphi^1 = 0, \quad \psi^3_\omega + \psi^3\cot\omega + \psi^1 = 0.$$

2. $$\tilde{\varphi}^2\varphi^1_\omega - \tilde{\psi}^2\psi^1_\omega - \nu\varphi^1_{\omega\omega} - \varphi^i\varphi^i + \psi^i\psi^i - 2\tilde{h} - \kappa\tilde{h}_\omega = 0,$$
$$\tilde{\varphi}^2\varphi^2_\omega - \tilde{\psi}^2\psi^2_\omega - \nu\varphi^2_{\omega\omega} - 2(\varphi^1_\omega + \kappa\varphi^2_\omega) + \tilde{h}_\omega = 0,$$
$$\tilde{\varphi}^2\varphi^3_\omega - \tilde{\psi}^2\psi^3_\omega - \nu\varphi^3_{\omega\omega} - \varphi^1\varphi^3 + \psi^1\psi^3 - 2\kappa\varphi^3_\omega - \varphi^3 = 0,$$
$$\tilde{\varphi}^2\psi^1_\omega - \tilde{\psi}^2\varphi^1_\omega - \tilde{\nu}\psi^1_{\omega\omega} = 0,$$
$$\tilde{\varphi}^2\psi^2_\omega - \tilde{\psi}^2\varphi^2_\omega - \tilde{\nu}\psi^2_{\omega\omega} + 2(\psi^1\varphi^2 - \psi^2\varphi^1) - 2\nu_m(\psi^1_\omega + \kappa\psi^2_\omega) = 0,$$
$$\tilde{\varphi}^2\psi^3_\omega - \tilde{\psi}^2\varphi^3_\omega - \tilde{\nu}\psi^3_{\omega\omega} + \psi^1\varphi^3 - \psi^3\varphi^1 - \nu_m(2\kappa\psi^3_\omega + \psi^3) = 0,$$
$$\tilde{\varphi}^2_\omega = 0, \quad \tilde{\psi}^2_\omega = 0,$$

where $\tilde\varphi^2=\varphi^2-\kappa\varphi^1$, $\tilde\psi^2=\psi^2-\kappa\psi^1$, $\tilde h=h+\frac12\psi^a\psi^a$, $\nu=1+\kappa^2$, $\tilde\nu=\nu_m(1+\kappa^2)$.

3-4. $$\omega^{-1}(\varphi^1\varphi^1_w-\psi^1\psi^1_\omega)-\varphi^1_{\omega\omega}+\omega^{-1}\varphi^1_\omega-\omega^{-2}(\varphi^i\varphi^i-\psi^i\psi^i)+ \omega(h+\tfrac12\omega^{-2}\psi^i\psi^i+\tfrac12(\psi^3)^2)_\omega=0,$$
$$\omega^{-1}(\varphi^1\varphi^2_w-\psi^1\psi^2_\omega)-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega+\varepsilon_1=0,$$
$$\omega^{-1}(\varphi^1\varphi^3_w-\psi^1\psi^3_\omega)-\varphi^3_{\omega\omega}-\omega^{-1}\varphi^3_\omega+\nu\varepsilon\omega^{-2}\varphi^2+\varepsilon\sigma\varphi^3+\varepsilon_2=0,$$
$$\omega^{-1}(\varphi^1\psi^1_w-\psi^1\varphi^1_\omega)-\nu_m(\psi^1_{\omega\omega}-\omega^{-1}\psi^1_\omega)-\varepsilon\delta\psi^1=0,$$
$$\omega^{-1}(\varphi^1\psi^2_w-\psi^1\varphi^2_\omega)-\nu_m(\psi^2_{\omega\omega}-\omega^{-1}\psi^2_\omega)+2\omega^{-2}(\psi^1\varphi^2-\psi^2\varphi^1)-\varepsilon\delta\psi^2=0,$$
$$\omega^{-1}(\varphi^1\psi^3_w-\psi^1\varphi^3_\omega)-\nu_m(\psi^3_{\omega\omega}+\omega^{-1}\psi^3_\omega)-\nu\varepsilon\omega^{-2}\psi^2-(\sigma+\delta)\varepsilon\psi^3=0,$$
$$\varphi^1_\omega+(\sigma+\tfrac32\delta)\varepsilon\omega=0,\quad \psi^1_\omega=0,$$
where $\varepsilon=\operatorname{sign}t$ and $\delta=1$ in case 3 and $\varepsilon=1$ and $\delta=0$ in case 4.

5-6. $$\varphi^3\vec\varphi_\omega-\psi^3\vec\psi_\omega+\sigma_{ij}\varphi^i\vec e_j-2\kappa K\vec\varphi-\varepsilon\vec\varphi_{\omega\omega}+(h_\omega+\vec\psi\cdot\vec\psi_\omega)\vec e_3+ \varepsilon_i\vec e_i+2\kappa\omega\sigma_{ij}(K\vec e_j\cdot\vec e_3)\vec e_i=\vec0,$$
$$\varphi^3\vec\psi_\omega-\psi^3\vec\varphi_\omega-\sigma_{ij}\psi^i\vec e_j-\nu_m\varepsilon\vec\varphi_{\omega\omega}-\delta\vec\psi=\vec0,$$
$$\varphi^3_\omega+\sigma_{ii}+\tfrac32\delta=0,\quad \psi^3_\omega=0,$$
where $\varphi^a=\vec e_a\cdot\vec\varphi$, $\psi^a=\vec e_a\cdot\vec\psi$;
$\varepsilon=\operatorname{sign}t$ and $\delta=1$ in case 5 and $\varepsilon=1$ and $\delta=0$ in case 6.

7. $$\varphi^1_\omega+\varphi^1(\theta^1+(\eta^3)^{-2})+\varphi^2(\theta^2-(\eta^3)^{-1}\varphi^3)+\psi^3(\eta^3)^{-1}\psi^2=0,$$
$$\varphi^2_\omega-\varphi^1(\theta^2-(\eta^3)^{-1}\varphi^3)+\varphi^2(\theta^1+(\eta^3)^{-2})-\psi^3(\eta^3)^{-1}\psi^1=0,$$
$$\varphi^3_\omega+\eta^3_w(\eta^3)^{-1}\varphi^3=0,$$
$$\psi^1_\omega+\psi^1(-\theta^1+\nu_m(\eta^3)^{-2})-\psi^2(\theta^2+(\eta^3)^{-1}\varphi^3)+\psi^3(\eta^3)^{-1}\varphi^2=0,$$
$$\psi^2_\omega+\psi^1(\theta^2+(\eta^3)^{-1}\varphi^3)+\psi^2(-\theta^1+\nu_m(\eta^3)^{-2})-\psi^3(\eta^3)^{-1}\varphi^1=0,$$
$$\psi^3_w-\psi^3\eta^3_w(\eta^3)^{-1}=0,\quad 2\theta^1+\eta^3_w(\eta^3)^{-1}=0.$$

8. $$\vec\varphi_\omega+\lambda^{-1}(\vec n^a\cdot\vec\varphi)\vec m^a_\omega=\vec0,\quad \vec\psi_\omega-\lambda^{-1}(\vec n^a\cdot\vec\psi)\vec m^a_\omega=\vec0,\quad \vec n^a\cdot\vec m^a_\omega=0.$$

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