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Subgroup Structure of the Poincaré Group *P(1,4)* and Symmetry Reduction of Five-Dimensional Equations of Mathematical Physics

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Abstract

Using the subgroup structure of the generalized Poincaré group $P(1,4)$, the symmetry reduction of the five-dimensional wave and Dirac equations and Euler–Lagrange–Born–Infeld, multidimensional Monge–Ampere, eikonal equations to differential equations with a smaller number of independent variables is done. Some classes of exact solutions of the investigated equations are constructed.

1 Introduction

The knowledge of the nonconjugate subgroups of local Lie groups of point transformations and construction in explicit form of the invariants of these subgroups is important in order to solve many problems of mathematics. In particular, in mathematical physics, the subgroup structure of the invariance groups of partial differential equations (PDEs) and the invariants of these subgroups allow us to solve many problems. Let me mention some of them.

1. The symmetry reduction of PDEs to differential equations with fewer independent variables [1–10].
2. The construction of PDEs with a given (nontrivial) symmetry group [6,8,9,11–14].
3. The construction of systems of coordinates in which the linear PDEs which are invariant under given groups admit partial or full separation of variables [15–20].

The development of theoretical and mathematical physics has required various extensions of the four-dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. One extension of the group $P(1,3)$ is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [21–24].

The purpose of the present paper is to give a survey of results obtained in [25–37] as well as some new results.

2 The subgroup structure of the group $P(1,4)$

The Lie algebra of the group $P(1,4)$ is given by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0, 1, 2, 3, 4)$ and P'_μ $\quad(\mu = 0, 1, 2, 3, 4)$ satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$, we use the method proposed in [38]. Continuous subgroups of the group $P(1,4)$ were found in [25–29].

One of nontrivial consequences of the description of the continuous subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [24], i.e., it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic quantum mechanics.

3 Invariants of subgroups of the group $P(1,4)$

In this Section, we will say something about the invariants of subgroups of the group $P(1,4)$. For all continuous subgroups of the group $P(1,4)$, we have constructed the invariants in the five-dimensional Minkowski space. The majority of these invariants are presented in [30, 31]. Among the invariants obtained, there are one-, two-, three- and four-dimensional ones.

Let us note that the invariants of continuous subgroups of the group $P(1,4)$ play an important role in solution of the problem of reduction for many equations of theoretical and mathematical physics which are invariant under the group $P(1,4)$ or its continuous subgroups.

In the following, we will consider the application of the subgroup structure of the group $P(1,4)$ and the invariants of these subgroups for the symmetry reduction and construction of classes of exact solutions for some important equations of theoretical and mathematical physics.

4 The nonlinear five-dimensional wave equation

Let us consider the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = F(u), \tag{4.1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3, x_4)$, F is a sufficiently smooth function. The invariance group of equation (4.1) is the generalized Poincaré group $P(1,4)$. Using the invariants of subgroups of the group $P(1,4)$, we have constructed ansatzes, which reduce the investigated equation to differential equations with a smaller number of independent variables. Taking into account the solutions of the reduced equations, some classes of exact solutions of the equations investigated are been found. The majority of these results have been published in [30–33].

Let us consider ansatzes of the form

$$u(x) = \varphi(\omega), \tag{4.2}$$

where $\omega(x)$ is a one-dimensional invariant of subgroups of the group $P(1,4)$. In many cases, these ansatzes reduce equation (4.1) to ordinary differential equations (ODEs) of the form

$$\frac{d^2\varphi}{d\omega^2} + \frac{d\varphi}{d\omega} k\omega^{-1} = \varepsilon F(\varphi), \tag{4.3}$$

where some of the new variables ω with the k and ε corresponding to them have the form

1. $\omega = \left(x_4^2 + x_2^2 + x_1^2 - x_0^2\right)^{1/2}, \qquad k = 3, \quad \varepsilon = -1;$

2. $\omega = \left(x_3^2 + x_1^2 + x_2^2 + x_4^2 - x_0^2\right)^{1/2}, \qquad k = 4, \quad \varepsilon = -1.$

Let $F(\varphi) = \lambda\varphi^n \quad (n \neq 1)$, then equations (4.3) have the form

$$\frac{d^2\varphi}{d\omega^2} + \frac{d\varphi}{d\omega} k\omega^{-1} = \varepsilon\lambda\varphi^n \quad (n \neq 1). \tag{4.4}$$

This equation is of the Fowler-Emden type. Particular solutions of equation (4.4) have the form:

$$\varphi = \alpha\omega^\nu,$$

where

$$\alpha = \left[\frac{2[1 + k + n(1-k)]}{\varepsilon\lambda(1-n)^2}\right]^{\frac{1}{n-1}}, \qquad \nu = \frac{2}{1-n}.$$

On the basis of the solutions of the reduced equations, we have obtained particular solutions of equation (4.1) with the right-hand side $F(u) = \lambda u^n \quad (n \neq 1)$.

Let us consider ansatzes of the form

$$u(x) = \varphi(\omega_1, \omega_2),$$

where $\omega_1(x)$ and $\omega_2(x)$ are invariants of subgroups of the group $P(1,4)$. These ansatzes reduce the equation at hand to two-dimensional PDEs. In 22 cases, we obtained ODEs instead of two-dimensional PDEs.

Let us note that the same situation for nonlinear relativistically invariant equations was noted and studied in [10].

Let me mention ansatzes of the form

$$u(x) = \varphi(\omega_1, \omega_2, \omega_3),$$

where $\omega_1(x)$, $\omega_2(x)$, $\omega_3(x)$ are invariants of subgroups of the group $P(1,4)$. Among the reduced equations, there are 10 two-dimensional PDEs.

5 Separation of variables in $M(1,4)$

Using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, we have constructed systems of coordinates in the space $M(1,4)$, in which a large class of linear PDEs invariant under the group $P(1,4)$ or under its Abelian subgroups admits partial or full separation of variables. Using this, we have studied the five-dimensional wave equation of the form

$$\Box u = -\kappa^2 u, \tag{5.1}$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

$u(x) = u(x_0, x_1, x_2, x_3, x_4)$ is a scalar $\mathbf{C}^2$ function, $\kappa = \text{const}$. As a result of partial (or full) separation of variables, we obtained PDEs with a smaller number of independent variables (or ODEs) which replace the original equation. Some exact solutions of the five-dimensional wave equation have been constructed. More details about these results can be found in [34].

6 The Dirac equation in $M(1,4)$

Let us consider the equation

$$\left(\gamma_k p^k - m\right) \psi(x) = 0, \tag{6.1}$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $p_k = i\dfrac{\partial}{\partial x_k}$, $k = 0, 1, 2, 3, 4$; γ_k are the (4×4) Dirac matrices. Equation (6.1) is invariant under the generalized Poincaré group $P(1,4)$.

Following [39] and using the subgroup structure of the group $P(1,4)$, the ansatzes which reduce equation (6.1) to systems of differential equations with a lesser number of independent variables are constructed. The corresponding symmetry reduction has been done. Some classes of exact solutions of the investigated equation have been found. The part of the results obtained is presented in [35].

7 The eikonal equation

We consider the equation

$$u^\mu u_\mu \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1, \tag{7.1}$$

where $u = u(x),\ x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \dfrac{\partial u}{\partial x_\mu}$, $u^\mu = g^{\mu\nu} u_\nu$, $\mu, \nu = 0, 1, 2, 3$.

From the results of [40], it follows that the symmetry group of equation (7.1) contains the group $P(1,4)$ as a subgroup. Using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, we have constructed ansatzes which reduce the investigated equation to differential equations with a smaller number of independent variables, and the corresponding symmetry reduction has been carried out. Having solved some of the reduced equations, we have found classes of exact solutions of the investigated equation. Some of these results are given in [30, 31, 36].

It should be noted that, among the ansatzes obtained, there are those which reduce the investigated equation to linear ODEs.

8 The Euler-Lagrange-Born-Infeld equation

Let us consider the equation

$$\Box u\,(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0, \tag{8.1}$$

where $u=u(x), x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_\mu\equiv\dfrac{\partial u}{\partial x^\mu}$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x^\mu\partial x^\nu}$, $u^\mu=g^{\mu\nu}u_\nu$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu=0,1,2,3$, $\Box$ is the d'Alembert operator.

The symmetry group [40] of equation (8.1) contains the group $P(1,4)$ as a subgroup.

On the basis of the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, the symmetry reduction of the investigated equation to differential equations with a lesser number of independent variables has been done. In many cases the reduced equations are linear ODEs. Taking into account the solutions of reduced equations, we have found multiparametric families of exact solutions of the considered equation. The part of these results can be found in [36].

9 The multidimensional homogeneous Monge-Ampère equation

Consider the equation

$$\det(u_{\mu\nu})=0, \tag{9.1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $\mu,\nu=0,1,2,3$.

The symmetry group of equation (9.1) was found in [40].

We have made the symmetry reduction of the investigated equation to differential equations with a smaller number of independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. Among the reduced equations, there are linear ODEs. Having solved some of the reduced equations, we have obtained classes of exact solutions of the investigated equation. Some of these results are presented in [36].

10 The multidimensional inhomogeneous Monge-Ampère equation

In this section, we consider the equation

$$\det(u_{\mu\nu})=\lambda\,(1-u_\nu u^\nu)^3,\quad \lambda\neq 0, \tag{10.1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $u^\nu=g^{\nu\alpha}u_\alpha$, $u_\alpha\equiv\dfrac{\partial u}{\partial x_\alpha}$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu,\alpha=0,1,2,3$.

Equation (10.1) is invariant [40] under the group $P(1,4)$.

We have constructed ansatzes which reduce the investigated equation to differential equations with a less number of independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. The corresponding symmetry reduction has been done. Some classes of exact solutions of the considered equation are found. The majority of these results was published in [37].

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