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Differential Invariants for a Nonlinear Representation of the Poincaré Algebra. Invariant Equations

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Abstract

We study scalar representations of the Poincaré algebra $p(1,n)$ with $n \geq 2$. We present functional bases of the first- and second-order differential invariants for a nonlinear representation of the Poincaré algebra $p(1,2)$ and describe new nonlinear Poincaré-invariant equations.

0. Introduction

The classical linear Poincaré algebra $p(1,n)$ can be represented by basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{1}$$

where μ, ν take values $0, 1, \ldots, n$; summation is implied over repeated indices (if they are small Greek letters) in the following way:

$$\begin{aligned} &x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2, \\ &g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1). \end{aligned} \tag{2}$$

We consider x_ν and x^ν equivalent with respect to summation. Algebra (1) is an invariance algebra of many important equations of mathematical physics, such as the nonlinear wave equation

$$\Box u = F(u)$$

or the eikonal equation

$$u_\alpha u_\alpha = 0,$$

and such invariance reflects compliance with the Poincaré relativity principle. Poincaré-invariant equations can be used for construction of meaningful mathematical models of relativistic processes. For more detail, see [11].

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In paper [8], scalar representations of the algebra $p(1,1)$ were studied, and it appeared that there exist nonlinear representations which are not equivalent to (1). Here, we investigate the same possibility for the algebra $p(1,n)$ with $n \geq 2$. We prove that there is one representation for $p(1,2)$ which is nonlinear and non-equivalent to (1). For $n > 2$, there are no scalar representations non-equivalent to (1).

To describe invariant equations with respect to our new representation, we need its differential invariants. A functional basis of these invariants is presented below together with examples of new nonlinear Poincaré-invariant equations.

1. Construction of a new representation for the scalar Poincaré algebra

The Poincaré algebra $p(1,2)$ is defined by the commutational relations

$$[J_{01}, J_{02}] = iJ_{12}, \quad [J_{01}, J_{12}] = iJ_{02}, \quad [J_{02}, J_{12}] = -iJ_{01}; \tag{3}$$

$$[P_\mu, J_{\mu\nu}] = iP_\nu, \qquad \mu, \nu = 0, 1, 2; \tag{4}$$

$$[P_\mu, P_\nu] = 0. \tag{5}$$

We look for new representations of the operators P_μ, $J_{\mu\nu}$ in the form

$$X = \xi^\mu(x_\mu, u)\partial_{x_\mu} + \eta(x_\mu, u)\partial_u \tag{6}$$

We get from (3), (4), (5) that up to equivalence with respect to local transformations of x_μ and u, we can take P_μ, $J_{\mu\nu}$ in the following form:

$$\begin{aligned} &p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \\ &J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + if_{\mu\nu}(u)\partial_u, \qquad \left(\partial_u \equiv \frac{\partial}{\partial u}\right). \end{aligned} \tag{7}$$

We designate $f_{01} = a$, $f_{02} = b$, $f_{12} = c$ and get from (2) the conditions on these functions:

$$ab_u - ba_u = c, \quad ac_u - ca_u = b, \quad bc_u - cb_u = -a.$$

Whence

$$c^2 = a^2 + b^2, \quad a = br, \quad c = b(1+r^2)^{1/2}, \quad b = \frac{1}{\left(\ln\left(r + \sqrt{1+r^2}\right)\right)_u},$$

where r is an arbitrary function of u.

Up to a transformation $u \to \varphi(u)$, we can consider the following nonlinear representation of the operators $J_{\mu\nu}$:

$$\begin{aligned} &J_{01} = -i\left(x_0\partial_1 + x_1\partial_0 + \sin u\partial_u\right), \\ &J_{02} = -i\left(x_0\partial_2 + x_2\partial_0 + \cos u\partial_u\right), \\ &J_{12} = -i\left(x_1\partial_2 - x_2\partial_1 + \partial_u\right). \end{aligned} \tag{8}$$

It is easily checked that the representation P_μ (1), $J_{\mu\nu}$ (8) is not equivalent to the representation (1).

To prove that there are no representations of $p(1,n)$, $n > 2$, which would be non-equivalent to (1), we take $P_\mu, J_{\mu\nu}$ in the form (6) and use the commutational relations of the algebra $p(1,n)$. Similarly to the previous case, we can take $P_\mu, J_{\mu\nu}$ in the form (7), and get from commutation relations for $J_{\mu\nu}$ that

$$f_{0a}^2 + f_{0b}^2 = f_{ab}^2, \quad f_{ab}^2 + f_{bc}^2 + f_{ac}^2 = 0, \quad a, b, c = 1, \ldots, n.$$

Whence all $f_{\mu\nu} = 0$, what was to be proved.

2. First-order differential invariants for a nonlinear representation

Definition. *The function $F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$, where $x = (x_0, x_1, \ldots, x_n)$, $\underset{k}{u}$ is the set of all k-th order partial derivatives of the function u, is called a differential invariant for the Lie algebra L with basis elements X_s of the form (6) if it is an invariant of the m-th prolongation of this algebra:*

$$\overset{m}{X}_s F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \lambda_s(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) F. \tag{9}$$

Theoretical studies of differential invariants and their applications can be found in [1–4].

Here, only absolute differential invariants are considered, for which all $\lambda_s = 0$. We look for a first-order absolute differential invariant in the form $F = F(u, \underset{1}{u})$. We use designations $u_0 = x, u_1 = y, u_2 = z$, and from (9) get the following defining conditions for F:

$$\begin{aligned}
&\sin u F_u - x F_y - y F_x + \cos u (x F_x + y F_y + z F_z) = 0, \\
&\cos u F_u - x F_z - z F_x - \sin u (x F_x + y F_y + z F_z) = 0, \\
&F_u - y F_z + z F_y = 0.
\end{aligned}$$

From the above equations, we get the only non-equivalent absolute differential invariant of the first order for the representation P_μ (1), $J_{\mu\nu}$ (8):

$$I_1 = \frac{u_0 - u_1 \cos u + u_2 \sin u}{u_0^2 - u_1^2 - u_2^2}. \tag{10}$$

The expressions

$$u_0 - u_1 \cos u + u_2 \sin u \tag{11}$$

and

$$u_0^2 - u_1^2 - u_2^2 = u_\mu u_\mu \tag{12}$$

are relative differential invariants for the representation P_μ (1), $J_{\mu\nu}$ (8).

3. Second-order differential invariants for a nonlinear representation

Here, we adduce a functional basis of second-order differential invariants for the representation P_μ (1), $J_{\mu\nu}$ (8). These invariants are found using the system of partial differential equations, obtained from the definition of differential invariants (9). The basis we present contains six more invariants in addition to I_1 (10).

$$I_2 = \frac{F_1}{(u_0^2 - u_1^2 - u_2^2)^{3/2}}, \tag{13}$$

where

$$\begin{aligned} F_1 = \lambda(u_0 - u_1 \cos u + u_2 \sin u) = u_{00} - 2u_{01}\cos u + 2u_{02}\sin u + u_{11}\cos^2 u - \\ -2u_{12}\sin u \cos u + u_{12}\sin^2 u - u_0 u_1 \sin u - u_0 u_2 \cos u - u_2^2 \sin u \cos u + \\ +u_1 u_2(\cos^2 u - \sin^2 u) + u_1^2 \sin u \cos u. \end{aligned} \tag{14}$$

L is an operator of invariant differentiation for the algebra $p(1,2)$, P_μ (1), $J_{\mu\nu}$ (8):

$$L = \partial_0 - \cos u \partial_1 + \sin u \partial_2. \tag{15}$$

Its first Lie prolongation has the form

$$\overset{1}{L} = L - (u_1 \cos u + u_2 \sin u) u_\alpha \partial_{u_\alpha}.$$

$$I_3 = \frac{F_2}{(u_0^2 - u_1^2 - u_2^2)^2}, \tag{16}$$

where

$$F_2 = \overset{1}{L}(u_0^2 - u_1^2 - u_2^2). \tag{17}$$

The remaining invariants from our chosen basis do not contain trigonometric functions. To construct them, we use second-order differential invariants of the standard linear scalar representation of $p(1,2)$ (1) [5]. The notations used for these invariants are as follows:

$$\begin{aligned} &r = u_0^2 - u_1^2 - u_2^2, \quad S_1 = \Box u, \quad S_2 = u_{\mu\nu}u_{\mu\nu}, \quad S_3 = u_\mu u_{\mu\nu} u_\nu, \\ &S_4 = u_{\mu\nu}u_{\mu\alpha}u_{\nu\alpha}, \quad S_5 = u_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha. \end{aligned} \tag{18}$$

It is easy to see that the expressions $r, S_1, S_2, S_3, S_4, S_5$, where $\mu, \nu, \alpha = 0, 1, 2$, are functionally independent. The absolute differential invariants of the nonlinear representation of $p(1,2)$ I_4, I_5, I_6, I_7 look as follows:

$$\begin{aligned} &I_4 = (S_3 - rS_1) r^{-3/2}, \quad I_5 = (S_2 r^2 - S_3^2) r^{-3}, \quad I_6 = (S_5 r - S_3^2) r^{-3}, \\ &I_7 = \left(rS_4 - 3(S_1 S_5 - S_1^2 S_3 + \frac{1}{3} r S_1^3 \right) r^{-5/2}. \end{aligned} \tag{19}$$

Proof of the fact that the invariants $I_1, I_2, \ldots, I_7$ present a functional basis of absolute differential invariants of the nonlinear representation of $p(1,2)$ consists of the following steps:

1. Proof of functional independence.
2. Proof of completeness of the set of invariants.

The first step is made by direct verification. The second requires calculation of the rank of the basis of the non-linear representation of $p(1,2)$. The rank of the set $\langle J_{01}, J_{02}, J_{12} \rangle$ (8) is equal to 2, and $F\left(u, \underset{1}{u}, \underset{2}{u}\right)$ depends on 9 variables. So, a complete set has to consist of 7 invariants.

4. Examples of invariant equations and their symmetry

The equation

$$u_0 - u_1 \cos u + u_2 \sin u = 0 \tag{20}$$

is invariant with respect to the algebra $p(1,2)$ with basis operators P_μ (1), $J_{\mu\nu}$ (8). The following theorem describes its maximal symmetry:

Theorem 1. *Equation (20) is invariant with respect to an infinite-dimensional algebra generated by operators*

$$X = \xi^0(x,u)\partial_0 + \xi^1(x,u)\partial_1 + \xi^2(x,u)\partial_2 + \eta(x,u)\partial_u,$$

where

$$\begin{aligned}
&\eta = \eta(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u),\\
&\xi^0 = \xi^0(x_0, u, x_1 + x_0 \cos u, x_2 - x_0 \sin u),\\
&\xi^1 = \varphi^1(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u) + \eta x_0 \sin u - \xi_0 \cos u,\\
&\xi^2 = \varphi^2(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u) + \eta x_0 \cos u + \xi_0 \sin u;
\end{aligned}$$

η, ξ^0, φ^1, φ^2 *are arbitrary functions of their arguments.*

The theorem is proved by means of the Lie algorithm [6, 7, 12].

The infinite-dimensional algebra described above contains as subalgebras the Poincaré algebra $p(1,2)$ (operators P_μ of the form (1), $J_{\mu\nu}$ of the form(8)) and also its extension - a nonlinear representation of the conformal algebra $c(1,2)$. For details on nonlinear representations of $c(1,2)$, see [10]. A basis of this algebra is formed by operators P_μ (1), $J_{\mu\nu}$ (8),

$$D = x_\mu \partial_\mu, \quad K_\nu = 2x_\nu x_\mu \partial_\mu - x^2 \partial_\nu + i\eta^\nu(x,u)\partial_u; \quad \mu,\nu = 0,1,2, \tag{21}$$

where

$$\eta^0 = 2(x_1 \sin u + x_2 \cos u), \quad \eta^1 = -2(x_2 - x_0 \sin u), \quad \eta^2 = 2(x_1 + x_0 \cos u).$$

Equation (20) has the general solution

$$u = \Phi(x_1 + x_0 \cos u, x_2 - x_0 \sin u).$$

The transformation

$$\widetilde{u} = u, \quad \widetilde{x}_0 = x_0, \quad \widetilde{x}_1 = x_1 + x_0 \cos u, \quad \widetilde{x}_2 = x_2 - x_0 \sin u$$

applied to (20) yields the equation

$$\widetilde{u}_{\widetilde{x}_0} = 0. \tag{22}$$

A simplest linear equation (22) appears to be invariant with respect to the following nonlinear representation of $p(1,2)$:

$$\widetilde{P}_0 = i(\cos u\partial_1 - \sin u\partial_2 + \partial_0), \quad \widetilde{P}_1 = -i\partial_1, \quad \widetilde{P}_2 = -i\partial_2,$$

$$\widetilde{J}_{01} = -i\Big((x_0 \sin^2 u - x_1 \cos u)\partial_1 - \sin u(x_1 - x_0 \cos u)\partial_2 + \\ +(x_1 - x_0 \cos u)\partial_0 + \sin u\partial_u\Big),$$

$$\widetilde{J_{02}} = -i\Big((x_0 \cos^2 u - x_2 \cos u)\partial_2 + \cos u(x_2 + x_0 \cos u)\partial_1 + \\ +(x_2 + x_0 \sin u)\partial_0 - \cos u\partial_u\Big),$$

$$\widetilde{J}_{12} = -i(x_1\partial_2 - x_2\partial_1 + \partial_u).$$

Examples of explicit solutions for equation (20):

$$u = \frac{\cos^{-1} c}{(x_1^2 + x_2^2)^{-1/2}} + \tan^{-1}\left(\frac{x_1}{x_2}\right); \qquad u = \frac{\cos^{-1} x_1}{c - x_0};$$

$$u = \tan^{-1}\left(\frac{c - x_1}{x_2}\right); \qquad u = 2\tan^{-1}\left(\frac{c + x_0 + x_1}{x_2}\right).$$

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