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Evolution Equations Invariant under the Conformal Algebra

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Abstract

The evolution equation of the following form $u_0 = f(u, \triangle u)$ which is invariant under the conformal algebra are investigated. The symmetry property of this equation is used for the construction of its exact solutions.

Analyzing the symmetry of the nonlinear heat equation

$$u_0 = \partial_1(u_1 f(u)) \tag{1}$$

in paper [1], L. Ovsyannikov showd that, in case

$$F(u) = \lambda u^{-\frac{4}{3}},$$

the widest invariance algebra consists of the operators:

$$\langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 4x_0\partial_0 + 3u\partial_u, D_2 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u\rangle$$

We shall consider a generalization of equation (1) in the form:

$$u_0 = F(u, u_{11}) \tag{2}$$

and define functions F with which equation (2) is invariant with respect to the algebra:

$$A = \langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u\rangle. \tag{3}$$

Theorem 1. *Equation (2) is invariant with respect to algebra (3) under the condition:*

$$F(u, u_{11}) = u f(u^3 u_{11}), \tag{4}$$

where f is an arbitrary function.

Proof. Coordinates ξ^0, ξ^1, η of the infinitesimal operator

$$X = \xi^0\partial_0 + \xi^a\partial_a + \eta\partial_u,$$

of algebra (3) have a form

$$\xi^0 = 0, \quad \xi^1 = x_1^2, \quad \eta = x_1 u,$$

Using the invariance condition of equation (2) with respect to algebra (3) (see, for example, [2]) we have:

$$u\frac{\partial F}{\partial u} - 3u_{11}\frac{\partial F}{\partial u_{11}} = F. \tag{5}$$

The general solution of equation (5) has the form (4). Theorem is prooved.

When the function F is given by formula (4), equation (2) has the form:

$$u_0 = uf(u^3 u_{11}). \tag{6}$$

We research the Lie's symmetry of equation (6).

Theorem 2. *Equation (6) is invariant with respect to the algebra:*

$$1.\quad A = \langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u \rangle \tag{7}$$

if f is an arbitrary smooth function,

$$2.\quad A_1 = \langle A, D_2 = -\frac{4}{3}x_0\partial_0 + u\partial_u, G = u\partial_1, K_1 = x_1 u\partial_1 + u^2\partial_u \rangle \tag{8}$$

if $f = \lambda_1 u u_{11}^{\frac{1}{3}}$, ($\lambda_1$ is an arbitrary constant),

$$3.\quad A_2 = \langle A, Q_1 = e^{-\frac{4}{3}\lambda_2 x_0}\partial_1, Q_2 = e^{-\frac{4}{3}\lambda_2 x_0}(\partial_0 + \lambda_2 u\partial_u) \rangle, \tag{9}$$

if $f = \lambda_1 u u_{11}^{\frac{1}{3}} + \lambda_2$, ($\lambda_2$ is an arbitrary constant $\lambda_2 \neq 0$),

Theorem 2 is proved by the standard Lie's method [2].

Remark 1. *From results of Theorem 2, it is follows that equation (6) has the widest symmetry in case where it has a form:*

$$u_0 = \lambda u^2 u_{11}^{\frac{1}{3}}. \tag{10}$$

We use the symmetry of equation (10) to find its exact solutions. The invariant solutions of this equation have such a form:

$$W = \varphi(\omega), \tag{11}$$

where ω, W are first integrals of the system of differential equations:

$$\begin{cases} \dot{x_0} = 4\kappa_1 x_0 + d_0, \\ \dot{x_1} = a_1 x_1^2 + (a_2 x_1 + gu + 2\kappa x_1 + d_1, \\ \dot{u} = a_1 x_1 u + a_2 u^2 + (\kappa_2 - 3\kappa_1)u; \end{cases} \tag{12}$$

Depending on values of the parameters $\kappa_1, \kappa_2, d_0, d_1, a_1, a_2, g$, we receive different solutions system (12).

Substituting anzatses built with the help of invariants of algebra (8) in equation (10), we obtain reduced equations. The solutions of system (12), corresponding anzatzes, and reduced equations are given in Table 1.

Remark 2. *All the received results can be generated by the formulae:*

$$\begin{cases} x_0' = e^{-\frac{4}{3}\theta_2} x_0 + \theta_1 \\ x_1' = \dfrac{e^{2\theta_3} x_1 + \theta_7 e^{\theta_2} u + \theta_5}{1 + \theta_5 \theta_4 x_1 e^{2\theta_3} + \theta_6 e_2^{\theta} u + \theta_4} \\ u' = \dfrac{e^{\theta_3 + \theta_2} u}{1 + \theta_5 \theta_4 x_1 e^{2\theta_3} + \theta_6 e_2^{\theta} u + \theta_4}, \end{cases} \tag{13}$$

where $\theta_1, \ldots, \theta_7$ are arbitrary constants. The way of generation is described in [3].

We shall consider the case of variables $x = (x_0, \vec{x}) \in \mathrm{R}_{1+n}$. We generalize equation (1) in such a way

$$u_0 = F(u, \triangle u), \tag{14}$$

and the operator K is replaced by the conformal operators

$$K_a = 2x_a D - \vec{x}^2 \partial_a, \tag{15}$$

where $D = x_b \partial_b + k(u)\partial_u$, $k = k(u)$ is an arbitrary smooth function.

Theorem 3. *Equation (3) is invariant with respect to the conformal algebra $AC(n)$ with the basic operators*

$$\begin{aligned} &1.\ \ \partial_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \partial_b - x_b \partial_a, \quad D = x_a \partial_a + \frac{2-n}{2} u \partial_u, \\ &K_a = 2x_a D - \vec{x}^2 \partial_a; \quad a, b = 1, \ldots, n, \end{aligned} \tag{16}$$

if

$$F = u\Phi(u^{\frac{2+n}{2-n}} \triangle u), \qquad n \neq 2. \tag{17}$$

$$2.\ \ \partial_a, \quad J_{12} = x_1 \partial_2 - x_2 \partial_1, \quad D = x_a \partial_a + 2\partial_u, K_a = 2x_a D - \vec{x}^2 \partial_a; \quad a = 1, 2, \tag{18}$$

if

$$F = \Phi(e^u \triangle u), \qquad n = 2. \tag{19}$$

Proof. Using the invariance condition of equation (14) with respect to operators (15) to find function F, we have the following system

$$\begin{cases} k'F - kF_1 - (k' - 2)\triangle u F_2 = 0, \\ (2k' + n - 2)F_2 = 0. \end{cases} \tag{20}$$

From the second equation of system (20), we receive

$$k = \frac{2-n}{2} u, \quad n \neq 2, \tag{21}$$

$$k = \mathrm{const}, \quad n = 2. \tag{22}$$

Table 1.

N	ω	Anzatzes	Reduced equations
1	x_0	$u = \varphi(\omega)$	$\varphi' = 0$
2	x_0	$u = x_1\varphi(\omega)$	$\varphi' = 0$
3	x_0	$u = \sqrt{x_1}\varphi(\omega)$	$\varphi' = -\frac{\lambda}{\sqrt[3]{4}}\varphi^{\frac{7}{3}}$
4	x_0	$u = \sqrt{x_1^2 + 1}\varphi(\omega)$	$\varphi' = \lambda\sqrt[3]{k^2}\varphi^{\frac{7}{3}}$
5	x_0	$u = \sqrt{x_1^2 - 1}\varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{k^2}\varphi^{\frac{7}{3}}$
6	x_0	$\frac{x_1}{u} + \frac{k}{u^2} = \varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{2k}$
7	$x_1 + mx_0$	$u = \varphi(\omega)$	$\varphi' = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
8	$\frac{1}{x_1} + mx_0$	$u = x_1\varphi(\omega)$	$\varphi' = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
9	$\ln x_1 + mx_0$	$u = \sqrt{x_1}\varphi(\omega)$	$\varphi' = \lambda\varphi^2\left(\varphi'' - \frac{1}{4}\varphi\right)^{\frac{1}{3}}$
10	$\mathrm{arctg}\, x_1 + mx_0$	$u = \sqrt{x_1^2 + 1}\varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{k}\varphi^2(\varphi'' + \varphi)^{\frac{1}{3}}$
11	$\mathrm{arcth}\, x_1 + mx_0$	$u = \sqrt{x_1^2 - 1}\varphi(\omega)$	$\varphi' = \lambda\varphi^2\left(\varphi'' - \frac{1}{4}\varphi\right)^{\frac{1}{3}}$
12	$\frac{1}{u} + mx_0$	$\frac{x_1}{u} + \frac{k}{u^2} = \varphi(\omega)$	$m\varphi' = \lambda\varphi(\varphi\varphi'' - 2\varphi')^{\frac{1}{3}}$
13	$\frac{1}{u} + mx_0$	$\frac{u}{x_1} = \varphi(\omega)$	$m\varphi' = -\lambda(2k - \varphi'')^{\frac{1}{3}}$
14	$x_1 + kx_0u + mx_0$	$u = \varphi(\omega)$	$k_1 + m\omega = -\lambda\omega^2\varphi''^{\frac{1}{3}}$
15	$x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
16	$\frac{1}{x_1} + m\ln x_0$	$u = x_0^{-\frac{3}{4}}x_1\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
17	$\ln x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\left(-\frac{1}{4}\varphi - \varphi''\right)^{\frac{1}{3}}$
18	$\mathrm{arctg}\, x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1^2 + 1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda k^{\frac{2}{3}}\varphi^2(\varphi + \varphi'')^{\frac{1}{3}}$
19	$\mathrm{arcth}\, x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1^2 - 1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda k^{\frac{2}{3}}\varphi^2(\varphi'' - \varphi)^{\frac{1}{3}}$
20	$\frac{1}{u} + m\ln x_0$	$x_0^{-\frac{3}{2}}x_1u^{-1} = \varphi(\omega)$	$\varphi = \frac{2}{3}\lambda\omega(2\varphi' + \omega\varphi'')^{\frac{1}{3}}$
21	$\frac{x_1}{u} + m\ln x_0$	$x_0^{-\frac{3}{2}}u^{-1} = \varphi(\omega)$	$m - \frac{3}{2}\omega\varphi' = -\lambda\omega(\omega\varphi'' + 2\varphi')^{\frac{1}{3}}$

Substituting (21) and (22) into the first equation of system (20), we have

$$1)\ \ F - uF_1 + \frac{2+n}{2-n}\triangle uF_2 = 0, \qquad \text{if} \quad n \neq 2. \tag{23}$$

$$2)\ \ -kF_1 + 2\triangle uF_2 = 0, \qquad \text{if} \quad n = 2. \tag{24}$$

Without restricting the generality, we assume $k = 2$. Solving equations (23) and (24), we obtain formulas (17), (19), and algebra (16), (18). Theorem is proved.

The following theorems are proved by the standard Lie's method.

Theorem 4. *The widest Lie's algebra of invariance of equation (17) for $n \neq 2$ consists of the operators:*

1. *(16), ∂_0, when Φ is an arbitrary smooth function;*
2. *(16), ∂_0, $D_0 = 2mx_0\partial_0 + x_a\partial_a$, when $\Phi(w) = \lambda w^m$, (λ, m are arbitrary constants).*

Theorem 5. *The widest Lie's algebra of invariance of equation (19) for $n = 2$ consists of the operators:*

1. *(18), ∂_0, when Φ is an arbitrary smooth function;*
2. *(18), ∂_0, $D_0 = 2mx_0\partial_0 + x_a\partial_a$, when $\Phi(w) = \lambda w^m$, (λ, m are arbitrary constants).*

References

[1] Ovsyannikov L.W., Group properties of the nonlinear heat equation, *Doklady of USSR*, 1959, V.125, N 3, 492–495.

[2] Ovsyannikov L.V., Group Analysis of Differential Equations, Mosckow, Nauka, 1978.

[3] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.