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Representations of Subalgebras of a Subdirect Sum of the Extended Euclid Algebras and Invariant Equations

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Abstract

Representations of subdirect sums of the extended Euclid algebras $A\tilde{E}(3)$ and $A\tilde{E}(1)$ in the class of Lie vector fields are constructed. Differential invariants of these algebras are obtained.

Description of general classes of partial differential equations invariant with respect to a given group is one of the central problems of the group analysis of differential equations. As is well known, to get the most general solution of this problem, we have to construct the complete set of functionally independent differential invariants of some fixed order for all possible realizations of the local transformation group under study. Fushchych and Yehorchenko [2–4] found the complete set of first- and second-order differential invariants for the standard representations of the groups $P(1,n)$, $E(n)$, $G(1,n)$. Rideau and Winternitz [5, 6] have obtained new realizations of the Poincaré and Galilei group in two-dimensional space-time. New (nonlinear) realizations of the Poincaré groups $P(1,2)$, $P(2,2)$ and Euclid group $E(3)$ were found by Yehorchenko [7] and Fushchych, Zhdanov & Lahno [8, 9, 10].

In this paper, we consider the problem of constructing the complete set of the second-order differential invariants of subalgebras of a subdirect sum of extended Euclid algebras. These algebras are invariance algebras of a number of important differential equations (for example, the Boussinesq equation, equations for the polytropic gas [11]).

1. Let $V = X \times U \cong R^4 \times R^1$ be the space of real variables x_0, $x = (x_1, x_2, x_3)$ and u, G be a local transformation group acting in V and having the generators

$$Q = \tau(x_0, x, u)\partial_{x_0} + \xi^a(x_0, x, u)\partial_{x_a} + \eta(x_0, x, u)\partial_u. \tag{1}$$

By definition the operators $\langle Q_1, \ldots, Q_N\rangle$ form the Lie algebra AG and fulfill the commutation relations

$$[Q_a, Q_b] = C^c_{ab} Q_c, \quad a, b, c = 1, \ldots, N. \tag{2}$$

The problem of classification of realizations of the transformation group G is reduced to classifying realizations of its Lie algebra AG within the class of Lie vector fields.

Introduce the binary relation on the set of realizations of the Lie algebra AG. Two realizations are called equivalent if there exists a nondegenerate change of variables

$$x_0' = h(x_0, x, u), \quad x_a' = g_a(x_0, x, u), \quad u' = f(x_0, x, u), \quad a = 1, 2, 3, \tag{3}$$

transforming them one into another. Note that the introduced equivalence relation does not affect the form of relations (2) [1]. Furthermore, it divides the set of all possible representations into equivalence classes.

We consider covariant realizations of subalgebras of a subdirect sum of the extended Euclid algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$. Saying about a subdirect sum of these algebras, we mean two algebras. If L is a direct sum of the algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$, then its basis operators satisfy the following commutation relations:

$$\begin{aligned} &[P_0, P_a] = [P_0, J_{ab}] = [P_a, P_b] = 0;\\ &[P_a, J_{bc}] = \delta_{ab}P_c - \delta_{ac}P_b;\\ &[J_{ab}, J_{cd}] = \delta_{ad}J_{bc} + \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac}; \end{aligned} \tag{4}$$

$$\begin{aligned} &[P_0, D_1] = P_0, \quad [P_a, D_2] = P_a,\\ &[P_0, D_2] = [D_1, D_2] = [P_a, D_1] = [J_{ab}, D_1] = [J_{ab}, D_2] = 0, \end{aligned} \tag{5}$$

where $a, b, c, = 1, 2, 3,\ \delta_{ab}$ is Kronecker symbol.

Next, if K is a subdirect sum of the algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$ and K is not isomorphic to L, then its basis is formed by the operators P_0, P_a, J_{ab}, D that satisfy the commutation relations (4), and

$$[P_0, D] = kP_0, \quad [P_a, D] = P_a, \quad [J_{ab}, D] = 0, \tag{6}$$

where $a, b = 1, 2, 3,\ k \neq 0,\ k \in R$.

Lemma 1. *An arbitrary covariant representation of the algebra $AE(1) \oplus AE(3)$ in the class of vector fields is reduced by transformations (3) to the following representation:*

$$P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \quad a, b = 1, 2, 3. \tag{7}$$

The proff of Lemma 1 follows from the results of Theorem 1 [8].

Theorem 1. *Nonequivalent covariant representations of the Lie algebra L in the class of vector fields are exhausted by representations (7) of the translation and rotation generators and one of the following representations of dilatation operators:*

$$D_1 = x_0\partial_{x_0}, \quad D_2 = x_a\partial_{x_a}; \tag{8}$$

$$D_1 = x_0\partial_{x_0}, \quad D_2 = x_a\partial_{x_a} + 2u\partial_u; \tag{9}$$

$$D_1 = x_0\partial_{x_0} - u\partial_u, \quad D_2 = x_a\partial_{x_a} + ku\partial_u,\ k \neq 0; \tag{10}$$

$$D_1 = x_0\partial_{x_0} - u\partial_u, \quad D_2 = x_a\partial_{x_a}; \tag{11}$$

$$D_1 = x_0\partial_{x_0} + u\partial_u, \quad D_2 = u\partial_{x_0} + x_a\partial_{x_a}. \tag{12}$$

Theorem 2. *Nonequivalent covariant representations of the algebra K in the class of vector fields are exhausted by representations (7) of the translation and rotation generators and one of the following representations of dilatation operators:*

$$D = kx_0\partial_{x_0} + x_a\partial_{x_a}, \quad k \neq 0, \tag{13}$$

$$D = kx_0\partial_{x_0} + x_a\partial_{x_a} + u\partial_u, \quad k \neq 0. \tag{14}$$

To prove these theorems, it is sufficient to complete the representation of the algebra $AE(1) \oplus AE(3)$ obtained in Lemma 1 by dilatation operators of the form (1) and to verify that the commutation relations (5) or (6) are true.

2. Consider the problem of description of second-order partial differential equations of the most general form

$$F\left(x, u, \underset{1}{u}, \underset{2}{u}\right) = 0, \tag{15}$$

invariant with respect to the obtained realizations of the algebras L and K in the class of the first-order differential operators.

As is known [1], this problem can be reduced to obtaining the differential invariants of the given algebras, i.e., solving the following first-order partial differential equations:

$$\underset{2}{X_i}\,\Psi\left(x_0, x, \underset{1}{u}, \underset{2}{u}\right) = 0, \quad i = 1, \ldots, N,$$

where $\underset{2}{X_i}$ $(i = 1, \ldots, N)$ are basis operators of the algebras L or K.

Lemma 2. *The functions*

$$\begin{array}{lll} u, & u_0, & u_{00} \\ S_1 = u_a u_a, & S_2 = u_{aa}, & S_3 = u_a u_{ab} u_b, \\ S_4 = u_{ab} u_{ab}, & S_5 = u_a u_{ab} u_{bc} u_c, & S_6 = u_{ab} u_{bc} u_{ca}, \\ S_7 = u_a u_{0a}, & S_8 = u_{0a} u_{0a}, & S_9 = u_{0a} u_{0b} u_{ab} \end{array} \tag{16}$$

form the fundamental system of second–order differential invariants of the algebra $AE(1) \oplus AE(3)$. Here, $a, b, c = 1, 2, 3,$ we mean summation from 1 to 3 over the repeated indices, $u_a = \dfrac{\partial u}{\partial x_a}$, $u_{ab} = \dfrac{\partial^2 u}{\partial x_a \partial x_b}$.

The lemma is proved in the same way as it is done in the paper by Fushchych and Yegorchenko [4].

Theorem 3. *The functions Λ_j $(j = 1, 2, \ldots, 10)$ given below form the basis of the fundamental system of differential invariants of the second-order of the algebra L:*

$$\begin{array}{llll} 1)\ \Lambda_1 = u_{00} u_0^{-2}, & \Lambda_2 = S_2 S_1^{-1}, & \Lambda_3 = S_3 S_1^{-2}, & \Lambda_4 = S_4 S_1^{-2}, \\ \Lambda_5 = S_5 S_1^{-3}, & \Lambda_6 = S_6 S_1^{-3}, & \Lambda_7 = S_7 u_0^{-1} S_1^{-1}, & \Lambda_8 = S_8 u_0^{-2} S_1^{-1}, \\ \Lambda_9 = S_9 u_0^{-2} S_1^{-2}, & \Lambda_{10} = u, & & \end{array}$$

if the generators D_1, D_2 are of the form (8).

$$\begin{array}{lllll} 2)\ \Lambda_1 = S_1 u^{-1}, & \Lambda_2 = S_2, & \Lambda_3 = S_3 u^{-1}, & \Lambda_4 = S_4, & \Lambda_5 = S_5 u^{-1}, \\ \Lambda_6 = S_6, & \Lambda_7 = S_7 u_0^{-1}, & \Lambda_8 = S_8 u u_0^{-2}, & \Lambda_9 = S_9 u u_0^{-2}, & \Lambda_{10} = u_{00} u u_0^{-2}, \end{array}$$

if the generators D_1, D_2 are of the form (9).

$$\begin{array}{lll} 3)\ \Lambda_1 = u^{4-2k} u_0^{-2} S_1^k, & \Lambda_2 = u^2 u_0^{-2} S_2^k, & \Lambda_3 = u^{8-3k} u_0^{-4} S_3^k, \\ \Lambda_4 = u^{8-2k} u_0^{-4} S_4^k, & \Lambda_5 = u^{12-4k} u_0^{-6} S_5^k, & \Lambda_6 = u^{12-3k} u_0^{-6} S_6^k, \\ \Lambda_7 = u^{-k+4} u_0^{-k-2} S_7^k, & \Lambda_8 = u^4 u^{-2k-2} S_8^k, & \Lambda_9 = u^{-k+8} u^{-2k-2} S_9^k, \\ \Lambda_{10} = u u_0^{-2} u_{00}, & & \end{array}$$

if the generators D_1, D_2 are of the form (10).

4) $\Lambda_1 = u^{-3}u_{00}$, $\Lambda_2 = uS_2S_1^{-1}$, $\Lambda_3 = uS_3S_1^{-2}$, $\Lambda_4 = u^2S_4S_1^{-2}$, $\Lambda_5 = u^2S_5S_1^{-3}$, $\Lambda_6 = u^3S_6S_1^{-3}$, $\Lambda_7 = u^{-1}S_7S_1^{-1}$, $\Lambda_8 = u^{-2}S_8S_1^{-1}$, $\Lambda_9 = u^{-1}S_9S_1^{-2}$, $\Lambda_{10} = u^{-2}u_0$,

if the generators D_1, D_2 are of the form (11).

5)
$$\Lambda_1 = u_0^{-6}u_{00}^2\exp(2u_0^{-1})S_1,$$
$$\Lambda_2 = u_0^{-6}u_{00}\exp(2u_0^{-1})[u_{00}S_1 + u_0^2S_2 - 2u_0S_7],$$
$$\Lambda_3 = u_0^{-14}u_{00}^3\exp(4u_0^{-1})[u_{00}S_1^2 - 2u_0S_1S_7 + u_0^2S_3],$$
$$\Lambda_4 = u_0^{-12}u_{00}^2\exp(4u_0^{-1})[2u_0^2(S_1S_8 + S_7^2) + u_{00}^2S_1^2 + u_0^4S_4 - 4u_0u_{00}S_1S_7 - 4u_0^3S_1^{-1}S_3S_7 + 2u_0^2u_{00}S_3],$$
$$\Lambda_5 = u_0^{-12}u_{00}^4\exp(6u_0^{-1})[u_0^{-4}S_5 - 4u^{-5}S_3S_7 + u_0^{-6}(3S_1S_7^2 + S_1^2S_8 + 2u_{00}S_1S_3) - 4u_0^{-7}u_{00}S_1^2S_7 + u_0^{-8}u_{00}^2S_1^3,$$
$$\Lambda_6 = u_0^{-18}u_{00}^3\exp(6u_0^{-1})[u_0^6S_6 - 2u_0^5(S_3^2S_7S_1^{-2} + 2S_5S_7S_1^{-1}) + 3u_0^4(S_1S_9 + S_3S_8 + 2S_7^2S_3S_1^{-1} + u_{00}S_5) - 2u_0^3(3S_1S_7S_8 + S_7^3 + 6u_{00}S_3S_7) + 3u_0^2u_{00}(S_1^2S_8 + 3S_1S_7^2 + u_{00}S_1S_3) - 6u_0u_{00}^2S_1^2S_7 + u_{00}^3S_1^3],$$
$$\Lambda_7 = u_0^{-7}u_{00}\exp(2u_0^{-1})(u_{00}S_1 - u_0S_7),$$
$$\Lambda_8 = u_0^{-6}\exp(2u_0^{-1})(u_{00}^2S_1 - 2u_0u_{00}S_7 + u_0^2S_8),$$
$$\Lambda_9 = u_0^{-12}u_{00}\exp(4u_0^{-1})[u_0^4S_9 - 2u_0^3(S_7S_8 + u_{00}S_3S_7S_1^{-1}) + u_0^2(3u_{00}S_7^2 + 2u_{00}S_1S_8 + u_{00}^2S_3) - 4u_0u_{00}^2S_1S_7 + u_{00}^3S_1^2],$$
$$\Lambda_{10} = uu_0^{-3}u_{00},$$

if the generators D_1, D_2 are of the form (12).

Here $S_1, S_2, \ldots, S_9$ are of the form (16).

Theorem 4. *The functions Ω_j $(j = 1, 2, \ldots, 11)$ given below form the basis of the fundamental system of differential invariants of the second-order of the algebra K:*

1) $\Omega_1 = S_1^ku_0^{-2}$, $\Omega_2 = S_2^ku_0^{-2}$, $\Omega_3 = S_3^ku_0^{-4}$, $\Omega_4 = S_4^ku_0^{-4}$, $\Omega_5 = S_5^ku_0^{-6}$, $\Omega_6 = S_6^ku_0^{-6}$, $\Omega_7 = S_7^ku_0^{-(2+k)}$, $\Omega_8 = S_8^ku_0^{-2(1+k)}$, $\Omega_9 = S_9^ku_0^{-2(k+2)}$, $\Omega_{10} = u_{00}u_0^{-2}$, $\Omega_{11} = u$,

if the generator D is of the form (13).

2) $\Omega_1 = S_1$, $\Omega_2 = S_2u$, $\Omega_3 = S_3u$, $\Omega_4 = S_4u^2$, $\Omega_5 = S_5u^2$, $\Omega_6 = S_6u^3$, $\Omega_7 = S_7u^k$, $\Omega_8 = S_8u^{2k}$, $\Omega_9 = S_9u^{2k+1}$, $\Omega_{10} = u_{00}u^{2k-1}$, $\Omega_{11} = u_0u^{k-1}$,

if the generator D is of the form (14).

It follows from Theorems 3, 4 that for the case of the algebra L (15), reads as

$$F(\Lambda_1, \Lambda_2, \ldots, \Lambda_{10}) = 0,$$

and for the case of the algebra K, (15) takes the form

$$\Phi(\Omega_1, \Omega_2, \ldots, \Omega_{11}) = 0.$$

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