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The Higher Dimensional Bateman Equation and Painlevé Analysis of Nonintegrable Wave Equations

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Abstract

In performing the Painlevé test for nonintegrable partial differential equations, one obtains differential constraints describing a movable singularity manifold. We show, for a class of wave equations, that the constraints are in the form of Bateman equations. In particular, for some higher dimensional wave equations, we derive the exact relations, and show that the singularity manifold condition is equivalent to the higher dimensional Bateman equation. The equations under consideration are: the sine-Gordon, Liouville, Mikhailov, and double sine-Gordon equations as well as two polynomial field theory equations.

1 Introduction

The Painlevé analysis, as a test for integrability of PDEs, was proposed by Weiss, Tabor and Carnevale in 1983 [20]. It is an generalization of the singular point analysis for ODEs, which dates back to the work of S. Kovalevsky in 1888. A PDE is said to possess the Painlevé property if solutions of the PDE are single-valued in the neighbourhood of non-characteristic, movable singularity manifolds (Ward [17], Steeb and Euler [15], Ablowitz and Clarkson [1]). Weiss, Tabor and Carnevale [20] proposed a test of integrability (which may be viewed as a necessary condition of integrability), analogous to the algorithm given by Ablowitz, Ramani and Segur [2] to determine whether a given ODE has the Painlevé property. One seeks a solution of a given PDE (in rational form) in the form of a Laurent series (also known as the Painlevé series)

$$u(\boldsymbol{x}) = \phi^{-m}(\boldsymbol{x}) \sum_{j=0}^{\infty} u_j(\boldsymbol{x}) \phi^j(\boldsymbol{x}), \tag{1.1}$$

where $u_j(\boldsymbol{x})$ are analytic functions of the complex variables $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$ (we do not change the notation for the complex domain), with $u_0 \neq 0$, in the neighbourhood of a non-characteristic, movable singularity manifold defined by $\phi(\boldsymbol{x}) = 0$ (the pole manifold), where $\phi(\boldsymbol{x})$ is an analytic function of $x_0, x_1, \ldots, x_{n-1}$. The PDE is said to pass the Painlevé test if, on substituting (1.1) in the PDE, one obtains the correct number of arbitrary functions as required by the Cauchy-Kovalevsky theorem, given by the expansion coefficients in (1.1), whereby ϕ should be one of arbitrary functions. The positions in the Painlevé expansion where arbitrary functions are to appear, are known as the resonances. If a PDE passes the Painlevé test, it is usually (Newell *et al* [13]) possible to construct

Bäcklund transformations and Lax pairs (Weiss [18], Steeb and Euler [15]), which proves the necessary condition of integrability.

Recently much attention was given to the construction of exact solutions of nonintegrable PDEs by the use of a truncated Painlevé series (Cariello and Tabor [3], Euler *et al.* [10], Webb and Zank [17], Euler [5]). On applying the Painlevé test to nonintegrable PDEs, one usually obtains conditions on ϕ at resonances; the singular manifold conditions. With a truncated series, one usually obtains additional constraints on the singularity manifolds, leading to a compatibility problem for the solution of ϕ.

In the present paper, we show that the general solution of the Bateman equation, as generalized by Fairlie [11], solves the singularity manifold condition at the resonance for a particular class of wave equations. For the n-dimensional ($n \geq 3$) sine-Gordon, Liouville, and Mikhailov equations, the n-dimensional Bateman equation is the general solution of the singularity manifold condition, whereas, the Bateman equation is only a special solution of the polynomial field theory equations which were only studied in two dimensions. For the n-dimensional ($n \geq 2$) double sine-Gordon equation, the Bateman equation also solves the constraint at the resonance in general.

2 The Bateman equation for the singularity manifold

The Bateman equation in two dimensions has the following form:

$$\phi_{x_0x_0}\phi_{x_1}^2 + \phi_{x_1x_1}\phi_{x_0}^2 - 2\phi_{x_0}\phi_{x_1}\phi_{x_0x_1} = 0. \tag{2.2}$$

In the Painlevé analysis of PDEs, (2.2) was first obtained by Weiss [19] in his study of the double sine-Gordon equation. As pointed out by Weiss [19], the Bateman equation (2.2) may be linearized by a Legendre transformation. Moreover, it is invariant under the Moebius group. The general implicit solution of (2.2) is

$$x_0 f_0(\phi) + x_1 f_1(\phi) = c, \tag{2.3}$$

where f_0 and f_1 are arbitrary smooth functions and c an arbitrary constant. Fairlie [11] proposed the following generalization of (2.2) for n dimensions:

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \cdots & \phi_{x_0x_{n-1}} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \cdots & \phi_{x_1x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0x_{n-1}} & \phi_{x_1x_{n-1}} & \cdots & \phi_{x_{n-1}x_{n-1}} \end{pmatrix} = 0. \tag{2.4}$$

We call (2.4) the n-dimensional Bateman equation. It admits the following general implicit solution

$$\sum_{j=0}^{n-1} x_j\, f_j(\phi) = c, \tag{2.5}$$

where f_j are n arbitrary functions.

Let us consider the following direct n-dimensional generalization of the well-known sine-Gordon, Liouville, and Mikhailov equations, as given respectively by

$$\begin{aligned}
&\Box_n u + \sin u = 0,\\
&\Box_n u + \exp(u) = 0, \qquad (2.6)\\
&\Box_n u + \exp(u) + \exp(-2u) = 0.
\end{aligned}$$

By a direct n-dimensional generalization, we mean that we merely consider the d'Alembert operator $\Box$ in the n-dimensional Minkowski space, i.e.,

$$\Box_n := \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

It is well known that the above given wave equations are integrable if $n = 2$, i.e., time and one space coordinates. We call PDEs integrable if they can be solved by an inverse scattering transform and there exists a nontrivial Lax pair (see, for example, the book of Ablowitz and Clarkson [1] for more details). For such integrable equations, the Painlevé test is passed and there are no conditions at the resonance, so that ϕ is an arbitrary function.

Before we state our proposition for the singularity manifold of the above given wave equations, we have to introduce some notations and a lemma. We call the $(n+1)\times(n+1)$-matrix, of which the determinant is the general Bateman equation, the Bateman matrix and denote this matrix by B, i.e.,

$$B := \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \cdots & \phi_{x_0x_{n-1}} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \cdots & \phi_{x_1x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0x_{n-1}} & \phi_{x_1x_{n-1}} & \cdots & \phi_{x_{n-1}x_{n-1}} \end{pmatrix}. \qquad (2.7)$$

Definition. *Let*

$$M_{x_{j_1}x_{j_2}\dots x_{j_r}}$$

denote the determinant of the submatrix that remains after the rows and columns containing the derivatives $\phi_{x_{j_1}}, \phi_{x_{j_2}}, \dots, \phi_{x_{j_r}}$ *have been deleted from the Bateman matrix (2.7). If*

$$j_1, \dots, j_r \in \{0, 1, \dots, n-1\}, \quad j_1 < j_2 < \cdots < j_r, \quad r \le n-2, \quad \textit{for } n \ge 3,$$

then $M_{x_{j_1}x_{j_2}\dots x_{j_r}}$ *are the determinants of Bateman matrices, and we call the equations*

$$M_{x_{j_1}x_{j_2}\dots x_{j_r}} = 0 \qquad (2.8)$$

the minor Bateman equations of (2.4).

Note that the n-dimensional Bateman equation (2.4) has $n!/[r!(n-r)!]$ minor Bateman equations. Consider an example. If $n = 5$ and $r = 2$, then there exist 10 minor Bateman equations, one of which is given by $M_{x_2x_3}$, i.e.,

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_4} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \phi_{x_0x_4} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \phi_{x_1x_4} \\ \phi_{x_4} & \phi_{x_0x_4} & \phi_{x_1x_4} & \phi_{x_4x_4} \end{pmatrix} = 0. \qquad (2.9)$$

Note that the subscript of M, namely x_2 and x_3, indicates that the derivatives of ϕ w.r.t. x_2 or x_3 do not appear in the minor Bateman equation.

We can now state the following

Lemma. *The Bateman equation (2.4) is equivalent to the following sum of minor Bateman equations*

$$\sum_{j_1,j_2,\ldots,j_r=1}^{n-1} M_{x_{j_1}x_{j_2}\ldots x_{j_r}} - \sum_{j_1,j_2,\ldots,j_{r-1}=1}^{n-1} M_{x_0 x_{j_1}x_{j_2}\ldots x_{j_{r-1}}} = 0, \tag{2.10}$$

where $j_1,\ldots,j_r \in \{1,\ldots,n-1\}$, $\quad j_1 < j_2 < \cdots < j_r$, $\quad r \le n-2$, $\quad n \ge 3$.

Proof. It is easy to show that the general solution of the n-dimensional Bateman equation satisfies every minor Bateman equation in n dimensions identically. Thus, equations (2.4) and (2.10) have the same general solution and are therefore equivalent. □

Theorem 1. *For $n \ge 3$, the singularity manifold condition of the direct n-dimensional generalization of the sine-Gordon, Liouville and Mikhailov equations (2.6), is given by the n-dimensional Bateman equation (2.4).*

The detailed proof will be presented elsewhere. Let us sketch the proof for the sine-Gordon equation. By the substitution

$$v(\boldsymbol{x}) = \exp[iu(\boldsymbol{x})],$$

the n-dimensional sine-Gordon equation takes the following form:

$$v\Box_n v - (\bigtriangledown_n v)^2 + \frac{1}{2}\left(v^3 - v\right) = 0, \tag{2.11}$$

where

$$(\bigtriangledown_n v)^2 := \left(\frac{\partial v}{\partial x_0}\right)^2 - \sum_{j=1}^{n-1}\left(\frac{\partial v}{\partial x_j}\right)^2.$$

The dominant behaviour of (2.11) is 2, so that the Painlevé expansion is

$$v(\boldsymbol{x}) = \sum_{j=0}^{\infty} v_j(\boldsymbol{x})\phi^{j-2}(\boldsymbol{x}).$$

The resonance is at 2 and the first two expansion coefficients have the following form:

$$v_0 = -4\left(\bigtriangledown_n\phi\right)^2, \qquad v_1 = 4\Box_n\phi.$$

We first consider $n = 3$. The singularity manifold condition at the resonance is then given by

$$\det\begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_2} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \phi_{x_0x_2} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \phi_{x_1x_2} \\ \phi_{x_2} & \phi_{x_0x_2} & \phi_{x_1x_2} & \phi_{x_2x_2} \end{pmatrix} = 0,$$

which is exactly the 2-dimensional Bateman equation as defined by (2.4).

Consider now $n \geq 4$. At the resonance, we then obtain the following condition

$$\sum_{j_1,j_2,\ldots,j_{n-3}=1}^{n-1} M_{x_{j_1}x_{j_2}\ldots x_{j_{n-3}}} - \sum_{j_1,j_2,\ldots,j_{n-4}=1}^{n-1} M_{x_0x_{j_1}x_{j_2}\ldots x_{j_{n-4}}} = 0, \tag{2.12}$$

where

$$j_1,\ldots,j_{n-3} \in \{1,\ldots,n-1\}, \quad j_1 < j_2 < \cdots < j_{n-3},$$

i.e., $M_{x_{j_1}x_{j_2}\ldots x_{j_{n-3}}}$ and $M_{x_0x_{j_1}x_{j_2}\ldots x_{j_{n-4}}}$ are the determinants of all possible 4×4 Bateman matrices. By the Lemma give above, equation (2.12) is equivalent to the n-dimensional Bateman equation (2.4).

The proof for the Liouville and Mikhailov equations is similar.

The wave equations studied above have the common feature that they are integrable in two dimensions. Let us consider the double sine-Gordon equation in n dimensions:

$$\Box_n u + \sin\frac{u}{2} + \sin u = 0. \tag{2.13}$$

It was shown by Weiss [19] that this equation does not pass the Painlevé test for $n = 2$, and that the singularity manifold condition is given by the Bateman equation (2.2). For n dimensions, we can state the following

Theorem 2. *For $n \geq 2$, the singularity manifold condition of the n-dimensional double sine-Gordon equation (2.13) is given by the n-dimensional Bateman equation (2.4).*

The proof will be presented elsewhere.

In Euler *et al.* [4], we studied the above wave equations with explicitly space- and time-dependence in one space dimension.

3 Higher order singularity manifold conditions

It is well known that in one and more space dimensions, polynomial field equations such as the nonlinear Klein-Gordon equation

$$\Box_2 u + m^2 u + \lambda u^n = 0 \tag{3.14}$$

cannot be solved exactly for $n = 3$, even for the case $m = 0$. In light-cone coordinates, i.e.,

$$x_0 \longrightarrow \frac{1}{2}(x_0 - x_1), \qquad x_1 \longrightarrow \frac{1}{2}(x_0 + x_1),$$

(3.14) takes the form

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^n = 0, \tag{3.15}$$

where we let $m = 0$ and $\lambda = 1$. The Painlevé test for the case $n = 3$ was performed by Euler *et al.* [10]. We are now interested in the relation between the Bateman equation and the singularity maifold condition of (3.15) for the case $n = 3$ as well as $n = 2$.

First, we consider equation (3.15) with $n = 3$. Performing the Painlevé test (Euler *et al.* [10]), we find that the dominant behaviour is -1, the resonance is 4, and the first three expansion coefficients in the Painlevé expansion are

$$
\begin{aligned}
&u_0^2 = 2\phi_{x_0}\phi_{x_1},\\
&u_1 = -\frac{1}{3u_0^2}\left(u_0\phi_{x_0x_1} + u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1}\right),\\
&u_2 = \frac{1}{3u_0^2}\left(u_{0x_0x_1} - 3u_0u_1^2\right),\\
&u_3 = \frac{1}{u_0^2}\left(u_2\phi_{x_0x_1} + u_{2x_1}\phi_{x_0} + u_{2x_0}\phi_{x_1} + u_{1x_0x_1} - 6u_0u_1u_2\right).
\end{aligned}
$$

At the resonance, we obtain the following singularity manifold condition:

$$\Phi\sigma - \left(\phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0}\right)^2 = 0, \tag{3.16}$$

where Φ is the two-dimensional Bateman equation given by (2.2) and

$$
\begin{aligned}
\sigma = {} & (24\phi_{x_0}\phi_{x_1}^6\phi_{x_0x_0x_0}\phi_{x_0x_0} - 54\phi_{x_0}^2\phi_{x_1}^5\phi_{x_0x_0}\phi_{x_0x_0x_1} - 18\phi_{x_0}^2\phi_{x_1}^5\phi_{x_0x_1}\phi_{x_0x_0x_0}\\
& +18\phi_{x_0}^3\phi_{x_1}^4\phi_{x_0x_1}\phi_{x_0x_0x_1} + 36\phi_{x_0}^3\phi_{x_1}^4\phi_{x_0x_0}\phi_{x_0x_1x_1} - 3\phi_{x_0}^2\phi_{x_1}^6\phi_{x_0x_0x_0x_0}\\
& +36\phi_{x_0}^4\phi_{x_1x_1}\phi_{x_1}^3\phi_{x_0x_0x_1} - 6\phi_{x_0}^4\phi_{x_0x_0}\phi_{x_1}^3\phi_{x_1x_1x_1} + 18\phi_{x_0}^4\phi_{x_1}^3\phi_{x_0x_1}\phi_{x_0x_1x_1}\\
& -6\phi_{x_0}^3\phi_{x_1x_1}\phi_{x_1}^4\phi_{x_0x_0x_0} + 24\phi_{x_0}^6\phi_{x_1}\phi_{x_1x_1}\phi_{x_1x_1x_1} - 54\phi_{x_0}^5\phi_{x_1}^2\phi_{x_1x_1}\phi_{x_0x_1x_1}\\
& -18\phi_{x_0}^5\phi_{x_1}^2\phi_{x_0x_1}\phi_{x_1x_1x_1} - 3\phi_{x_0}^6\phi_{x_1}^2\phi_{x_1x_1x_1x_1} + 12\phi_{x_0}^5\phi_{x_1}^3\phi_{x_0x_1x_1x_1}\\
& -18\phi_{x_0}^4\phi_{x_1}^4\phi_{x_0x_0x_1x_1} + 12\phi_{x_0}^3\phi_{x_1}^5\phi_{x_0x_0x_0x_1} + 48\phi_{x_1}\phi_{x_0x_1}\phi_{x_0}^5\phi_{x_1x_1}^2\\
& -30\phi_{x_0}^3\phi_{x_1}^3\phi_{x_0x_0}\phi_{x_0x_1}\phi_{x_1x_1} + 3\phi_{x_0}^2\phi_{x_0x_0}^2\phi_{x_1}^4\phi_{x_1x_1} - 2\phi_{x_0}^3\phi_{x_1}^3\phi_{x_0x_1}^3\\
& +3\phi_{x_0}^4\phi_{x_1}^2\phi_{x_0x_0}\phi_{x_1x_1}^2 - 15\phi_{x_0}^4\phi_{x_1}^2\phi_{x_0x_1}^2\phi_{x_1x_1} - 20\phi_{x_0}^6\phi_{x_1x_1}^3\\
& +48\phi_{x_0}\phi_{x_1}^5\phi_{x_0x_1}\phi_{x_0x_0}^2 - 20\phi_{x_1}^6\phi_{x_0x_0}^3 - 15\phi_{x_0}^2\phi_{x_1}^4\phi_{x_0x_1}^2\phi_{x_0x_0})/(3\phi_{x_0}^2\phi_{x_1}^2).
\end{aligned}
$$

Clearly, the general solution of the two-dimensional Bateman equation solves (3.16).

For the equation

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^2 = 0, \tag{3.17}$$

the singularity manifold condition is even more complicated. However, also in this case, we are able to express the singularity manifold condition in terms of Φ. The dominant behaviour of (3.17) is -2 and the resonance is at 6. The first five expansion coefficients in the Painlevé expansion are as follows:

$$
\begin{aligned}
&u_0 = -6\phi_{x_0}\phi_{x_1},\\
&u_1 = \frac{1}{\phi_{x_0}\phi_{x_1} + u_0}\left(u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1} + u_0\phi_{x_0x_1}\right),\\
&u_2 = -\frac{1}{2u_0}\left(u_{0x_0x_1} + u_1^2 - u_{1x_1}\phi_{x_0} - u_{1x_0}\phi_{x_1} - u_1\phi_{x_0x_1}\right),\\
&u_3 = -\frac{1}{2u_0}\left(u_{1x_0x_1} + 2u_1u_2\right),
\end{aligned}
$$

$$u_4 = -\frac{1}{\phi_{x_1}\phi_{x_0} + u_0}\left(u_3\phi_{x_0x_1} + u_{2x_0x_1} + 2u_1u_3 + u_{3x_1}\phi_{x_0} + u_{3x_0}\phi_{x_1} + u_2^2\right),$$
$$u_5 = -\frac{1}{6\phi_{x_0}\phi_{x_1} + 2u_0}\left(2u_1u_4 + 2u_4\phi_{x_0x_1} + 2u_{4x_0}\phi_{x_1} + 2u_{4x_1}\phi_{x_0} + 2u_2u_3 + u_{3x_0x_1}\right).$$

At the resonance, the singularity manifold condition is a PDE of order six, which consists of 372 terms (!) all of which are derivatives of ϕ with respect to x_0 and x_1. This condition may be written in the following form:

$$\sigma_1\Phi + \sigma_2\Psi + \left(\phi_{x_0}\Psi_{x_1} - \phi_{x_1}\Psi_{x_0} - \sigma_3\Psi - \sigma_4\Phi\right)^2 = 0, \qquad (3.18)$$

where Φ is the two-dimensional Bateman equation (2.2), and

$$\Psi = \phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0}.$$

The σ's are huge expressions consisting of derivatives of ϕ with respect to x_0 and x_1. We do not present these expressions here. Thus, the general solution of the Bateman equation satisfies the full singularity manifold condition for (3.17).

Solution (2.5) may now be exploited in the construction of exact solutions for the above wave equations, by truncating their Painlevé series. A similar method, as was used in the papers of Webb and Zank [17] and Euler [5], may be applied. This will be the subject of a future paper.

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