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On Exact Solutions of the Lorentz-Dirac-Maxwell Equations

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Abstract

Exact solulions of the Lorentz-Dirac-Maxwell equations are constructed.

Motion of a classical spinless particle moving in electromagnetic field is described by the system of ordinary differential equations (Lorentz-Dirac) and partial differential equations (Maxwell) [1]:

$$m\dot{u}_\mu = eF_{\mu\nu}u^\nu + \frac{2}{3}e^2\left(\ddot{u}_\mu + u_\mu \dot{u}_\nu \dot{u}^\nu\right), \tag{1}$$

where $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x^\mu} - \dfrac{\partial A_\mu}{\partial x^\nu}$ is the tensor of electromagnetic field

$$\frac{\partial F_{\mu\nu}}{\partial x^\nu} = 0, \qquad \frac{\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}}{\partial x^\nu} = 0, \tag{2}$$

$$u_\mu \equiv \dot{x}_\mu = \frac{dx_\mu}{d\tau}, \qquad u_\mu u^\mu = 1. \tag{3}$$

Some exact solutions of system (1), (2) can be found in [2].

In the present paper, we have obtained new classes of exact solutions of the Lorentz-Dirac-Maxwell system using $P(1,3)$ symmetry properties of (1), (2).

1. Let us show that if system (1) is invariant with respect to the algebra

$$\langle \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3}, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \rangle, \tag{4}$$

then its particular solutions can be looked for in the form

$$x_1 = a\tau, \qquad x_1 = R\cos d\tau, \qquad x_2 = R\sin d\tau, \qquad x_3 = b\tau, \tag{5}$$

where a, b, d, R are constants.

Indeed, equation (1) admits algebra (4) if and only if

$$\begin{aligned}
&\frac{\partial \vec{E}}{\partial x_0} = \frac{\partial \vec{H}}{\partial x_0} = \frac{\partial \vec{E}}{\partial x_3} = \frac{\partial \vec{H}}{\partial x_3} = 0,\\
&\frac{\partial E_a}{\partial x_2}x_1 - \frac{\partial E_a}{\partial x_1}x_2 = \delta_{a2}E_1 - \delta_{a1}E_2,\\
&\frac{\partial H_a}{\partial x_2}x_1 - \frac{\partial H_a}{\partial x_1}x_2 = \delta_{a2}H_1 - \delta_{a1}H_2.
\end{aligned} \tag{6}$$

The general solutions of equations (6) read:

$$\begin{aligned} &H_1 = f_1x_1 + f_2x_2, \qquad H_2 = f_1x_2 - f_2x_1, \qquad H_3 = f_5,\\ &E_1 = f_3x_1 + f_4x_2, \qquad E_2 = f_3x_2 - f_4x_1, \qquad E_3 = f_6, \end{aligned} \tag{7}$$

where $f_i = f_i(x_1^2 + x_2^2)$.

Substituting (7) into (2), we find functions f_i

$$f_i = \frac{\lambda_i}{x_1^2 + x_2^2}, \quad i = \overline{1,4}, \qquad f_i = \lambda_i, \quad i = \overline{5,6}. \tag{8}$$

Substituting expessions (5) into (1), where $F_{\mu\nu}$ is given by (7), (8), we obtain the additional condition for the constants a, b, d, R, λ_i:

$$\begin{aligned} &a^2 - b^2 - R^2d^2 = 1,\\ &aR^2d^4 + \frac{3}{2e}\{\lambda_6 b - \lambda_4 d\} = 0,\\ &R^2d^3 = -R^4d^5 + \frac{3}{2e}\{a\lambda_4 - b\lambda_1\},\\ &\frac{m}{e}R^2d^2 + a\lambda_3 + b\lambda_2 + R^2d\lambda_5 = 0. \end{aligned} \tag{9}$$

2. To construct another class of exact solutions of equations (1), (2), we require the invariance of (1) with respect to the algebra

$$\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + \alpha\frac{\partial}{\partial x_0}\rangle, \qquad \alpha \neq 0. \tag{10}$$

In this case, solutions (1) can be looked for in the form:

$$x_0 = a\tau, \qquad x_1 = R\cos\frac{a\tau}{\alpha}, \qquad x_2 = R\sin\frac{a\tau}{\alpha}, \qquad x_3 = 0, \tag{11}$$

where a, R, α are constants.

Indeed, a requirement of the invariance of (1) with respect to algebra (10) yields the following form of the functions E_i, H_i:

$$\begin{aligned} &E_1 = f_1\cos\frac{x_0}{\alpha} + f_2\sin\frac{x_0}{\alpha}, \qquad E_2 = f_2\sin\frac{x_0}{\alpha} - f_1\cos\frac{x_0}{\alpha}, \qquad E_3 = f_6,\\ &H_1 = f_3\cos\frac{x_0}{\alpha} + f_4\sin\frac{x_0}{\alpha}, \qquad H_2 = f_3\sin\frac{x_0}{\alpha} - f_4\cos\frac{x_0}{\alpha}, \qquad H_3 = f_5, \end{aligned} \tag{12}$$

where $f_i = f_i(x_3)$.

The electromagnetic field $\{E_i, H_i\}$ (12) satisfies the Maxwell equations if the functions f_i are of the form

$$\begin{aligned} &f_1 = \lambda_1\cos\frac{x_3}{\alpha} + \lambda_2\sin\frac{x_3}{\alpha}, \qquad f_2 = \lambda_3\cos\frac{x_3}{\alpha} + \lambda_4\sin\frac{x_3}{\alpha},\\ &f_3 = -\lambda_1\sin\frac{x_3}{\alpha} + \lambda_2\cos\frac{x_3}{\alpha}, \qquad f_4 = -\lambda_3\sin\frac{x_3}{\alpha} + \lambda_4\cos\frac{x_3}{\alpha},\\ &f_5 = \lambda_5, \qquad f_6 = \lambda_6, \end{aligned} \tag{13}$$

where $\lambda_i = \text{const}$.

After substituting expressions (11) into (2), where E_i, H_i are determined from (12), (13), we obtain the following expressions for constants:

$$a^2\left(1-\frac{R^2}{\alpha^2}\right)=1, \qquad \lambda_5-\frac{R}{\alpha}\lambda_3, \qquad m\frac{a}{\alpha^2}R=e\lambda_1,$$
$$R\left(\frac{a}{\alpha}\right)^3+\left(\frac{a}{\alpha}\right)^5R^5+\frac{3}{2e}a\lambda_2=0. \tag{14}$$

Thus, exact solutions of (1), (2) are given by formulae (11)–(14).

References

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