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To the Classification of Integrable Systems in 1+1 Dimensions

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Abstract

The aim of this article is to classify completely integrable systems of the following form $u_t = u_3 + f(u, v, u_1, v_1, u_2, v_2,)$, $v_t = g(u, v, u_1, v_1)$. Here, $u_i = \partial u/\partial x^i$, $u_t = \partial u/\partial t$. The popular symmetry approach to the classification of integrable partial differential systems requires large calculations. That is why we applied the simpler "Chinese" method that deals with canonical conserved densities. Moreover, we proved and applied some additional integrability conditions. These conditions follow from the assumption that the Noether operator exists.

1 Introduction

This article contains our recent results on a classification of the following completely integrable systems

$$u_t = u_3 + f(u, v, u_1, v_1, u_2, v_2), \qquad v_t = g(u, v, u_1, v_1) \tag{1}$$

Here, $u_t = \partial u/\partial t$, $u_1 = \partial u/\partial x$, $u_2 = \partial^2 u/\partial x^2$ and so on. We used the so-called "Chinese" method of classification [1] that was developed in the recent years (see [2], [3], for example).

Let us consider the evolutionary partial differential system

$$u_t = K(u_1, u_2, \ldots, u_q) \tag{2}$$

with two independent variables t, x and the m-dependent $u = \{u^1, u^2, \ldots, u^m\}$. Let K' be a Frechet derivative of the operator K and K'^+ be the formally conjugate operator for K':

$$(K')^\alpha_\beta(D) = \frac{\partial K^\alpha}{\partial u^\beta_n} D^n, \qquad (K'^+)^\alpha_\beta(D) = (-D)^n \frac{\partial K^\beta}{\partial u^\alpha_n}.$$

Here D is the total differentiation operator with respect to x and the summation over the index n is implied.

The "Chinese" method deals with the following equation

$$\left[D_t + \theta + K'^+(D + \rho)\right] a = 0, \tag{3}$$

where D_t is the derivative along trajectories of system (2), the functions θ and ρ satisfy the continuity equation

$$D_t \rho = D\theta \tag{4}$$

and the vector function a satisfies the normalization condition $(c, a) = 1$ with a constant vector c.

It is proved in [3] that if system (2) admits the Lax representation and satisfies some additional conditions, then equation (3) generates a sequence of *local* conservation laws for system (2). The notion *local function* F means that F depends on $t, x, u^\alpha, u_1^\alpha, \ldots, u_n^\alpha$ only and $n < \infty$. (Local function does not depend on any integrals in the form $\int h(t,x,u)\,dx$.) The conserved densities ρ_k and the currents θ_k follow from the formal series expansions

$$\rho = \sum_{k=0}^{\infty} \rho_k z^{k-n}, \quad \theta = \sum_{k=l}^{\infty} \theta_k z^{k-n}, \quad a = \sum_{k=0}^{\infty} a_k z^k, \tag{5}$$

where z is a parameter and n is a positive integer. Substituting expansions (5) into equation (3), one can obtain ρ_k as differential polynomials K. Then equation (4) provides the infinity of the local conservation laws $D_t\rho_k = D\theta_k$. As $\rho_i = \rho_i(K)$, constraints for the function K are obtained. The explicit form of these constraints are $\delta(D_t\rho_i)/\delta u^\alpha = 0$, where $\delta/\delta u^\alpha$ is variational derivative. The conserved densities ρ_i arising from equation (3) are called canonical densities.

If system (3) has a formal local solution in the form (5) satisfying equation (4), then we call system (2) formally integrable.

2 Systems with a Noether operator

According to the definition [4], a Noether operator N satisfies the following equation

$$(D_t - K')N = N(D_t + K'^{+}) \tag{6}$$

Theorem. *If the formally integrable system (2) admits the Noether operator N, then the equation*

$$[D_t + \tilde{\theta} - K'(D + \tilde{\rho})]\tilde{a} = 0 \tag{7}$$

generates the same canonical densities ρ_i as equation (3).

Proof. Let us set $\omega = \int \rho\,dx + \theta\,dt$ and $\gamma = a\exp(\omega)$, where a is a solution of equation (3). Then the following obvious equation is valid

$$[D_t + K'^{+}(D)]\gamma = \mathrm{e}^{\omega}[D_t + \theta + K'^{+}(D + \rho)]a = 0.$$

Therefore, equation (6) implies

$$[D_t + \theta - K'(D + \rho)]\tilde{a} = 0, \tag{8}$$

where $\tilde{a} = \mathrm{e}^{-\omega}N(D)\mathrm{e}^{\omega}a = N(D+\rho)a$. It was proved in [3] that one may require the constraint $(\tilde{a}, \tilde{c}) = 1$. This is equivalent to the gauge transformation

$$\rho(u,z) = \tilde{\rho}(u,z) + D\xi(u,z), \qquad \theta(u,z) = \tilde{\theta}(u,z) + D_t\xi(u,z),$$

where ξ is a holomorphic function of z. This completes the proof.

The known today Noether operators take the following form

$$N^{\alpha\beta} = \sum_{k=0}^{p} N_k^{\alpha\beta} D^k + \sum_{i=1}^{r} A_i^{\alpha} D^{-1} B_i^{\beta} \tag{9}$$

and we consider below this case only.

Proposition. *If a Noether operator takes the form (9), then the vector function* $\tilde{a} = N(D+\rho)a$ *can be represented by the Laurent series in a parameter* z.

Proof. If ρ and a are given by series (5), then the expression $(D+\rho)^k a = D^k a + kD^{k-1}\rho a + \cdots$ is the Laurent series obviously. Let us consider the last term in expression (9) and denote $\mathrm{e}^{-\omega} D^{-1} \mathrm{e}^{\omega}(B_i, a) \equiv h$. This is equivalent to the following equation for h:

$$(D+\rho)h = (B_i, a) = \sum_{k=0}^{\infty} (B_i, a_k) z^k.$$

Setting $h = \sum h_i z^i$, we obtain $h_i = 0$ for $i < n$, $h_n = \rho_0^{-1}(B_i, a_0)$, $h_{n+1} = \rho_0^{-1}[(B_i, a_1) - \rho_1 h_n - \delta_n^1 D h_n]$ and so on. So, the last term in expression (9) gives the Taylor series and this completes the proof.

As equation (8) contains $\tilde{a}$ in the first power, we can multiply $\tilde{a}$ by any power of z. Hence, we can consider $\tilde{a}$ the Taylor series. It was mentioned above that we can submit the vector $\tilde{a}$ to a normalization condition $(\tilde{c}, \tilde{a}) = 1$ by the gauge transformation. It is important to stress that this gauge transformation does not change the densities $\rho_0, \ldots, \rho_{n-1}$ as $\xi(z)$ is a holomorphic function. Hence, the conserved densities ρ_i and $\tilde{\rho}_i$ obtained from equations (3) and (7), respectively, satisfy the following conditions

$$\rho_i = \tilde{\rho}_i \text{ for } i < n, \quad \rho_i - \tilde{\rho}_i \in \mathrm{Im} D \quad \text{for} \quad i \geq n. \tag{10}$$

These conditions give very strong constraints for system (2) and we must explain why they are relevant. It is well known that systems integrable by the inverse scattering transform method possess the Hamiltonian structures. It is also known that any Hamiltonian (or implectic) operator is a Noether operator for the associated evolution system [4]. Hence, for a wide class of integrable systems, conditions (10) are valid.

To use conditions (10), we must choose the correct normalization vector $\tilde{c}$ for $\tilde{a}$. Let us write the matrix operator K' in the form $K' = K_q D^q + K_{q-1} D^{q-1} + \cdots$, then the vector a_0 is a eigenvector of the matrix K_q^T

$$K_q^T a_0 = \lambda a_0, \qquad \lambda = (-1)^{q+1} \theta_l / \rho_0^q, \tag{11}$$

where $l = n(1-q)$ [3]. As the series expansions for $\tilde{\rho}$, $\tilde{\theta}$ and $\tilde{a}$ take the same form (5) and $\tilde{\theta}_l = \theta_l$, $\tilde{\rho}_0 = \rho_0$ according to (10), then we easily obtain

$$K_q \tilde{a}_0 = (-1)^{q+1} \lambda \tilde{a}_0. \tag{12}$$

Equations (11) and (12) define the normalization of the vector $\tilde{a}$, but some ambiguity is always possible [3].

3 Classification results

For system (1), we set $u^1 = u,\ u^2 = v$. Then

$$K_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda = \tilde{\lambda} = 1.$$

One can see now that $a_0 = \tilde{a}_0 = (1,0)^T$. This means that we can choose $c = \tilde{c} = (1,0)^T$ or equivalently $a = (1,b)^T,\ \tilde{a} = (1,\tilde{b})^T$. The series expansions (5) take now the following form

$$\rho = z^{-1} + \sum_{i=0}^{\infty} \rho_i z^i, \qquad \theta = z^{-3} + \sum_{i=0}^{\infty} \theta_i z^i,$$

$\tilde{\rho}$ and $\tilde{\theta}$ has the same form. Substituting these series into equations (3) and (7), we obtain the recursion relations for ρ_i and $\tilde{\rho}_i$. We can not present these relations here because they take large room. Here are the first terms of the sequences of ρ_i and $\tilde{\rho}_i$:

$$\rho_0 = -\tilde{\rho}_0 = \frac{1}{3}\frac{\partial f}{\partial u_2}, \qquad \rho_1 = \tilde{\rho}_1 = \rho_0^2 - \frac{1}{3}\frac{\partial f}{\partial u_1},$$

$$\rho_2 = -\tilde{\rho}_2 = \frac{1}{3}\left(\theta_0 - \rho_0^3 + 3\rho_0\rho_1 + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v_2}\frac{\partial g}{\partial u_1}\right).$$

We find with the help of a computer that sufficiently many integrability conditions (4) and (10) (8 or 12 sometimes) are satisfied in the following four cases only: (I) system (1) admits nontrivial higher conserved densities; (II) the system is reducible to the triangular or linear form with the help of a contact transformation; (III) the system is linear or triangular. The last case is not interesting and we omit it.

We present here the complete classification the systems (I) and (II).

List I. Systems admitting higher conserved densities

$$u_t = u_3 - 3u_2^2/(4u_1) + v_2u_1 + c_1u_1, \quad v_t = u + c_2v_1. \tag{13}$$

$$u_t = u_3 - 3u_2^2/(4u_1) + v_1u_1 + c_1u_1, \quad v_t = u_1 + c_2v_1. \tag{14}$$

$$\begin{aligned} &u_t = w_2 + 6c_0\mathrm{e}^u u_1 - 1/2w^3 + c_1w + c_2u_1, \quad w = u_1 - v, \\ &v_t = 3c_0\mathrm{e}^u u_1^2 + 4c_0\mathrm{e}^u v_1 + c_0\mathrm{e}^u v^2 - 2c_0c_1\mathrm{e}^u + c_2v_1, \quad c_0 \neq 0. \end{aligned} \tag{15}$$

$$u_t = u_3 + uv_2 + u_1v_1 + c_1u_1, \quad v_t = u + c_0v_1. \tag{16}$$

$$u_t = u_3 + u_1v + uv_1 + c_1u_1 + c_2v_1, \quad v_t = u_1 + c_0v_1. \tag{17}$$

$$u_t = u_3 + u_1v_1 + c_1u_1 + c_2v_1, \quad v_t = u_1 + c_0v_1. \tag{18}$$

$$u_t = u_3 + u_1v_2 + c_1u_1 + c_2v_2, \quad v_t = u + c_0v_1. \tag{19}$$

$$u_t = u_3 + 3u_1v + 2uv_1 + v_1v_2 + c_1v_1 + 2c_2vv_1 - 2v^2v_1, \quad v_t = u_1 + vv_1 + c_2v_1. \tag{20}$$

$$u_t = u_3 + 2u_1v_1 + uv_2 + c_1u_1, \quad v_t = c_2u^2 + c_3v_1. \tag{21}$$

$$u_t = u_3 + 2u_1v + uv_1 + c_1u_1, \quad v_t = 2c_2uu_1 + c_3v_1. \tag{22}$$

$$\begin{aligned} u_t &= u_3 - u_1v_2/v - 3u_2v_1/(2v) + 3u_1v_1^2/(2v^2) + u^2u_1/v+ \\ &\quad +c_1u_1/(2v) + 2c_2v^2u_1 + 3c_2uvv_1, \quad v_t = 2uu_1. \end{aligned} \tag{23}$$

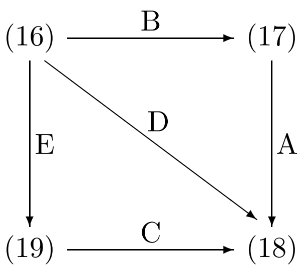
$$\begin{aligned} u_t &= u_3 - u_1v_2/v - 3u_2v_1/(2v) + 3u_1v_1^2/(2v^2) + 3c_1uv_1^2/(2v^2)+ \\ &\quad +u^2u_1/v - c_1(u_1v_1 + uv_2 - u^3)/v + c_1^2uv_1/(2v) - c_1^2u_1 - c_2u, \\ v_t &= 2uu_1 + 2c_1u^2 - 2c_2v. \end{aligned} \tag{24}$$

$$u_t = u_3 + 3/2u1^2 + c_1v_1^2 + c_2u_1, \quad v_t = u_1v_1 + c_3v_1. \tag{25}$$

$$u_t = u_3 + 3uu1 + 2c_1v_1v_2 + c_2u_1, \quad v_t = uv_1. \tag{26}$$

$$u_t = u_3 + 3uu1 + 2c_1vv_1 + c_2u_1, \quad v_t = uv_1 + u_1v. \tag{27}$$

System (14) follows from system (13) under the substitution $v_1 \to v$. System (15) is triangular if $c_0 = 0$, and moreover the transformation $(u, v) \to (v, w)$ gives in this case the pair of independent equations. Systems (16)–(19) are connected by the contact transformations A, B, C, D and E according to the following diagram



A: $(u, v) \to (u_1, v_1)$

B: $(u, v_1) \to (u + c_2, v)$

C: $(u, v_1) \to (u, v)$

D: $(u, v) \to (u_1 + c_2, v + c_2t)$

E: $(u, v) \to (u_1 + c_2, v_1 + c_2t)$

System (22) follows from system (21) under the substitution $v_1 \to v$. The systems (25)–(27) are connected each other according to the following diagram

$$(25) \xrightarrow{\;F\;} (26) \xrightarrow{\;G\;} (27)$$

where the maps F and G take the following form: $F : (u_1, v, c_2) \to (u - c_3, v, c_2 + 3c_3)$, $G : (u, v_1) \to (u, v)$.

List II. Systems reducible to the triangular form

$$\begin{aligned} u_t &= w_2 - 3w_1^2/(4w) - h(u, v) + f_1(w), \\ v_t &= h_uu_1 + h_vv_1 + f_2(w), \quad w = v + u_1. \end{aligned} \tag{28}$$

The reduction to the triangular form: $(u, v) \to (u, w)$.

$$\begin{aligned} u_t &= u_3 + v_2 + h(u, v) - \xi(u)v_1 + 3/2\xi' u_1^2, \\ v_t &= \xi^2v_1 - h_vv_1 - h_uu_1 + \xi' u_1v_1 - 3/2\xi\xi' u_1^2 - 1/2\xi'' u_1^3 - h\xi + c. \end{aligned} \tag{29}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=v+u_1+\int\xi(u)\,du$.

$$\begin{aligned} u_t &= u_3 - 2/3(v_2+u_1v^2+c_1v_1\mathrm{e}^{-u}-c_2v_1\mathrm{e}^{u}) - 3/2(c_1^2\mathrm{e}^{-2u}+c_2^2\mathrm{e}^{2u})u_1 \\ &\quad +2(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})u_1v+u_1^2v-1/2u_1^3+h(u,v)+c_0u_1, \\ v_t &= 3/2u_1^2v(c_2\mathrm{e}^{u}-c_1\mathrm{e}^{-u})+3/2(h_uu_1+h_vv_1-c_1h\mathrm{e}^{-u}+c_2h\mathrm{e}^{u})-2/3v^2v_1 \\ &\quad +(u_1v_1+2vv_1)(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})-1/2v_1(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})^2+(c_0-c_1c_2)v_1. \end{aligned} \tag{30}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=u_1-2v/3+c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u}$.

$$u_t=w_2-3w_1^2/(4w)-c_1v^2+c_2w,\quad v_t=c_3\sqrt{w}-c_1v_1,\quad w=u_1+v^2. \tag{31}$$

The reduction to the linear form: $(u,v)\to(y,v)$, where $y=\sqrt{w}$.

$$u_t=w_2-h(u,v)-3{w_1}^2/(2w)+c_1w^3+c_2/w,\quad v_t=h_vv_1+h_uu_1+c_3w, \tag{32}$$

where $w=u_1+v$. The reduction to the triangular form: $(u,v)\to(u,w)$.

$$u_t=w_2-\frac{3}{2}\frac{w{w_1}^2}{w^2+c}-h(u,v)+f_1(w),\quad v_t=h_vv_1+h_uu_1+f_2(w),\quad w=u_1+v. \tag{33}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$u_t=u_3+u_1\phi(v,v_2-u),\quad v_t=u_1,\quad \frac{\partial\phi}{\partial v}=\phi\frac{\partial\phi}{\partial v_2}. \tag{34}$$

One can check that $D_t(\phi)=0$, hence, $\phi=F_1(x)$ on solutions of system (34). Denoting $v_2-u=w$, we can integrate the equation $\phi_v=\phi\phi_w$ in the following implicit form $w+v\phi=H(\phi)$, where H is arbitrary function. This implies the following equations

$$u=v_{xx}+vF_1(x)+F_2(x),\quad v_t=v_{xxx}+(vF_1)_x+F_{2,x}, \tag{34$'$}$$

where F_1 is an arbitrary function and $F_2=H(F_1)$.

$$\begin{aligned} u_t &= w_2+kw_1+h(u,v)+f_1(w)+k^2u_1,\quad w=u_1+v-ku, \\ v_t &= kh-h_uu_1-h_vv_1+kc_1w+kc_2+k^2v_1. \end{aligned} \tag{35}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$\begin{aligned} u_t &= u_3+c_3u_1+c_4w+c_5w^2+kw^3+c_6,\quad w=v_2-u-c_2v_1+c_1v, \\ v_t &= u_1+c_2u-c_1c_2v+(c_3+c_2^2-c_1)v_1+c_7. \end{aligned} \tag{36}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$\begin{aligned} u_t &= u_3+v_2+6c_0(c_1uu_1-vu_1-c_0u^2u_1)-2c_0uv_1+c_1v_1+c_2u_1-h(u,v), \\ v_t &= (c_1^2+c_2)v_1-6c_0vv_1+2c_0(u_1v_1-c_0u^2v_1+c_1uv_1) \\ &\quad +2c_0uh-c_1h+h_uu_1+h_vv_1. \end{aligned} \tag{37}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=u_1+v-c_1u+c_0u^2$.

In formulas (13)–(37), c_i and k are arbitrary constants, h, ξ, ϕ and f_i are arbitrary functions, $h_u=\partial h/\partial u$.

Conclusion

We believe that any system from the list (I) is integrable in the frame of the inverse scattering transform method.

Let us note that some equations from the list (I) admit the reduction to a single integrable equation. For example, excluding the function u from system (13) and setting $c_2 = 0$ for simplicity, we obtain

$$v_{tx}v_{tt} - v_{tx}v_{txxx} + \frac{3}{4}v_{txx}^2 - v_{xx}v_{tx}^2 - c_1 v_{tx}^2 = 0. \tag{13'}$$

For system (16), the same operation gives

$$z_{tt} - z_x z_{tx} + (2z_x c_0 - z_t)z_{xx} - z_{txxx} + c_0 z_{xxxx} = 0, \tag{16'}$$

where $v = z - (c_1 + c_0)x - c_0(c_1 + 2c_0)t$. And system (20) is reduced to the following form

$$\begin{aligned} &z_{tt} + (3c_2 - 4z_x)z_{tx} + z_x z_{xxxx} - z_{txxx} + 2z_{xx}z_{xxx} + \\ &\qquad + 3z_x z_{xx}(2z_x - 3c_2) - 2z_t z_{xx} = 0, \end{aligned} \tag{20'}$$

where $v = z_x - c_2$. Let us also notice another forms for systems (23) and (24) that arise under the exponential substitution $v \to \mathrm{e}^v$.

$$\begin{aligned} &u_t = u_3 - u_1 v_2 + 1/2 u_1 v_1^2 - 3/2 u_2 v_1 + c_2 \mathrm{e}^{2v}(2u_1 + 3uv_1) + \mathrm{e}^{-v}(u^2 + c_1/2)u_1, \\ &v_t = 2uu_1 \mathrm{e}^{-v}. \end{aligned} \tag{23'}$$

$$\begin{aligned} &u_t = u_3 - 3/2u_2 v_1 - u_1 v_2 + c_1(uv_1^2/2 - uv_2 - u_1 v_1 - c_1 u_1 + c_1 uv_1/2) + \\ &\qquad + 1/2 u_1 v_1^2 - c_2 u + \mathrm{e}^{-v}(c_1 u^3 + u^2 u_1), \quad v_t = 2(uu_1 + c_1 u^2)\mathrm{e}^{-v} - 2c_2. \end{aligned} \tag{24'}$$

For the linearizable system (34), we get

$$u_t = u_3 + u_1(u - v_2)/v, \quad v_t = u_1.$$

This system is equivalent to system (34′), where $F_2 = 0$.

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