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Generalization of Translation Flows of an Ideal Incompressible Fluid: a Modification of the "Ansatz" Method

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Abstract

New exact solutions of the Euler equations describing flows of an ideal homogeneous incompressible fluid are obtained by means of a modification of the "ansatz" method.

In recent years, several new methods for finding exact solutions of the partial differential equations have been developed. Often these methods are generalizations of older ones and are reduced to either appending additional differential equations (the method of differential constraints, side conditions, conditional symmetry, and so on) or to assuming a general form for the solution (the "ansatz" method called often the direct method, and the generalization of the usual "separation of variables" technique). Both approaches are closely connected with each other.

In our paper [1], the Euler equations (the EEs)

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{0}, \quad \operatorname{div}\vec{u} = 0 \tag{1}$$

which describe flows of an ideal homogeneous incompressible fluid were considered with the following additional condition:

$$u^1_1 = u^3 = 0. \tag{2}$$

All the solutions of system (1)–(2) were found. They can be interpreted as a particular case of translation flows. In this paper, we construct more general classes of exact solutions for EEs (1) by means of a modification of the "ansatz" method.

Let us transform the variables in (1):

$$\begin{aligned} &\vec{u} = O(t)\vec{w}(\tau,\vec{y}) - O(t)O_t^T(t)\vec{x}, \quad p = q(\tau,\vec{y}) + \frac{1}{2}|O_t^T(t)\vec{x}|^2, \\ &\tau = t, \quad \vec{y} = O^T(t)\vec{x}, \end{aligned} \tag{3}$$

where $O = O(t)$ is an orthogonal matrix function depending on t, i.e., transformation (3) defines time-depending space rotation. It can be noted that the non-Lie invariance of hydrodynamics equations under transformations of the type (3) was investigated, for instance, in [2, 3]. As a result of transformation (3), we obtain equations in new unknown functions $\vec{w}$ and q and new independent variables τ and $\vec{y}$:

$$\vec{w}_\tau + (\vec{w}\cdot\nabla)\vec{w} + \nabla q - 2\vec{\gamma}\times\vec{w} - \vec{\gamma}_t\times\vec{y} = \vec{0}, \tag{4}$$

$$\operatorname{div}\vec{w} = 0, \tag{5}$$

where the vector function $\vec{\gamma} = \vec{\gamma}(t)$ is defined by means of the formula

$$\vec{\gamma}\times\vec{z} = O_t^T O\vec{z} \quad \forall z, t.$$

(The matrix $O_t^T O$ being antisymmetric, the vector $\vec{\gamma}$ exists.)

In fact, instead of equation (4), we investigate its differential consequence

$$(\text{rot}\ \vec{w})_\tau + (\vec{w}\cdot\nabla)\text{rot}\ \vec{w} - (\text{rot}\ \vec{w}\cdot\nabla)\vec{w} + 2(\vec{\gamma}\cdot\nabla)\vec{w} - 2\vec{\gamma}_\tau = \vec{0}. \tag{6}$$

Equation (4) will be used only to find the expression for p. To simplify solutions of (4)–(5), we make transformations generated by a Lie symmetry operator of the form

$$\tilde{R}(\vec{n}) = n^a\partial_{y_a} + n^a_\tau\partial_{w^a} - (\vec{n}_{\tau\tau} - 2\gamma\times\vec{n}_\tau - \vec{\gamma}_\tau\times\vec{n})\cdot\vec{y}\,\partial_q, \tag{7}$$

where $\vec{n}$ is an arbitrary smooth vector function of τ. The vector function $\vec{w}$ is to be found in the form

$$\begin{aligned} w^1 &= v^1(\tau, y_2, y_3) + \alpha^1(\tau)y_1,\\ w^2 &= v^2(\tau, y_1, y_3) + \alpha^2(\tau)y_2,\\ w^3 &= \beta^i(\tau)y_i + \alpha^3(\tau)y_3, \end{aligned} \tag{8}$$

where $v^1_{22}v^2_{11} \neq 0$, i.e., it is to satisfy the additional conditions

$$w^1_{1a} = w^2_{2a} = w^3_{ab} = 0, \quad w^1_{22}w^2_{11} \neq 0.$$

Note that the functions v^i depend on the different "similarity" variables.

Here and below, we sum over repeated indices. Subscript of a function denotes differentiation with respect to the corresponding variables. The indices a, b take values in $\{1,2,3\}$ and the indices i, j in $\{1,2\}$.

It follows from (5) that $\alpha^3 = -(\alpha^1+\alpha^2)$. Substituting (8) into (6), we obtain the equations to find the functions v^i, β^i, α^i and γ^a:

$$\begin{aligned}
&\beta^2_\tau - v^2_{3\tau} - v^2_{31}(v^1+\alpha^1y_1) - (\beta^iy_i - (\alpha^1+\alpha^2)y_3)v^2_{33} - (\beta^2 - v^2_3)\alpha^1 + \beta^1v^1_2 -\\
&\quad v^2_1v^1_3 + 2\gamma^1\alpha^1 + 2\gamma^2v^1_2 + 2\gamma^3v^1_3 - 2\gamma^1_t = 0,\\
&v^1_{3\tau} - \beta^1_\tau + (v^2+\alpha^2y_2)v^1_{32} + (\beta^iy_i - (\alpha^1+\alpha^2)y_3)v^1_{33} - \beta^2v^2_1 - (v^1_3 - \beta^1)\alpha^2 +\\
&\quad v^1_2v^2_3 + 2\gamma^1v^2_1 + 2\gamma^2\alpha^2 + 2\gamma^3v^2_3 - 2\gamma^2_t = 0,\\
&v^2_{1\tau} - v^1_{2\tau} + (v^1+\alpha^1y_1)v^2_{11} - (v^2+\alpha^2y_2)v^1_{22} + v^2_3\beta^1 - v^1_3\beta^2 +\\
&\quad (v^2_1 - v^1_2)(\alpha^1+\alpha^2) + (\beta^iy_i - (\alpha^1+\alpha^2)y_3)(v^2_{13} - v^1_{23}) +\\
&\quad 2\gamma^1\beta^1 + 2\gamma^2\beta^2 - 2\gamma^3(\alpha^1+\alpha^2) - 2\gamma^3_t = 0.
\end{aligned} \tag{9}$$

Unlike the "ansatz" method, we do not demand realizing the reduction conditions in system (9). The differential consequences of system (9) are the equations

$$v^1_{22}v^2_{1111} = v^2_{11}v^1_{2222},$$

i.e.,

$$\frac{v^1_{2222}}{v^1_{22}} = \frac{v^2_{1111}}{v^2_{11}} := h = h(t, x_3), \tag{10}$$

and

$$(v^1_{22}v^2_{11})_3 = 0. \tag{11}$$

Consider the particular cases.

Case I. $h > 0$. Let $k := h^{1/2}$. Then equation (10) gives that

$$\begin{aligned} v^1 &= f^1 e^{ky_2} + f^2 e^{-ky_2} + f^3 y_2 + f^4, \\ v^2 &= g^1 e^{ky_1} + g^2 e^{-ky_1} + g^3 y_1 + g^4, \end{aligned} \tag{12}$$

where $f^m = f^m(\tau, y_3)$, $g^m = g^m(\tau, y_3)$, $m = \overline{1,4}$, $f^i f^i \neq 0$, $g^i g^i \neq 0$. It follows from (11) that

$$k_3 = 0, \quad (f^i g^j)_3 = 0.$$

Therefore, there exist the functions $\mu^i = \mu^i(\tau)$, $\nu^i = \nu^i(\tau)$, $f = f(\tau, y_3)$, and $g = g(\tau, y_3)$ such that

$$f^i = \mu^i f, \quad g^i = \nu^i g.$$

Substituting expression (12) for v^i into system (9) and using a linear independence of the functions

$$y_i e^{k(\pm y_2 \pm y_1)}, \quad e^{k(\pm y_1 \pm y_2)}, \quad y_i^2, \quad y_1 y_2, \quad y_i, \quad \text{and} \quad 1,$$

we obtain the complicated system for the rest of functions:

$$\begin{aligned}
&\nu^i(\beta^2 g_3 - (-1)^i f^3 g) = 0, \quad \mu^i(\beta^1 f_3 - (-1)^i k g^3 f) = 0, \\
&\nu^i((k_\tau + \alpha^1 k) g - (-1)^i \beta^1 g_3) = 0, \quad \mu^i((k_\tau + \alpha^2 k) f - (-1)^i \beta^2 f_3) = 0, \\
&\nu^i(\beta^2 g_{33} + (\beta^2 - 2\gamma^1) k^2 g + (-1)^i \gamma^3 k g_3) = 0, \\
&\mu^i(\beta^1 f_{33} + (\beta^1 + 2\gamma^2) k^2 f - (-1)^i \gamma^3 k f_3) = 0, \\
&\nu^i_\tau g + \nu^i(g_\tau - (-1)^i k f^4 g - (\alpha^1 + \alpha^2) y_3 g_3 + \alpha^2 g) = 0, \\
&\mu^i_\tau f + \mu^i(f_\tau - (-1)^i k g^4 f - (\alpha^1 + \alpha^2) y_3 f_3 + \alpha^1 f) = 0, \\
&-(f^3 g^3)_3 - \beta^2 g^4_{33} + 2\gamma^3 f^3_3 = 0, \quad (f^3 g^3)_3 + \beta^1 f^4_{33} + 2\gamma^3 g^3_3 = 0, \\
&f^3_{3\tau} - (\alpha^1 + \alpha^2) y_3 f^3_{33} + \beta^2 f^4_{33} = 0, \quad g^3_{3\tau} - (\alpha^1 + \alpha^2) y_3 g^3_{33} + \beta^1 g^4_{33} = 0, \\
&\beta^1(f^3_3 - 2g^3_3) = 0, \quad \beta^2(2f^3_3 - g^3_3) = 0, \quad \beta^i f^3_{33} = \beta^i g^3_{33} = 0, \\
&\beta^2_\tau - g^4_{3\tau} - (f^4 g^3)_3 + (\alpha^1 + \alpha^2) y_3 g^4_{33} + \alpha^1 g^4_3 + (\beta^1 + 2\gamma^2) f^3 + \\
&\quad 2\gamma^3 f^4_3 - 2\gamma^1_\tau - \alpha^1(\beta^2 - 2\gamma^1) = 0, \\
&f^4_{3\tau} - \beta^1_\tau + (g^4 f^3)_3 - (\alpha^1 + \alpha^2) y_3 f^4_{33} - \alpha^1 f^4_3 - (\beta^2 - 2\gamma^2) g^3 + \\
&\quad 2\gamma^3 g^4_3 + \alpha^2(\beta^1 + 2\gamma^2) - 2\gamma^2_t = 0, \\
&g^3_\tau - f^3_\tau - (\alpha^1 + \alpha^2) y_3 (g^3_3 - f^3_3) + \beta^1 g^4_3 - \beta^2 f^4_3 + (\alpha^1 + \alpha^2)(g^3 - f^3) + \\
&2\gamma^i \beta^i - 2\gamma^3(\alpha^1 + \alpha^2) - 2\gamma^3_t = 0.
\end{aligned} \tag{13}$$

Here we do not sum over the index i.

We integrate system (13), substitute the obtained expressions for the functions f^m, g^m, $m = \overline{1,4}$, k, α^i, β^i, and γ^a into (12) and (8). Then, integrating equation (4) to find the function q, we get solutions of (4)–(5) that are simplified by means of transformations generated by operators of the form (8)

Depending on different means of integrating system (13), the following solutions of (4)–(5) can be obtained in such a way:

1. $(\nu^1\nu^2)^2 + (\mu^1\mu^2)^2 \neq 0$, $\vec{\gamma} \neq \vec{0}$:

$$\begin{aligned}
w^1 &= C_{11}ke^{ky_2} + C_{12}ke^{-ky_2} - k_\tau k^{-1}y_1,\\
w^2 &= C_{21}ke^{ky_1} + C_{22}ke^{-ky_1} - k_\tau k^{-1}y_2,\\
w^3 &= -2\gamma^2 y_1 + 2\gamma^1 y_2 + 2k_\tau k^{-1}y_3,\\
q &= -k^2(C_{11}e^{ky_2} - C_{12}e^{-ky_2} + 2\gamma^3 k^{-2})(C_{21}e^{ky_1} - C_{22}e^{-ky_1} - 2\gamma^3 k^{-2}) +\\
&\quad \tfrac{1}{2}k_{\tau\tau}k^{-1}(y_1^2 + y_2^2 - 2y_3^2) - (k_\tau k^{-1})^2|\vec{y}|^2 - 2(\gamma^1 y_2 - \gamma^2 y_1)^2 +\\
&\quad (\gamma^2_\tau + 4\gamma^2 k_\tau k^{-1})y_1y_3 - (\gamma^1_\tau + 4\gamma^1 k_\tau k^{-1})y_2y_3.
\end{aligned}$$

2. $(\nu^1\nu^2)^2 + (\mu^1\mu^2)^2 \neq 0$, $\vec{\gamma} = \vec{0}$. Then the matrix O can be considered to be equal to the unit matrix, and $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$\begin{aligned}
u^1 &= ke^{\zeta(\omega)}(C_{11}e^{kx_2} + C_{12}e^{-kx_2}) - k_t k^{-1}x_1,\\
u^2 &= ke^{-\zeta(\omega)}(C_{21}e^{kx_1} + C_{22}e^{-kx_1}) - k_t k^{-1}x_2,\\
u^3 &= 2k_t k^{-1}x_3,\\
p &= -k^2(C_{11}e^{kx_2} - C_{12}e^{-kx_2})(C_{21}e^{kx_1} - C_{22}e^{-kx_1}) +\\
&\quad \tfrac{1}{2}k_{tt}k^{-1}(x_1^2 + x_2^2 - 2x_3^2) - (k_t k^{-1})^2|\vec{x}|^2,
\end{aligned}$$

where $\omega = k^{-2}(t)x_3$, k is an arbitrary function of t which does not vanish, ζ is an arbitrary function of ω.

3. $\mu^1\mu^2 = \nu^1\nu^2 = 0$, $\beta^i = \gamma^3 = 0$. Then $\gamma^i = 0$ and, as above, we can assume that $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$\begin{aligned}
u^1 &= C_1 k \exp\{(-1)^i kx_2 + H(\tau, \omega)\} + (-1)^i kF(\omega) - k_t k^{-1}x_1,\\
u^2 &= C_2 k \exp\{(-1)^j kx_1 - H(\tau, \omega)\} - (-1)^j kF(\omega) - k_t k^{-1}x_2,\\
u^3 &= 2k_t k^{-1}x_3,\\
p &= -C_1C_2(-1)^{i+j}k^2 e^{(-1)^i kx_2 + (-1)^j kx_1} + \tfrac{1}{2}k_{tt}k^{-1}(x_1^2 + x_2^2 - 2x_3^2) - (k_t k^{-1})^2|\vec{x}|^2,
\end{aligned}$$

where $\omega = k^{-2}(\tau)x_3$, k is an arbitrary function of t which does not vanish.

$$H = (-1)^{i+j}F(\omega)\int k^2(t)dt + G(\omega).$$

F and G are arbitrary functions of ω, i and j assumed to be fixed from $\{1; 2\}$.

4. $\mu^1\mu^2 = \nu^1\nu^2 = 0$, $\beta^i\beta^i + (\gamma^3)^2 \neq 0$. The solution obtained in this case is very complicated, and we omit it.

Case 2. $h < 0$. Let $k := (-h)^{1/2}$. Then equation (10) gives that

$$v^1 = f^1 \cos(ky_1) + f^2 \sin(ky_1) + f^3 y_1 + f^4,$$
$$v^2 = g^1 \cos(ky_2) + g^2 \sin(ky_2) + g^3 y_2 + g^4,$$

where $f^m = f^m(\tau, y_3)$, $g^m = g^m(\tau, y_3)$, $m = \overline{1,4}$, $f^i f^i \neq 0$, $g^i g^i \neq 0$. In a way being analogous to Case 1, we obtain the following solutions of equations (4)–(5):

1. $\vec{\gamma} \neq \vec{0}$:

$$w^1 = C_{11} k \cos(ky_2) + C_{12} k \sin(ky_2) - k_\tau k^{-1} y_1,$$
$$w^2 = C_{21} k \cos(ky_1) + C_{22} k \sin(ky_1) - k_\tau k^{-1} y_2,$$
$$w^3 = -2\gamma^2 y_1 + 2\gamma^1 y_2 + 2k_\tau k^{-1} y_3,$$
$$q = k^2 (C_{11} \sin(ky_2) - C_{12} \cos(ky_2) - 2\gamma^3 k^{-2})(C_{21} \sin(ky_1) - C_{22} \cos(ky_1) + 2\gamma^3 k^{-2}) +$$
$$\tfrac{1}{2} k_{\tau\tau} k^{-1} (y_1^2 + y_2^2 - 2y_3^2) - (k_\tau k^{-1})^2 |\vec{y}|^2 - 2(\gamma^1 y_2 - \gamma^2 y_1)^2 +$$
$$(\gamma^2_\tau + 4\gamma^2 k_\tau k^{-1}) y_1 y_3 - (\gamma^1_\tau + 4\gamma^1 k_\tau k^{-1}) y_2 y_3,$$

where $\vec{\gamma}$ is an arbitrary vector function of τ, $k = C|\gamma^3|^{\frac{1}{2}}$ if $\gamma^3 \neq 0$ and k is an arbitrary function of τ if $\gamma^3 = 0$.

2. $\vec{\gamma} = \vec{0}$. As above, we can consider that $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$u^1 = k e^{\zeta(\omega)} (C_{11} \cos(kx_2) + C_{12} \sin(kx_2)) - k_t k^{-1} x_1,$$
$$u^2 = k e^{-\zeta(\omega)} (C_{21} \cos(kx_1) + C_{22} \sin(kx_1)) - k_t k^{-1} x_2,$$
$$u^3 = 2k_t k^{-1} x_3,$$
$$p = k^2 (C_{11} \sin(kx_2) - C_{12} \cos(kx_2))(C_{21} \sin(kx_1) - C_{22} \cos(kx_1)) +$$
$$\tfrac{1}{2} k_{tt} k^{-1} (x_1^2 + x_2^2 - 2x_3^2) - (k_t k^{-1})^2 |\vec{x}|^2,$$

where $\omega = k^{-2}(t) x_3$, $\zeta = \zeta(\omega)$ and $k = k(\tau)$ are arbitrary functions of their arguments, $k \neq 0$.

Case $h = 0$ is impossible.

References

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