Symmetry in Nonlinear Mathematical Physics 1997, V.1, 164–171.

Group Analysis of Ordinary Differential Equations of the Order $n > 2$

L.M. BERKOVICH and S.Y. POPOV

Samara State University, 443011, Samara, Russia
E-mail: berk@info.ssu.samara.ru

Abstract

This paper deals with three strategies of integration of an n-th order ordinary differential equation, which admits the r-dimensional Lie algebra of point symmetries. These strategies were proposed by Lie but at present they are not well known. The "first" and "second" integration strategies are based on the following main idea: to start from an n-th order differential equation with r symmetries and try to reduce it to an $(n-1)$-th order differential equation with $r-1$ symmetries. Whether this is possible or not depends on the structure of the Lie algebra of symmetries. These two approaches use the normal forms of operators in the space of variables ("first") or in the space of first integrals ("second"). A different way of looking at the problem is based on the using of differential invariants of a given Lie algebra.

1. Introduction

The experience of an ordinary differential equation (ODE) with one symmetry which could be reduced in order by one and of a second order differential equation with two symmetries which could be solved may lead us to the following question: *Is it possible to reduce a differential equation with r symmetries in order by r?* In full generality, the answer is "no". This paper deals with three integration strategies which are based on the group analisys of an n-th order ordinary differential equation (ODE-n, $n > 2$) with r symmetries ($r > 1$). These strategies were proposed by S. Lie (see [1–2]) but at present they are not well known. We studied the connection between the structure of a Lie algebra of point symmetries and the integrability conditions of a differential equation. We refer readers to the literature where these approaches are described (see [3], [6], [7]).

Suppose we have an n-th order ordinary differential equation (ODE-n, $n > 2$)

$$y^{(n)} = \omega\left(x, y, y', \ldots, y^{(n-1)}\right), \tag{1.1}$$

which admits r point symmetries $X_1, X_2, \ldots, X_r$. It is well known (due to Lie) that $r \leq n + 4$.

Definition. *The infinitesimal generator*

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.2}$$

Present work was partially financed by RFBR, the grant 96–01–01997.

is called a point symmetry of ODE-n (1.1) if

$$X^{(n-1)}\omega\left(x, y, y', \ldots, y^{(n-1)}\right) \equiv \eta^{(n)}\left(x, y, y', \ldots, y^{(n)}\right) \quad (\text{mod } y^{(n)} = \omega) \tag{1.3}$$

holds; here,

$$\begin{aligned} X^{(n-1)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y')\frac{\partial}{\partial y'} + \cdots \\ + \eta^{(n-1)}\left(x, y, y', \ldots, y^{(n-1)}\right)\frac{\partial}{\partial y^{(n-1)}} \end{aligned} \tag{1.4}$$

is an extension (prolongation) X up to the n-th derivative.

Consider the differential operator

$$A = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \cdots + \omega\left(x, y, y', \ldots, y^{(n-1)}\right)\frac{\partial}{\partial y^{(n-1)}}. \tag{1.5}$$

It is not difficult to see that A can formally be written as

$$A \equiv \frac{d}{dx} \quad (\text{mod } y^{(n)} = \omega).$$

Proposition 1. *Differential equation (1.1) admits the infinitesimal generator $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ iff $[X^{(n-1)}, A] = -(A(\xi(x,y)))A$ holds.*

Concept of proof. Let $\varphi_i(x, y, y', ..., y^{(n-1)}),\ i = \overline{1,n}$ be the set of functionally independent first integrals of ODE-n (1.1); then $\{\varphi_i\}_{i=1}^n$ are functionally independent solutions of the partial differential equation

$$A\varphi = 0. \tag{1.6}$$

It's easy to show that ODE-n (1.1) admits the infinitesimal generator (1.2) iff $X^{(n-1)}\varphi_i$ is a first integral of ODE-n (1.1) for all $i = \overline{1,n}$.

So, on the one hand, we have that the partial differential equations (1.6) and

$$\left[X^{(n-1)}, A\right]\varphi = 0 \tag{1.7}$$

are equivalent iff X is a symmetry of (1.1). On the other hand, we have that (1.6) and (1.7) are equivalent iff

$$\left[X^{(n-1)}, A\right] = \lambda\left(x, y, y', \ldots, y^{(n-1)}\right)A \tag{1.8}$$

holds.

Comparing the coefficient of $\frac{\partial}{\partial x}$ on two sides of (1.8) yields $\lambda = -A(\xi(x,y))$ ■

2. First integration strategy: normal forms of generators in the space of variables

Take one of the generators, say, X_1 and transform it to its normal form $X_1 = \dfrac{\partial}{\partial s}$, i.e., introduce new coordinates t (independent) and s (dependent),where the functions $t(x,y)$, $s(x,y)$ satisfy the equations $X_1 t = 0$, $X_1 s = 1$. This procedure allows us to transform the differential equation (1.1) into

$$s^{(n)} = \Omega\left(t, s', \dots, s^{(n-1)}\right), \tag{2.1}$$

which, in fact, is a differential equation of order $n-1$ (we take s' as a new dependent variable). Now we interest in the following question:

Does (2.1) really inherit $r-1$ symmetries from (1.1), which are given by (2.2)?

$$Y_i = X_i^{(n-1)} - \overline{\eta}(t,s)\frac{\partial}{\partial s}. \tag{2.2}$$

The next theorem answers this question.

Theorem 1. *The infinitesimal generators* $Y_i = X_i^{(n-1)} - \overline{\eta}(t,s)\dfrac{\partial}{\partial s}$ *are the symmetries of ODE-(n-1) (2.1) if and only if*

$$[X_1, X_i] = \lambda_i X_1, \qquad \lambda_i = \text{const}, \quad i = \overline{2,r}, \tag{2.3}$$

hold.

So, if we want to follow this first integration strategy for a given algebra of generators, we should choose a generator X_1 at the first step (as a linear combination of the given basis), for which we can find as many generators X_a satisfying (2.3) as possible; choose Y_a and try to do everything again.

At each step, we can reduce the order of a given differential equation by one.

Example 1. The third-order ordinary differential equation $4y^2y''' = 18yy'y'' - 15y'^3$ admits the symmetries

$$X_1 = \frac{\partial}{\partial x}; \quad X_2^{(2)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''}; \quad X_3^{(2)} = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''}.$$

Remark. All examples presented in this paper only illustrate how one can use these integration strategies.

Note the relations $[X_1, X_2] = X_1$, $[X_1, X_3] = 0$.

Transform X_1 to its normal form by introducing new coordinates: $t = y; s = x$. Now we have the ODE-2

$$s''' = \frac{3s''^2}{s'} + \frac{18ts's'' + 15s'^2}{4t^2s'}$$

with symmetries

$$Y_2 = s'\frac{\partial}{\partial s'} + s''\frac{\partial}{\partial s''}; \quad Y_3 = t\frac{\partial}{\partial t} - s''\frac{\partial}{\partial s''}; \quad [Y_2, Y_3] = 0.$$

Transform Y_2 to its normal form: $v = \log t$; $u = \log s'$

$$u'' = 2u'^2 + \frac{11}{2}u' + \frac{9}{2},$$

$$x = c_2 \int \frac{dy}{y^{\frac{11}{8}} \left(\cos\left(\frac{\sqrt{23}}{4}\log c_1 y\right)\right)^{1/2}} + c_3.$$

3. Second integration strategy: the normal form of a generator in the space of first integrals

We begin with the assumptions:

a) $r = n$;

b) $X_i, i = \overline{1,n}$, act transitively in the space of first integrals, i.e., there is no linear dependence between $X_i^{(n-1)}$, $i = \overline{1,n}$, and A.

We'll try to answer the following question:

Does a solution to the system of equations

$$X_1^{(n-1)}\varphi = \left(\xi_1\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial y} + \ldots + \eta_1^{(n-1)}\frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 1, \tag{3.1}$$

$$X_i^{(n-1)}\varphi = \left(\xi_i\frac{\partial}{\partial x} + \eta_i\frac{\partial}{\partial y} + \ldots + \eta_i^{(n-1)}\frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 0, \qquad i = \overline{2,n}, \tag{3.2}$$

$$A\varphi = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \ldots + \omega\frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 0, \tag{3.3}$$

exist?

A system of n homogeneous linear partial differential equations in $n+1$ variables $(x, y, \ldots, y^{(n-1)})$ (3.2)–(3.3) has a solution if all commutators between $X_i^{(n-1)}$, $i = \overline{2,n}$, and A are linear combinations of the same operators. It's easy to check that these integrability conditions are fulfilled iff $X_i, i = \overline{2,n}$, generate an $(n-1)$-dimensional Lie subalgebra in the given Lie algebra of point symmetries.

Let φ be a solution to system (3.1)–(3.3), then

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]\varphi = X_1^{(n-1)}\left(X_i^{(n-1)}\varphi\right) - X_i^{(n-1)}\left(X_1^{(n-1)}\varphi\right) = 0 \tag{3.4}$$

necessarily holds. On the other hand, we have

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]\varphi = C_{1i}^1 X_1^{(n-1)}(\varphi) + C_{1i}^k X_k^{(n-1)}(\varphi) = C_{1i}^1, \qquad i,k = \overline{2,n}. \tag{3.5}$$

(3.4) and (3.5) do not contradict each other if and only if

$$C_{1i}^1 = 0, \qquad i = \overline{2,n}. \tag{3.6}$$

All preceding reasonings lead us to the necessary condition of existence of the function φ. This condition is also sufficient. Now we prove it. Let $u \neq \text{const}$ be a solution to system (3.2)–(3.3), then we have

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]u = X_1^{(n-1)}\left(X_i^{(n-1)}u\right) - X_i^{(n-1)}\left(X_1^{(n-1)}u\right) =$$
$$= -X_i^{(n-1)}\left(X_1^{(n-1)}u\right) = 0, \quad i = \overline{2,n},$$

that is, $X_1^{(n-1)}u$ is a nonzero solution to system (3.2)–(3.3). Hence, $X_1^{(n-1)}u = f(u)$. It is not difficult to check that the function $\int \frac{du}{f(u)}$ is a solution to system (3.1)–(3.3).

Suppose that the integrability conditions for system (3.1)–(3.3) are fulfilled. Now we consider this system as a system of linear algebraic equations in $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \ldots, \frac{\partial\varphi}{\partial y^{(n-1)}}$. We can solve this system using Cramer's rule:

$$\boldsymbol{\Delta} = \begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \dots & \omega \end{vmatrix} \neq 0; \quad \frac{\partial\varphi}{\partial x} = \boldsymbol{\Delta}^{-1} \begin{vmatrix} 1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ 0 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 0 & y' & y'' & \dots & \omega \end{vmatrix};$$

$$\frac{\partial\varphi}{\partial y} = \boldsymbol{\Delta}^{-1} \begin{vmatrix} \xi_1 & 1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ \xi_2 & 0 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & 0 & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & 0 & y'' & \dots & \omega \end{vmatrix}; \ldots; \frac{\partial\varphi}{\partial y^{(n-1)}} = \boldsymbol{\Delta}^{-1} \begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & 1 \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & 0 \\ 1 & y' & y'' & \dots & 0 \end{vmatrix}.$$

The differential form

$$d\varphi = \boldsymbol{\Delta}^{-1} \begin{vmatrix} dx & dy & dy' & \dots & dy^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \dots & \omega \end{vmatrix}$$

is a differential of the solution φ to system (3.1)–(3.3).

Theorem 2. *Suppose point symmetries $X_i, i = \overline{1,n}$, act transitively in the space of first integrals; then there exists a solution to the system $X_1^{(n-1)}\varphi = 1$; $X_i^{(n-1)}\varphi = 0$, $i = \overline{2,n}$; $A\varphi = 0$ if and only if X_i, $i = \overline{2,n}$, generate an $(n-1)$-dimensional ideal in the given Lie algebra of point symmetries. This solution is as follows:*

$$\varphi = \int \frac{\begin{vmatrix} dx & dy & dy' & \dots & dy^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \dots & \omega \end{vmatrix}}{\begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \dots & \omega \end{vmatrix}}. \tag{3.7}$$

Now we can use $\varphi\left(x, y, y', \ldots, y^{(n-1)}\right)$ instead of $y^{(n-1)}$ as a new variable. In new variables, we have

$$y^{(n-1)} = y^{(n-1)}\left(x, y, y', \ldots, y^{(n-2)}; \varphi\right), \tag{3.8}$$

$$X_i^{(n-2)} = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \cdots + \eta_i^{(n-2)} \frac{\partial}{\partial y^{(n-2)}}, \qquad i = \overline{2, n}, \tag{3.9}$$

$$A = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \cdots + y^{(n-1)}\left(x, y, y', \ldots, y^{(n-2)}; \varphi\right) \frac{\partial}{\partial y^{(n-2)}}. \tag{3.10}$$

System (3.8)–(3.10) is exactly what we want to achieve. Now we can establish an iterative procedure.

Example 2. $2y'y''' = 3y''^2$. This equation admits the 3-dimensional Lie algebra of point symmetries with the basis.

$$X_1 = \frac{\partial}{\partial y}; \quad X_2^{(2)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''}; \quad X_3 = \frac{\partial}{\partial x}$$

and commutator relations: $[X_1, X_2] = 0,\ [X_1, X_3] = 0,\ [X_2, X_3] = -X_3$.
The given ODE-3 is equivalent to the equation $\{y, x\} = 0$, where

$$\{y, x\} = 1/2\frac{y'''}{y'} - 3/4\frac{y''^2}{y'^2}$$

is Schwarz's derivative.

$$\mathbf{\Delta} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ x & 0 & -y' & -2y'' \\ 1 & 0 & 0 & 0 \\ 1 & y' & y'' & \dfrac{3y''^2}{2y'} \end{vmatrix} = y''^2/2 \neq 0, \quad \varphi_1 = \int \frac{\begin{vmatrix} dx & dy & dy' & dy'' \\ x & 0 & -y' & -2y'' \\ 1 & 0 & 0 & 0 \\ 1 & y' & y'' & \dfrac{3y''^2}{2y'} \end{vmatrix}}{\mathbf{\Delta}} = y - \frac{2y'^2}{y''}.$$

Now we have the ODE-2: $y'' = \dfrac{2y'^2}{y - \varphi_1}$, which admits the generators

$$X_2^{(1)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'}; \quad X_3 = \frac{\partial}{\partial x};$$

$$\mathbf{\Delta_1} = \begin{vmatrix} x & 0 & -y' \\ 1 & 0 & 0 \\ 1 & y' & \dfrac{2y'^2}{y - \varphi_1} \end{vmatrix} = -y'^2. \quad \varphi_2 = \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & 0 & 0 \\ 1 & y' & \dfrac{2y'^2}{y - \varphi_1} \end{vmatrix}}{\mathbf{\Delta_1}} = \log\frac{(y - \varphi_1)^2}{y'},$$

$$y' = \frac{(y - \varphi_1)^2}{\exp\varphi_2}.$$

A general solution of the differential equation is given by the next function: $y = \dfrac{ax + b}{cx + d}$.

4. Third integration strategy: differential invariants

Definition. *Differential invariants of order k (DI-k) are functions*

$$\psi\left(x, y, y', ..., y^{(k)}\right), \qquad \frac{\partial \psi}{\partial y^{(k)}} \not\equiv 0,$$

that are invariant under the action of $X_1, \ldots, X_r$, that is, satisfy r equations ($i = \overline{1, r}$):

$$X_i^{(k)}\psi = \left(\xi_i(x, y)\frac{\partial}{\partial x} + \eta_i(x, y)\frac{\partial}{\partial y} + \cdots + \eta_i^{(k)}\left(x, y, y', \ldots, y^{(k)}\right)\frac{\partial}{\partial y^{(k)}}\right)\psi = 0. \quad (4.1)$$

How can one find differential invariants? To do it, we must know two lowest order invariants φ and ψ.

Theorem 3. *If $\psi\left(x, y, y', \ldots, y^{(l)}\right)$ and $\varphi\left(x, y, y', \ldots, y^{(s)}\right)$ $(l \leq s)$ are two lowest order differential invariants, then*

1) $s \leq r$; 2) List of all functionally independent differential invariants is given by the following sequence:

$$\psi, \varphi, \frac{d\varphi}{d\psi}, \ldots, \frac{d^n\varphi}{d\psi^n}, \ldots$$

Let X_i, $i = \overline{1, r}$, be symmetries of (1.1), then we have the list of functionally independent DI up to the n-th order:

$$\psi, \varphi, \frac{d\varphi}{d\psi}, \ldots, \frac{d^{(n-s)}\varphi}{d\psi^{(n-s)}}.$$

Express all derivatives $y^{(k)}$, $k \geq s$, in terms ψ, φ, and derivatives $\varphi^{(m)}$ beginning with the highest order. That will give ODE-$(n - s)$:

$$H\left(\psi, \varphi, \varphi', \ldots, \frac{d^{(n-i)}\varphi}{d\psi^{(n-s)}}\right) = 0.$$

Unfortunately, this equation does not inherit any group information from (1.1). If it's possible to solve (4.2), then we have ODE-s

$$\varphi\left(x, y, y', \ldots, y^{(s)}\right) = f\left(\psi(x, y, \ldots, y^{(l)})\right) \qquad (4.2)$$

with r symmetries.

Example 3. ODE-3 $yy'y''' = y'^2y'' + yy''^2$ admits the 2-dimensional Lie algebra of point symmetries $X_1 = \dfrac{\partial}{\partial x}$; $X_2 = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

DI-0 does not exist. DI-1: $\psi = \dfrac{y'}{y^2}$, DI-2: $\varphi = \dfrac{y''}{y^3}$. Now we can find DI-3:

$$\frac{d\varphi}{d\psi} = \varphi' = \frac{y'''y - 3y''y'}{y(yy'' - 2y'^2)}.$$

Express y''', y'', y' in terms of ψ, φ, φ'. This procedure leads us to ODE-1: $\varphi' = \dfrac{\varphi}{\psi}$. Hence, we obtain $\varphi = C\psi$ or $y'' = Cyy'$, that is, we obtain ODE-2 with 2 symmetries, which can be solved as

$$x = \int \frac{2dy}{Cy^2 + a} + b.$$

In conclusion, we have to note that we have discussed only some simple strategies. A different way of looking at the problem is described in [9] and based on using both point and nonpoint symmetries.

References

[1] Lie S., Klassification und Integration von gewöhnlichen Differential gleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten, *Math. Annalen*, 1888, **32**, 213–281; Gesamlte Abhundungen, V.5, B.G. Teubner, Leipzig, 1924, 240–310.

[2] Lie S., Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen, B.G. Teubner, Leipzig, 1891.

[3] Chebotarev M.G., Theory of Lie Groups, M.-L., Gostekhizdat, 1940 (in Russian).

[4] Pontrjagin L.S., Continuous Groups, M.-L., Gostekhizdat, 1954 (in Russian).

[5] Ovsyannikov L.V., Group Analysis of Differential Equations, Academic Press, N.Y., 1982.

[6] Stephani H., Differential Equations. Their solutions using symmetries, Edit. M. Maccallum, Cambridge, Cambridge Univ. Press, 1989.

[7] Olver P.J., Applications of Lie Groups to Differential Equations (second edition), Springer-Verlag, 1993.

[8] Ibragimov N.H., Group analysis of ordinary differential equations and principle of invariance in mathematical physics, *Uspekhi Mat. Nauk*, 1992, V.47, N 4, 83–144 (in Russian).

[9] Berkovich L.M., The method of an exact linearization of n-order ordinary differential equations, *J. Nonlin. Math. Phys.*, 1996, V.3, N 3–4, 341–350.

[10] Berkovich L.M. and Popov S.Y., Group analysis of the ordinary differential equations of the order greater than two, *Vestnik SamGU*, 1996, N 2, 8–22 (in Russian).