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Equivalence Transformations and Symmetry of the Schrödinger Equation with Variable Potential

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Abstract

We study symmetry of the Schrödinger equation with potential as a new dependent variable, i.e., transformations which do not change the form of a class of equations, which are called equivalence trasformations. We consider the systems comprising a Schrödinger equation and a certain condition for the potential. Symmetry properties of the equation with convection term are investigated.

This talk is based on the results obtained by the authors in collaboration with Prof. W. Fushchych and dedicated to his memory.

1. Introduction

Consider the following generalization of the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x},|\psi|)\psi + V_a(t,\vec{x})\frac{\partial\psi}{\partial x_a} = 0, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial x_a \partial x_a}$, $a = \overline{1,n}$, $\psi = \psi(t,\vec{x})$ is an unknown complex function, $W = W(t,\vec{x},|\psi|)$ and $V_a = V_a(t,\vec{x})$ are potentials of interaction (for convenience, we set $m = \frac{1}{2}$).

In the case where $V_a = 0$ in (1), we have the standard Schrödinger equation. Symmetry properties of this equation were thoroughly investigated (see, e.g., [1]–[4]). For arbitrary $W(t,\vec{x},|\psi|)$, equation (1) admits only the trivial group of identical transformations $\vec{x} \to \vec{x'} = \vec{x}$, $t \to t' = t$, $\psi \to \psi' = \psi$ [1], [3].

In [5]–[7], the method of extension of the symmetry group of equation (1) was suggested. The idea lies in the fact that, in equation (1), we assume that $W(t,\vec{x},|\psi|)$ and $V_a(t,\vec{x})$ are new dependent variables. This means that equation (1) is regarded as a nonlinear equation even in the case where the potential W does not depend on ψ. Symmetry operators of this type generate transformations called equivalence transformations.

2. Symmetry of the Schrödinger equation with potential

Using the above idea, we obtain the invariance algebra of the Schrödinger equation with potential

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x},|\psi|)\psi = 0. \tag{2}$$

Theorem 1. *Equation (2) is invariant under the infinite-dimensional Lie algebra with basis operators of the form*

$$\begin{aligned}
&J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\\
&Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{2}\ddot{U}_a x_a\partial_W,\\
&Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} + \frac{i}{4}\ddot{A}x_c x_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) -\\
&\qquad - \frac{n\dot{A}}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) + \left(\frac{1}{4}\dddot{A}\, x_c x_c - 2W\dot{A}\right)\partial_W,\\
&Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W, \quad Z_1 = \psi\partial_\psi, \qquad Z_2 = \psi^*\partial_{\psi^*},
\end{aligned} \tag{3}$$

where $U_a(t), A(t), B(t)$ are arbitrary smooth functions of t, there is summation from 1 to n over the repeated index c and no summation over the repeated index a, $a,b = \overline{1,n}$, the upper dot means the derivative with respect to time.

Note that the invariance algebra (3) includes the operators of space ($U_a = 1$) and time ($A = 1/2$) translations, the Galilei operator ($U_a = t$), the dilation ($A = t$) and projective ($A = t^2/2$) operators.

Proof of Theorem 1. We seek the symmetry operators of equation (2) in the class of first-order differential operators of the form:

$$X = \xi^\mu(t,\vec{x},\psi,\psi^*)\partial_{x_\mu} + \eta(t,\vec{x},\psi,\psi^*)\partial_\psi + \eta^*(t,\vec{x},\psi,\psi^*)\partial_{\psi^*} + \rho(t,\vec{x},\psi,\psi^*,W)\partial_W. \tag{4}$$

Using the invariance condition [1], [8], [9] of equation (2) under the operator (4) and the fact that $W = W(t,\vec{x},|\psi|)$, i.e., $\psi\dfrac{\partial W}{\partial\psi} = \psi^*\dfrac{\partial W}{\partial\psi^*}$, we obtain the system of determining equations:

$$\begin{aligned}
&\xi^j_\psi = \xi^j_{\psi^*} = 0, \quad \xi^0_a = 0, \quad \xi^a_a = \xi^b_b, \quad \xi^a_b + \xi^b_a = 0, \quad \xi^0_0 = 2\xi^a_a,\\
&\eta_{\psi^*} = 0, \quad \eta_{\psi\psi} = 0, \quad \eta_{\psi a} = (i/2)\xi^a_0,\\
&\eta^*_\psi = 0, \quad \eta^*_{\psi^*\psi^*} = 0, \quad \eta^*_{\psi^* a} = -(i/2)\xi^a_0,\\
&i\eta_0 + \eta_{cc} - \eta_\psi W\psi + 2W\xi^n_n\psi + W\eta + \rho\psi = 0,\\
&-i\eta^*_0 + \eta^*_{cc} - \eta^*_{\psi^*}W\psi^* + 2W\xi^n_n\psi^* + W\eta^* + \rho\psi^* = 0,\\
&\rho_\psi = \rho_{\psi^*} = 0,
\end{aligned} \tag{5}$$

where the index j varies from 0 to n, $a,b = \overline{1,n}$, we mean summation from 1 to n over the repeated index c and no summation over the indices a,b.

We solve system (5) and obtain the following result:

$$\xi^0 = 2A, \quad \xi^a = \dot{A}x_a + C^{ab}x_b + U_a, \quad a = \overline{1,n},$$

$$\eta = \frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + B\right)\psi,$$

$$\eta^* = -\frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + E\right)\psi^*,$$

$$\rho = \frac{1}{2}\left(\frac{1}{2}\dddot{A}\, x_c x_c + \ddot{U}_c x_c + \dot{B}\right) - \frac{n}{2}i\ddot{A} - 2W\dot{A},$$

where A, U_a, B are arbitrary functions of t, $E = B - 2in\dot{A} + C_1$, $C^{ab} = -C^{ba}$ and C_1 are arbitrary constants. Theorem 1 is proved.

The operators Q_B generate the finite transformations

$$\begin{cases} t' = t, \quad \vec{x}' = \vec{x}, \\ \psi' = \psi\exp(iB(t)\alpha), \quad \psi^{*'} = \psi^*\exp(-iB(t)\alpha), \\ W' = W + \dot{B}(t)\alpha, \end{cases} \tag{6}$$

where α is a group parameter, $B(t)$ is an arbitrary smooth function.

Using the Lie equations, we obtain that the following transformations correspond to the operators Q_a:

$$\begin{cases} t' = t, \; x'_b = U_a(t)\beta_a\delta_{ab} + x_b, \\ \psi' = \psi\exp\left(\frac{i}{4}\dot{U}_a U_a\beta_a^2 + \frac{i}{2}\dot{U}_a x_a\beta_a\right), \\ \psi^{*'} = \psi^*\exp\left(-\frac{i}{4}\dot{U}_a U_a\beta_a^2 - \frac{i}{2}\dot{U}_a x_a\beta_a\right), \\ W' = W + \frac{1}{2}\ddot{U}_a x_a\beta_a + \frac{1}{4}\ddot{U}_a U_a\beta_a^2, \end{cases} \tag{7}$$

where $\beta_a (a = \overline{1,n})$ are group parameters, $U_a = U_a(t)$ are arbitrary smooth functions, there is no summation over the index a, δ_{ab} is a Kronecker symbol. In particular, if $U_a(t) = t$, then the operators Q_a are the standard Galilei operators

$$G_a = t\partial_{x_a} + \frac{i}{2}x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}). \tag{8}$$

For the operators Q_A, it is difficult to write out finite transformations in the general form. We consider several particular cases:
(a) $A(t) = t$.
Then $Q_A = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W$ is a dilation operator generating the transformations

$$\begin{cases} t' = t\exp(2\lambda), \quad x'_c = x_c\exp(\lambda), \\ \psi' = \exp\left(-\frac{n}{2}\lambda\right)\psi, \quad \psi^{*'} = \exp\left(-\frac{n}{2}\lambda\right)\psi^*, \\ W' = W\exp(-2\lambda), \end{cases} \tag{9}$$

where λ is a group parameter.

(b) $A(t) = t^2/2$.

Then $Q_A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) - \frac{n}{2}t(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2tW\partial_W$ is the operator of projective transformations:

$$\begin{cases} t' = \dfrac{t}{1-\mu t}, \; x'_c = \dfrac{x_c}{1-\mu t}, \\ \psi' = \psi(1-\mu t)^{n/2}\exp\left(\dfrac{ix_c x_c \mu}{4(1-\mu t)}\right), \\ \psi^{*'} = \psi^*(1-\mu t)^{n/2}\exp\left(\dfrac{-ix_c x_c \mu}{4(1-\mu t)}\right), \\ W' = W(1-\mu t)^2, \end{cases} \tag{10}$$

μ is an arbitrary parameter.

Consider the example. Let

$$W = \frac{1}{\vec{x}^2} = \frac{1}{x_c x_c}. \tag{11}$$

We find how new potentials are generated from potential (11) under transformations (6), (7), (9), (10).

(i) Q_B:

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{x_c x_c} + B(t)\alpha \to W'' = \frac{1}{x_c x_c} + B(t)(\alpha + \alpha') \to \cdots,$$

where $B(t)$ is an arbitrary smooth function, α and α' are arbitrary real parameters.

(ii) Q_a:

$$W = \frac{1}{x_c x_c} \to W',$$

$$W' = \frac{1}{(x_a - U_a(t)\beta_a)^2 + x_b x_b} + \frac{1}{4}\ddot{U}_a U_a \beta_a^2 + \frac{1}{2}\ddot{U}_a\beta_a(x_a - U_a\beta_a),$$

$$W' \to W'',$$

$$W'' = \frac{1}{(x_a - U_a(t)(\beta_a + \beta'_a))^2 + x_b x_b} + \frac{1}{4}\ddot{U}_a U_a(\beta_a^2 + \beta_a'^2) + \\ + \frac{1}{2}\ddot{U}_a(\beta_a + \beta'_a)(x_a - U_a(\beta_a + \beta'_a)) + \frac{1}{2}\ddot{U}_a U_a \beta_a \beta'_a \to \cdots,$$

where U_a are arbitrary smooth functions, β_a and β'_a are real parameters, there is no summation over a but there is summation over b ($b \neq a$). In particular, if $U_a(t) = t$, then we have the standard Galilei operator (8) and

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{(x_a - t\beta_a)^2 + x_b x_b} \to W'' = \frac{1}{(x_a - t(\beta_a + \beta'_a))^2 + x_b x_b} \to \cdots$$

(iii) Q_A for $A(t) = t$ or $A(t) = t^2/2$ do not change the potential, i.e.,

$$W = \frac{1}{x_c x_c} \to W' = \frac{1}{x_c x_c} \to W'' = \frac{1}{x_c x_c} \to \cdots$$

3. The Schrödinger equation and conditions for the potential

Consider several examples of the systems in which one of the equations is equation (2) with potential $W = W(t, \vec{x})$, and the second equation is a certain condition for the potential W. We find the invariance algebras of these systems in the class of operators

$$X = \xi^\mu(t, \vec{x}, \psi, \psi^*, W)\partial_{x_\mu} + \eta(t, \vec{x}, \psi, \psi^*, W)\partial_\psi + \eta^*(t, \vec{x}, \psi, \psi^*, W)\partial_{\psi^*} + \\ + \rho(t, \vec{x}, \psi, \psi^*, W)\partial_W.$$

(i) Consider equation (2) with the additional condition for the potential, namely, the Laplace equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x})\psi = 0, \\ \Delta W = 0. \end{cases} \tag{12}$$

System (12) admits the infinite-dimensional Lie algebra with the basis operators

$$\begin{aligned} &P_0 = \partial_t,\ P_a = \partial_{x_a},\ J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \\ &Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{2}\ddot{U}_a x_a\partial_W,\ a = \overline{1,n}, \\ &D = x_c\partial_{x_c} + 2t\partial_t - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W, \\ &A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) - \frac{n}{2}t(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2Wt\partial_W, \\ &Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W,\ Z_1 = \psi\partial_\psi,\ Z_2 = \psi^*\partial_{\psi^*}, \end{aligned} \tag{13}$$

where $U_a(t)$ ($a = \overline{1,n}$) and $B(t)$ are arbitrary smooth functions. In particular, algebra (13) includes the Galilei operator (8).

(ii) The condition for the potential is the wave equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x})\psi = 0, \\ \Box W = 0. \end{cases} \tag{14}$$

The maximal invariance algebra of system (14) is $\langle P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,\ Z_3, Z_4\rangle$, where P_0, P_a, J_{ab}, Z_1, Z_2 have the form (13) and

$$Z_3 = it(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \partial_W, \qquad Z_4 = it^2(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + 2t\partial_W.$$

(iii) Consider the important case in a $(1+1)$-dimensional space–time where the condition for the potential is the KdV equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \dfrac{\partial^2\psi}{\partial x^2} + W(t, x)\psi = 0, \\ \dfrac{\partial W}{\partial t} + \lambda_1 W\dfrac{\partial W}{\partial x} + \lambda_2\dfrac{\partial^3 W}{\partial x^3} = F(|\psi|), \quad \lambda_1 \neq 0. \end{cases} \tag{15}$$

For an arbitrary $F(|\psi|)$, system (15) is invariant under the Galilei operator and the maximal invariance algebra is the following:

$$\begin{aligned} &P_0 = \partial_t,\ P_1 = \partial_x,\ Z = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \\ &G = t\partial_x + \frac{i}{2}\left(x + \frac{2}{\lambda_1}t\right)(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{\lambda_1}\partial_W. \end{aligned} \tag{16}$$

For $F = C = \text{const}$, system (15) admits the extension, namely, it is invariant under the algebra $\langle P_0, P_1, G, Z_1, Z_2\rangle$, where G has the form (16).

The Galilei operator G generates the following transformations:

$$\begin{cases} t' = t, \ x' = x + \theta t, \quad W' = W + \dfrac{1}{\lambda_1}\theta, \\ \psi' = \psi \exp\left(\dfrac{i}{2}\theta x + \dfrac{i}{\lambda_1}\theta t + \dfrac{i}{4}\theta^2 t\right), \\ \psi^{*'} = \psi^* \exp\left(-\dfrac{i}{2}\theta x - \dfrac{i}{\lambda_1}\theta t - \dfrac{i}{4}\theta^2 t\right), \end{cases}$$

where θ is a group parameter. Here, it is important that $\lambda_1 \neq 0$, since otherwise, system (15) does not admit the Galilei operator.

4. The Schrödinger equation with convection term

Consider equation (1) for $W = 0$, i.e., the Schrödinger equation with convection term

$$i\frac{\partial \psi}{\partial t} + \Delta\psi = V_a \frac{\partial \psi}{\partial x_a}, \tag{17}$$

where ψ and V_a $(a = \overline{1,n})$ are complex functions of t and $\vec{x}$. For extension of symmetry, we again regard the functions V_a as dependent variables. Note that the requirement that the functions V_a be complex is essential for symmetry of (17).

Let us investigate symmetry of (17) in the class of first-order differential operators

$$X = \xi^\mu \partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a \partial_{V_a} + \rho^{*a}\partial_{V_a^*},$$

where $\xi^\mu, \eta, \eta^*, \rho^a, \rho^{*a}$ are functions of $t, \vec{x}, \psi, \psi^*, \vec{V}, \vec{V}^*$.

Theorem 2. *[10] Equation (17) is invariant under the infinite-dimensional Lie algebra with the basis operators*

$$\begin{aligned} &Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} - i\ddot{A}x_c(\partial_{V_c} - \partial_{V_c^*}) - \dot{A}(V_c\partial_{V_c} + V_c^*\partial_{V_c^*}), \\ &Q_{ab} = E_{ab}(x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}) - \\ &\qquad - i\dot{E}_{ab}(x_a\partial_{V_b} - x_b\partial_{V_a} - x_a\partial_{V_b^*} + x_b\partial_{V_a^*}), \\ &Q_a = U_a\partial_{x_a} - i\dot{U}_a(\partial_{V_a} - \partial_{V_a^*}), \ Z_1 = \psi\partial_\psi, \ Z_2 = \psi^*\partial_{\psi^*}, \ Z_3 = \partial_\psi, \ Z_4 = \partial_{\psi^*}, \end{aligned} \tag{18}$$

where A, E_{ab}, U_a are arbitrary smooth functions of t, we mean summation over the index c and no summation over indices a and b.

This theorem is proved by analogy with the previous one.

Note that algebra (18) includes as a particular case the Galilei operator of the form:

$$\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{19}$$

This operator generates the following finite transformations:

$$\begin{cases} x'_b = x_b + \beta_a t\delta_{ab}, \quad t' = t, \\ \psi' = \psi, \ \psi^{*'} = \psi^*, \quad V'_b = V_b - i\beta_a\delta_{ab}, \quad V_b^{*'} = V_a^* + i\beta_a\delta_{ab}, \end{cases}$$

where β_a is an arbitrary real parameter. Operator (19) is essentially different from the standard Galilei operator (8) of the Schrödinger equation, and we cannot derive operator (8) from algebra (18).

Consider now the system of equation (17) with the additional condition for the potentials V_a, namely, the complex Euler equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi = V_a\dfrac{\partial\psi}{\partial x_a}, \\ i\dfrac{\partial V_a}{\partial t} - V_b\dfrac{\partial V_a}{\partial x_b} = F(|\psi|)\dfrac{\partial\psi}{\partial x_a}, \end{cases} \tag{20}$$

Here, ψ and V_a are complex dependent variables of t and $\vec{x}$, F is a function of $|\psi|$. The coefficients of the second equation of the system provide the broad symmetry of this system.

Let us investigate the symmetry classification of system (20). Consider the following five cases:

1. F is an arbitrary smooth function.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a\rangle$, where

$$J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*},$$

$\widetilde{G}_a$ has the form (19).

2. $F = C|\psi|^k$, where C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}\rangle$, where

$$D^{(1)} = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*} - \frac{2}{1+k}(\psi\partial_\psi + \psi^*\partial_{\psi^*}).$$

3. $F = \dfrac{C}{|\psi|}$, where C is an arbitrary complex constant, $C \neq 0$.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, Z = Z_1 + Z_2\rangle$, where

$$Z = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}.$$

4. $F = C \neq 0$, where C is an arbitrary complex constant.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}, Z_3, Z_4\rangle$, where

$$Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*}.$$

5. $F = 0$.
In this case, system (20) admits the widest maximal invariance algebra, namely, $\langle P_0$, P_a, J_{ab}, $\widetilde{G}_a$, D, A, Z_1, Z_2, Z_3, $Z_4\rangle$, where

$$D = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*},$$
$$A = t^2\partial_t + tx_c\partial_{x_c} - (ix_c + tV_c)\partial_{V_c} + (ix_c - tV_c^*)\partial_{V_c^*}.$$

In conclusion, we note that the equivalence groups can be successfully used for construction of exact solutions of the nonlinear Schrödinger equation.

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