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On Specific Symmetries of the KdV Equation and on New Representations of Nonlinear *sl*(2)-algebras

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Abstract

On the one hand, we put in evidence new symmetry operators of the nonlinear Korteweg-de Vries equation by exploiting its Lax form expressed in terms of a pair of linear equations. A KdV supersymmetric version is also studied in order to determine its symmetry Lie superalgebra. On the other hand, nonlinear $sl(2)$-algebras are then visited and new unitary irreducible representations are characterized.

This talk is dedicated to the Memory of my Colleague ***Wilhelm Fushchych****, a man I have mainly appreciated* ***outside*** *our common interests in mathematical physics.*

1 Introduction

I would like to discuss in this talk *two* subjects which both are concerned with fundamental *symmetries* in theoretical *physics* and both are developed through *mathematical methods*. Moreover, these two subjects deal with *nonlinear* characteristics so that the invitation of the organisers of this Conference was welcome and I take this opportunity to thank them cordially.

The *first* subject (reported in Sections 2 and 3) deals with new symmetries and supersymmetries of the famous (nonlinear) Korteweg-de Vries equation [1] by exploiting its formulation(s) through the corresponding Lax form(s) [2, 3]. These results have already been collected in a not yet published recent work [4] to which I refer for details if necessary.

The *second* subject (developed in Sections 4 and 5) concerns the so-called nonlinear $sl(2, R)$-algebras containing, in particular, the linear $sl(2)$-case evidently, but also the quantum $sl_q(2)$-case. These two particular cases are respectively very interesting in connection with the so-important theory of angular momemtum [5, 6] (developed in quantum physics at all the levels, i.e., the molecular, atomic, nuclear and subnuclear levels) *and* with the famous quantum deformations [7] applied to one of the simplest Lie algebras with fundamental interest in quantum physics [8, 9]. But my main purpose is to study here the nonlinear $sl(2)$-algebras which are, in a specific sense, just *between* these two cases: in

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such categories we find the nonlinear cases corresponding to *finite* powers of Lie generators differing from the first power evidently but being possibly of second (the *quadratic* context) or third powers (the *cubic* context) for example. Both of these quadratic and cubic cases [10] have already been exploited in *physical* models or theories so that a very interesting amount of original informations is the knowledge of the unitary irreducible representations (unirreps) of these nonlinear algebras: such results can be found in different recent papers [10, 11, 12] that I want to refer to in the following by dealing more particularly here with the so-called Higgs algebra [13]. This algebra is an example of a *cubic* $sl(2)$-algebra with much physical attraction: it corresponds to physical descriptions in *curved* spaces (its original appearance [13]) or in *flat* spaces (its more interesting recent discover in quantum optics, for example [14]).

2 On the KdV equation and its symmetries

Let us remember the famous nonlinear Korteweg-de Vries (KdV) equation [1] introduced in 1885 in the form

$$u_t = 6uu_x - u_{xxx}, \tag{1}$$

where the unknown function $u(x,t)$ also admits time and space partial derivatives with the usual notations

$$u_t \equiv \frac{\partial u(x,t)}{\partial t}, \qquad u_x \equiv \frac{\partial u(x,t)}{\partial x} = \partial_x u(x,t), \tag{2}$$

the space ones going to the maximal third order (u_{xxx}). Its Lax form [2] is usually denoted by (L, A) where

$$L \equiv -\partial_x^2 + u, \qquad A \equiv -4\partial_x^3 + 6u\partial_x + 3u_x, \tag{3}$$

so that

$$\partial_t L = [A, L]. \tag{4}$$

This is equivalent to a pair of *linear* equations

$$L\psi(x,t) = \lambda\psi(x,t), \qquad \psi_t = A\psi \tag{5}$$

which can be rewritten as a system of the following form

$$\begin{aligned} L_1\psi(x,t) &= 0, \qquad L_1 \equiv L - \lambda, \\ L_2\psi(x,t) &= 0, \qquad L_1 \equiv \partial_t - A, \end{aligned} \tag{6}$$

with the compatibility condition

$$[L_1, L_2]\psi = 0. \tag{7}$$

By searching for *symmetry* operators X of such a system, we ask for operators X such that

$$[\triangle, X] = \lambda X \tag{8}$$

ensuring that, if ψ is a solution of $\triangle\psi = 0$, we know that $X\psi$ is still a solution of the same equation, i.e., $\triangle(X\psi) = 0$. We have solved [4] such an exercise with two equations (6) so that the general conditions (8) here reduce to

$$[L_1, X] = \lambda_1 L_1 \quad \text{and} \quad [L_2, X] = \lambda_2 L_1, \tag{9}$$

where λ_1 and λ_2 are arbitrary functions of x and t. This system leads to nine partial differential equations and to a resulting set of *three* (nontrivial) independent symmetry operators according to different u-values [4]. They are given by the explicit expressions

$$\begin{aligned} &X_1 \equiv \partial_x, \\ &X_2 \equiv \partial_x^3 - \frac{3}{2}u\partial_x + \lambda\partial_x - \frac{3}{4}u_x, \\ &X_3 \equiv t\partial_x^3 - \frac{1}{12}x\partial_x - \frac{3}{2}tu\partial_x + \lambda t\partial_x - \frac{3}{4}tu_x. \end{aligned} \tag{10}$$

They generate a (closed) invariance (Lie) algebra characterized by the commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = \frac{1}{3}X_1, \quad [X_2, X_3] = X_2 - \frac{4}{3}\lambda X_1. \tag{11}$$

In fact, such results are readily obtained if one requires that the X-operators are given by

$$X = \sum_{i=0}^{3} a_i(x,t)\partial_x^i, \tag{12}$$

where $i = 0$ refers to time and $i = 1, 2, 3$ refer to first, second and third respective space derivatives. Conditions (9) relate among themselves the corresponding arbitrary $a_i(x,t)$-functions so that the discussion appears to be restricted to the following contexts:

$$u_{xx} = 0 \quad \text{and} \quad u_{xx} \neq 0. \tag{13}$$

Another simple equation leads to the possible values

$$u = \lambda \neq 0 \quad \text{or} \quad 6ut = -x \tag{14}$$

so that these different contexts allow the existence of new symmetries of the KdV equation.

3 The supersymmetric context

Let us remember the Mathieu supersymmetric extension [3] of the KdV equation characterized by

$$u_t = 6uu_x - u_{xxx} - a\xi\xi_{xx} \tag{15}$$

and

$$\xi_t = -\xi_{xxx} + au_x\xi + (6-a)u\xi_x, \tag{16}$$

where a is a constant taken equal to 3 for obtaining a nontrivial Lax representation and ξ, ξ_t , ξ_x, ... are "fermionic" quantities as usual. Now the differential operators of the Lax pair take the explicit forms

$$L \equiv -\partial_x^2 + u + \theta\xi_x + \theta\xi\partial_x - \xi\partial_\theta - u\theta\partial_\theta \tag{17}$$

and

$$\begin{aligned} A \equiv & -4\partial_x^3 + 6u\partial_x + 3u_x + 3\theta\xi_{xx} - 3\theta u_x\partial_\theta + 9\theta\xi_x\partial_x \\ & -3\xi_x\partial_\theta + 6\theta\xi\partial_x^2 - 6\xi\partial_x\partial_\theta - 6\theta u\partial_x\partial_\theta. \end{aligned} \tag{18}$$

Here θ is the necessary Grassmannian variable permitting to distinguish *even* and *odd* terms in these developments, or let us say "bosonic" and "fermionic" symmetry operators.

By taking care of the grading in the operators as developed in the study of supersymmetric Schrödinger equations [15], we can determine the even and odd symmetries of the context after a relatively elaborated discussion. We refer the interested reader to the original work [4]. Here evidently we get invariance Lie *superalgebras* [16] for the KdV equation whose orders are 11, 9 or 4 in the complete discussion we skip here.

4 On nonlinear $sl(2)$-algebras

Let us remember the linear $sl(2)$-algebra subtending all the ingredients of the angular momentum theory [5, 6] , i.e., the set of commutation relations (defining this Lie algebra) between the three Cartan-Weyl generators $(J_\pm, J_3)$ given by

$$[J_3, J_\pm] = \pm J_\pm, \tag{19}$$

$$[J_+, J_-] = 2J_3. \tag{20}$$

Here J_+ and J_- are the respective raising and lowering operators acting on the real orthogonal basis $\{ \mid j,m> \}$ in the following well-known way:

$$J_+ \mid j,m >= \sqrt{(j-m)(j+m+1)} \mid j, m+1 >, \tag{21}$$

$$J_- \mid j,m >= \sqrt{(j+m)(j-m+1)} \mid j, m-1 > \tag{22}$$

while J_3 is the diagonal generator giving to $m(= -j, -j+1, ..., j-1, j)$ its meaning through the relation

$$J_3 \mid j,m >= m \mid j,m > . \tag{23}$$

Moreover, by recalling the role of the Casimir operator

$$C \equiv \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \tag{24}$$

such that

$$C \mid j,m >= j(j+1) \mid j,m >, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \tag{25}$$

it is easy to define our nonlinear $sl(2)$-algebras by asking that the commutation relation (20) has to be replaced by the following one:

$$[J_+, J_-] = f(J_3) = \sum_{p=0}^{N} \beta_p \, (2J_3)^{2p+1}. \tag{26}$$

In particular, if $N = p = 0$, $\beta_0 = 1$, we recover the linear (above-mentioned) context. If $N = 1, p = 0, 1$, $\beta_0 = 1, \beta_1 = 8\beta$ where β is a real continuous parameter, we obtain the remarkable Higgs algebra [13] already studied [10, 11, 12] in connection with specific physical contexts in curved or flat spaces [13, 14]. If $N \to \infty$ and if we choose ad hoc β_p-coefficients [12] , we easily recover the quantum $sl_q(2)$-algebra [8].

Here, let us more specifically consider finite values of $N(\neq 0)$ in order to include (in particular) the Higgs algebra. Our main purpose is to determine the unirreps of such a family of algebras by putting in evidence the effect of the deformed generators satisfying Eqs. (19) and (26). In terms of the old $sl(2)$-generators, we have obtained the whole answer [12] by noticing that relation (23) has to play a very important role in the discovery of new unirreps having a physical interest. In order to summarize our improvements, let us mention that equation (23) can be generalized through two steps.

The first step is to modify it in the following way as proposed by Abdesselam et al. [12], i.e.,

$$J_3 \mid j, m >= (m + \gamma) \mid j, m >, \tag{27}$$

where γ is a real scalar parameter. This proposal leads us to

$$\begin{aligned} J_+ \mid j, m >= ((j-m)(j+m+1+2\gamma)(1 + 2\beta(j(j+1) + m(m+1) + \\ 2\gamma(j+m+1+\gamma))))^{\frac{1}{2}} \mid j, m+1 >, \end{aligned} \tag{28}$$

$$\begin{aligned} J_- \mid j, m >= ((j-m+1)(j+m+2\gamma)(1 + 2\beta(j(j+1) + m(m-1) + \\ 2\gamma(j+m+\gamma))))^{\frac{1}{2}} \mid j, m-1 > . \end{aligned} \tag{29}$$

This context (interesting to consider at the limits $\gamma = 0$, or $\beta = 0$, or $\beta = \gamma = 0$) is such that three types of unirreps can be characterized, each of them corresponding to specific families. Restricting our discussion to the Higgs algebra corresponding to $N = 1$, $\beta_0 = 1$, $\beta_1 = \beta$, it is possible to show [12] that, if $\gamma = 0$, it corresponds a *class I* of unirreps permitting

$$\beta \geq -\frac{1}{4j^2} \qquad \forall j(\neq 0) \tag{30}$$

but also that, if $\gamma \neq 0$, we get two other families of unirreps characterized by

$$\gamma_+ = \frac{1}{2\beta}\sqrt{-\beta - 4\beta^2 j(j+1)}, \qquad (class\ II), \tag{31}$$

or by

$$\gamma_- = -\frac{1}{2\beta}\sqrt{-\beta - 4\beta^2 j(j+1)}, \qquad (class\ III), \tag{32}$$

both values being constrained by the parameter β according to

$$-\frac{1}{4j(j+1)} < \beta \leq \frac{1}{4j(j+1)+1}. \tag{33}$$

The second step is another improvement proposed by N. Debergh [12] so that Eqs. (23) or (27) are now replaced by

$$J_3 \mid j,m >= \left(\frac{m}{c} + \gamma\right) \mid j,m > \tag{34}$$

where c is a nonnegative and nonvanishing integer. The previous contexts evidently correspond to $c = 1$, but for other values, we get new unirreps of specific interest as we will see. Let us also mention that actions of the ladder operators are also c-dependent. We have

$$J_+ \mid j,m >= \sqrt{f(m)} \mid j,m+c > \tag{35}$$

and

$$J_- \mid j,m >= \sqrt{f(m-c)} \mid j,m-c >, \tag{36}$$

where the f-functions can be found elsewhere [12]. The Casimir operator of this deformed (Higgs) structure appears also as being c-dependent. In conclusion, this second step leads for $c = 2, 3, \ldots$ to new unirreps of the Higgs algebra.

5 On the Higgs algebra and some previous unirreps

The special context characterizing the Higgs algebra is summarized by the following commutation relation replacing Eq.(20)

$$[J_+, J_-] = 2J_3 + 8\beta J_3^3 \tag{37}$$

besides the unchanged ones

$$[J_3, J_\pm] = \pm J_\pm. \tag{38}$$

After Higgs [13, 17], we know that this is an invariance algebra for different physical systems (the harmonic oscillator in two dimensions, in particular) but described in *curved* spaces. It has also been recognized very recently as an interesting structure for physical models in *flat* spaces and I just want to close this talk by mentioning that multiphoton processes of scattering described in quantum optics [14, 18] are also subtended (in a 2-dimensional flat space) by such a nonlinear algebra. The important point to be mentioned here is that these developments deal with the very recent last step (34) and the corresponding unirreps, this fact showing their immediate interest.

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References

[1] See for example, Solitons, in: Topics in Current Physics, Eds. R.K. Bullough and P.J. Caudrey, Springer Verlag, Berlin, 1980 and references therein.

[2] Lax P.D., *Commun. Pure Appl. Math.*, 1968, V.21, 467;
Gardner C., Kruskal M. and Miura R., *J. Math. Phys.*, 1968, V.9, 1202.

[3] Mathieu P., *J. Math. Phys.*, 1988, V.29, 2499.

[4] Bagchi B., Beckers J. and Debergh N., New Symmetries of the KdV Equation and their Supersymmetric Generalizations, ULg-preprint, 1996.

[5] Edmonds A.R., Angular Momemtum in Quantum Mechanics, Princeton University Press, Princeton, 1957.

[6] Rose M.E., Elementary Theory of Angular Momemtum, Wiley, New-York, 1957.

[7] Drinfeld V.G., Quantum Groups, in: Proc. Int. Congr. of Mathematicians, Berkeley, CA, 1, 798, New-York, Academic 1986;
Jimbo M., *Lett. Math. Phys.*, 1985, V.10, 63;
Chari V. and Pressley A., A Guide to Quantum Groups, Cambridge University Press, 1994 and references therein.

[8] Biedenharn L.C., *J. Phys.*, 1989, V.A22, L873.

[9] Macfarlane A.J., *J. Phys.*, 1989, V.A22, 4581.

[10] Schoutens K., Sevrin A. and Van Nieuwenhuizen P., *Phys. Lett.*, 1991, V.B255, 549;
Rocek M., *Phys. Lett.*, 1991, V.B255, 554;
Quesne C., *Phys. Lett.*, 1994, V.A193, 245;
Daskaloyannis C., *J. Phys.*, 1992, V.A25, 2261.

[11] Zhedanov A.S., *Mod. Phys. Lett.*, 1992, V.A7, 507.

[12] Abdesselam B., Beckers J., Chakrabarti A. and Debergh N., J. Phys., 1996, V.A29, 3075;
Debergh N., *J. Phys.*, 1997, V.A30, 5239.

[13] Higgs P.W., *J. Phys.*, 1979, V.A12, 309.

[14] Debergh N., Supersymmetry in Quantum Optics through the Higgs Algebra, Liege preprint, August 1997 .

[15] Beckers J. and Debergh N., *Helv. Phys. Acta*, 1991, V.64, 24.

[16] Cornwell J.F., Group Theory in Physics, Vol.III, Academic Press, New York, 1989.

[17] Beckers J. and Debergh N., *J. Nonlin. Math. Phys.*, 1997, V.4, 401.

[18] Karassiov V.P., *J. Sov. Laser Research*, 1992, V.13, 188, 288; *Laser Phys.*, 1993, V.3, 895.