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Scientific Heritage of W. Fushchych

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Abstract

A very short discussion of the main scientific results obtained by Prof. W. Fushchych is presented.

Introduction

The scientific heritage of Prof. W. Fushchych is great indeed. All of us have obtained the list of his publications which includes more than 330 items. It is difficult to imagine that all this was produced by one man. But it is the case, and numerous students of Prof. Fushchych can confirm that he made a decisive contribution to the majority of his publications.

As one of the first students of Prof. W. Fushchych, I would like to say a few words about the style of his collaboration with us. He liked and appreciated any collaboration with him. He was a very optimistic person and usually believed in the final success of every complicated investigation, believed that his young collaborators are able to overcome all difficulties and to solve the formulated problem. In addition to his purely scientific contributions to research projects, such an emotional support was very important for all of us. He helped to find a way in science and life for great many of people including those of them who had never collaborated with him directly. His scientific school includes a lot of researchers, and all of them will remember this outstanding and kind person.

Speaking about scientific results obtained by my teacher, Prof. W. Fushchych, I have to restrict myself to the main ones only. In any case, our discussion will be fragmentary inasmuch it is absolutely impossible to go into details of such a large number of publications.

From the extremely rich spectrum of scientific interests of W. Fushchych, I selected the following directions:

1. Invariant wave equations.
2. Generalized Poincaré groups and their representations.
3. Non-Lie and hidden symmetries of PDE.
4. Symmetry analysis and exact solutions of nonlinear PDE.

I will try to tell you about contributions of Prof. W. Fushchych to any of the fields enumerated here. It is necessary to note that item 4 represents the most extended field of investigations of W. Fushchych, which generated the majority of his publications.

1. Invariant wave equations

1. Poincaré-invariant equations

W. Fushchych solved a fundamental problem of mathematical physics, which was formulated long ago and attacted much attention of such outstanding scientists as Wigner, Bargmann, Harish-Chandra, Gelfand and others. The essence of this problem is a description of multicomponent wave equations which are invariant with respect to the Poincaré group and satisfy some additional physical requirements.

In order to give you an idea about of this problem, I suggest to consider the Dirac equation

$$L\Psi = (\gamma^\mu p_\mu - m)\Psi = 0, \qquad p_\mu = -i\frac{\partial}{\partial x_\mu}, \qquad \mu = 0,1,2,3. \tag{1}$$

Here, γ_μ are 4×4 matrices satisfying the Clifford algebra:

$$\begin{aligned} &\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \\ &g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \qquad g_{\mu\nu} = 0, \quad \mu \neq \nu. \end{aligned} \tag{2}$$

Equation (1) is invariant with respect to the Poincaré group. Algebraic formulation of this statement is the following: there exist symmetry operators for (1)

$$P_\mu = i\frac{\partial}{\partial x_\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{3}$$

where

$$S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \qquad \mu,\nu = 0,1,2,3.$$

These operators commute with L of (1) and satisfy the Poincaré algebra $AP(1,3)$

$$[L, P_\mu] = [L, J_{\mu\nu}] = 0, \qquad [P_\mu, P_\nu] = 0, \tag{4}$$

$$[P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu). \tag{5}$$

It follows from (4) that generators P_μ, $J_{\mu\nu}$ transform solutions of (1) into solutions.

Of course, the Dirac equation is not the only one having this symmetry, and it is interesting to search for other equations invariant with respect to the Poincaré algebra. In papers of Bargmann, Harish-Chandra, Gelfand, Umezawa and many others, we can find a number of relativistic wave equations for particles of arbitrary spin. It happens, however, that all these equations are inconsistent inasmuch as they lead to violation of the causality principle for the case of a particle interacting with an external field. Technically speaking, these equations lose their hyperbolic nature if we take into account the interaction with an external field.

To overcome this difficulty, Fushchych proposed to search for Poincaré-invariant wave equations in the Schrödinger form

$$i\frac{\partial}{\partial x_0}\Psi = H\Psi, \qquad H = H(\vec{p}), \tag{6}$$

where H is a differential operator which has to be found starting with the requirement of Poincaré invariance of equation (6). In spite of the asymmetry between spatial and time variables, such an approach proved to be very fruitful and enables to find causal equations for arbitrary spin particles.

I will not enter into details but present a nice formulation of the Poincaré-invariance condition for equation (6):

$$\begin{cases} \Big[[H, x_a], [H, x_b]\Big] = -4iS_{ab}, \\ H^2 = \vec{p}^2 + m^2. \end{cases} \tag{7}$$

It is easy to verify that the Dirac Hamiltonian $H = \gamma_0\gamma_a\gamma_a + \gamma_0 m$ satisfies (7). The other solutions of these relations present new relativistic wave equations [3, 4].

It is necessary to note that relativistic wave equations found by W. Fushchych with collaborators present effective tools for solving physical problems related to the interaction of spinning particles with external fields. Here, I present the formula (obtained by using these equations) which describes the energy spectrum of a relativistic particle of spin s interesting with the Coulomb field:

$$E = n\left[1 + \frac{\alpha^2}{\left(n' + \frac{1}{2} + \left[\left(j + \frac{1}{2}\right)^2 - \alpha^2 - b_\lambda^{sj}\right]^{1/2}\right)^2}\right]^{1/2}. \tag{8}$$

Here $n' = 0, 1, 2, \ldots$, $j = \dfrac{1}{2}, \dfrac{3}{2}, \ldots$, b_λ^{sj} is a root of the specific algebraic equation defined by the value of spin s.

Formula (8) generalizes the famous Sommerfeld formula for the case of arbitrary spin s [3, 4] .

2. Galilei-invariant wave equations

In addition to the Poincaré group, the Galilei group has very important applications in physics. The Galilei relativity principle is valid for the main part of physical phenomena which take place on the Earth. This makes the problem of description of Galilei-invariant equations very interesting. In papers of W. Fushchych with collaborators, the problem is obtained a consistent solution.

Starting with the first–order equations

$$(\beta_\mu p^\mu - \beta_4 m)\Psi = 0 \tag{9}$$

and requiring the invariance with respect to the Galilei transformations

$$x_a \to R_{ab}x_b + V_a t + b_a, \qquad t_0 \to t_0 + b_0,$$

($\|R_{ab}\|$ are orthogonal matrices), we come to the following purely algebraic problem:

$$\begin{aligned} &\widetilde{S}_a\beta_0 - \beta_0 S_a = 0, \qquad \widetilde{S}_a\beta_4 - \beta_4 S_a = 0, \\ &\widetilde{\eta}_a\beta_4 - \beta_4\eta_a = -i\beta_a, \qquad \widetilde{\eta}_a\beta_b - \beta_b\eta_a = -i\delta_{ab}\beta_0, \\ &\widetilde{\eta}_a\beta_0 - \beta_0\eta_a = 0, \qquad a = 1, 2, 3; \end{aligned} \tag{10}$$

where S_a, η_a and $\widetilde{S}_a, \widetilde{\eta}_a$ are matrices satisfying the algebra $AE(3)$

$$[S_a, S_0] = i\varepsilon_{abc}S_c, \quad [S_a, \eta_b] = i\varepsilon_{abc}\eta_c, \quad [\eta_a, \eta_b] = 0. \tag{11}$$

The principal result of investigation of the Galilei-invariant equations (9) is that these equations describe correctly the spin-orbit and Darwin couplings of particles with all external fields. Prior to works of W. Fushchych, it was generally accepted that these couplings are purely relativistic effects. Now we understand that these couplings are compatible with the Galilei relativity principle [3, 4].

For experts in particle physics, I present the approximate Hamiltonian for a Galilean particle interacting with an external electromagnetic field:

$$\begin{aligned} H = \frac{p^2}{2m} + m + eA_0 + \frac{e}{2ms}\vec{S}\cdot\vec{H} + \\ + \frac{e}{4m^2}\left[-\frac{1}{2}\vec{S}\cdot(\pi\times E - \vec{E}\times\vec{\pi}) + \frac{1}{b}\Theta_{ab}\frac{\partial E_a}{\partial x_b} + \frac{1}{3}s(s+1)\,\mathrm{div}\,\vec{E}\right], \end{aligned} \tag{12}$$

where

$$\Theta_{ab} = 3[S_a, S_b]_+ - 2\delta_{ab}s(s+1).$$

The approximate Hamiltonian (12), obtained by using Galilei-invariant equations, coincides for $s = \dfrac{1}{2}$ with the related Hamiltonian obtained from the Dirac equation [3, 4].

3. Nonlinear equations invariant with respect to the Galilei and Poincaré groups

W. Fushchych made a very large contribution into the theory of nonlinear equations with a given invariance group. Here, I present some of his results connected with Galilei and Poincaré invariant equations.

Theorem 1 [4, 5]. *The nonlinear d'Alembert equation*

$$p_\mu p^\mu \Psi + F(\Psi) = 0$$

is invariant with respect the extended Poincaré group $\widetilde{P}(1,3)$ *iff*

$$F(\Psi) = \lambda_1 \Psi^r, \qquad r \neq 1,$$

or

$$F(\Psi) = \lambda_2 \exp(\Psi).$$

Here, Ψ *is a real scalar function.*

Theorem 2 [4, 5, 8]. *The nonlinear Dirac equation*

$$[\gamma^\mu p_\mu + F(\overline{\Psi}, \Psi)]\Psi = 0$$

is invariant with respect to the Poincaré group iff

$$F(\overline{\Psi}, \Psi) = F_1 + F_2\gamma_5 + F_3\gamma^\mu\overline{\Psi}\gamma_5\gamma_\mu\Psi + F_4 S^{\nu\lambda}\overline{\Psi}\gamma_5 S_{\nu\lambda}\Psi,$$

where $F_1, \ldots, F_4$ *are arbitrary functions of* $\overline{\Psi}\Psi$ *and* $\overline{\Psi}\gamma_5\Psi$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$.

Theorem 3 [3, 4]. *The Maxwell's equations for electromagnetic field in a medium*

$$\frac{\partial\vec{D}}{\partial x_0} = -\vec{p}\times\vec{H}, \qquad \frac{\partial\vec{B}}{\partial x_0} = \vec{p}\times\vec{E},$$
$$\vec{p}\cdot\vec{D} = 0, \qquad \vec{p}\cdot\vec{B} = 0$$

with constitutive equations

$$\vec{E} = \vec{\Phi}(\vec{D},\vec{H}), \qquad \vec{B} = \vec{F}(\vec{D},\vec{H})$$

are invariant with respect to the group $P(1,3)$ iff

$$D = M\vec{E} + N\vec{B}, \qquad \vec{H} = M\vec{B} - N\vec{E},$$

where $M = M(C_1,C_2)$ and $N = N(C_1,C_2)$ are arbitrary functions of the invariants of electromagnetic field

$$C_1 = \vec{E}^2 - \vec{B}^2, \qquad C_2 = \vec{B}\cdot\vec{E}.$$

Theorem 4 [4]. *The nonlinear Schrödinger equation*

$$\left(p_0 - \frac{p^2}{2m}\right)u + F(x,u,u^*) = 0$$

is invariant with respect to the Galilei algebra $AG(1,3)$ iff

$$F = \Phi(|u|)u,$$

to the extended Galilei algebra (including the dilation operator) $AG_1(1,3)$ iff

$$F = \lambda|u|^k u, \qquad \lambda, k \neq 0,$$

and to the Schrödinger algebra $AG_2(1,3)$ iff

$$F = \lambda|u|^{3/4}u.$$

I present only a few fundamental theorems of W. Fushchych concerning to the description of nonlinear equations with given invariance groups. A number of other results can be found in [1–10].

By summarizing, we can say that W. Fushchych made the essential contribution to the theory of invariant wave equations. His fundamental results in this field are and will be used by numerous researchers.

2. Generalized Poincaré groups and their representations

Let us discuss briefly the series of W. Fushchych's papers devoted to representations of generalized Poincaré groups.

A generalized Poincaré group is defined as a semidirect product of the groups $SO(1,n)$ and T

$$P(1,n) = SO(1,n) \ltimes T,$$

where T is an additive group of $(n+1)$-dimensional vectors $p_1, p_2, \ldots, p_n$ and $SO(1,n)$ is a connected component of the unity in the group of all linear transformations of T into T preserving the quadratic form

$$p_0^2 - p_1^2 - p_2^2 - \cdots - p_n^2.$$

Prof. W. Fushchych was one of the first who understood the importance of generalized Poincaré groups for physics. A straightforward interest to these groups can be explained, for example, by the fact that even the simplest of these, the group $P(1,4)$, includes the Poincaré, Galilei and Euclidean groups as subgroups. In other words, the group $P(1,4)$ unites the groups of motion of the relativistic and nonrelativistic quantum mechanics and the symmetry group of the Euclidean quantum field theory.

Using the Wigner induced representations method, W. Fushchych described for the first time all classes of unitary IRs of the generalized Poincaré group and the related unproper groups including reflections [3, 4].

The Lie algebra of the generalized Poincaré group $P(1,4)$ includes $\dfrac{n(n+3)+2}{2}$ basis elements $\{P_m, J_{mn}\}$ which satisfy the following commutation relations:

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),\\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\nu\rho}J_{\mu\sigma} - g_{\mu\sigma}J_{\nu\rho}),\\
&\mu,\nu,\rho,\sigma = 0,1,\ldots,n.
\end{aligned} \tag{13}$$

W. Fushchych found realizations of algebra (13) in different bases.

3. Non-Lie symmetries

In 1974, W. Fushchych discovered that the Dirac equation admits a specific symmetry which is characterized by the following property.

1. Symmetry operators are non-Lie derivatives (i.e., do not belong to the class of first order differential operators).

2. In spite of this fact, they form a finite-dimensional Lie algebra.

This symmetry was called a non-Lie symmetry. It was proved by W. Fushchych and his collaborators that a non-Lie symmetry is not a specific property of the Dirac equation. Moreover, it is admitted by great many of equations of quantum physics and mathematical physics. Among them are the Kemmer-Duffin-Petiau, Maxwell equations, Lamé equation, relativistic and nonrelativistic wave equations for spinning particles and so on.

In order to give you an idea about "non-Lie" symmetries, I will present you an example connected with the Dirac equation. In addition to generators of the Poincaré group, this equation admits the following symmetries [1, 2]:

$$Q_{\mu\nu} = \gamma_\mu\gamma_\nu + (1 - i\gamma_5)\frac{\gamma_\mu p_\nu - \gamma_\nu p_\mu}{2m}. \tag{14}$$

Operators (14) transform solutions of the Dirac equation into solutions. They are non-Lie derivatives inasmuch as their first term includes differential operators with matrix coefficients. In spite of this fact, they form a 6-dimensional Lie algebra defined over the field of real numbers. Moreover, this algebra can be united with the Lie algebra of the Poincaré group in frames of a 16-dimensional Lie algebra. This algebra is characterized by the following commutation relations

$$\begin{aligned}&[J_{\mu\nu}, Q_{\lambda\sigma}] = [Q_{\mu\nu}, Q_{\lambda\sigma}] = 2i\left(g_{\mu\sigma}Q_{\nu\sigma} + g_{\sigma\lambda}Q_{\mu\nu} - g_{\mu\lambda}Q_{\nu\sigma} - g_{\nu\sigma}Q_{\mu\lambda}\right),\\&[Q_{\mu\nu}, P_\lambda] = 0.\end{aligned} \tag{15}$$

Taking into account relations (15) and

$$Q^2_{\mu\nu}\Psi = \Psi, \tag{16}$$

we conclude that the Dirac equation is invariant with respect to the 16-parameter group of transformations. Generalizing symmetries (14) to the case of higher order differential operators, we come to the problem of description of complete sets of such operators (which was called higher order symmetries):

$$\Psi(x) \to B_\mu \frac{\partial\Psi}{\partial x_\mu} + D(\Lambda)\Psi(\Lambda^{-1}x - a).$$

In the papers of W. Fushchych, the complete sets of higher order symmetry operators for the main equations of classical field theory were found. Here, I present numbers of linearly independent symmetry operators of order n for the Klein-Gordon-Fock, Dirac, and Maxwell equations.

KGF equation:

$$N_n = \frac{1}{4!}(n+1)(n+2)(n+3)(n^2+3n+4).$$

The Dirac equation:

$$\widetilde{N}_n = 5N_n - \frac{1}{6}(2n+1)(13n^2+19n+18) - \frac{1}{2}[1-(-1)^n].$$

The Maxwell equation:

$$N_n = (2n+3)[2n(n-1)(n+3)(n+4) + (n+1)^2(n+2)^2]/12.$$

4. Symmetries and exact solutions of nonlinear PDE

In this fundamental field, W. Fushchych obtained a lot of excellent results. Moreover, he discovered new ways in obtaining exact solutions of very complicated systems of nonlinear PDE.

It is necessary to mention the following discoveries of W. Fushchych.

1. The ansatz method

It is proved that if a system of nonlinear differential equations

$$L(x, \Psi(x)) = 0$$

admits a Lie symmetry, it is possible to find exact solutions of this system in the form

$$\Psi = A(x)\varphi(\omega), \tag{17}$$

where $A(x)$ is a matrix, $\varphi(\omega)$ is an unknown function of group invariants $\omega = (\omega_1, \ldots, \omega_n)$.

Long ago, W. Fushchych understood that relation (17) can be treated as an ansatz which, in some sense, is a more general substance than a Lie symmetry. I should like to say that it is possible to use successfully substitutions (17) (and more general ones) even in such cases when an equation do not admit a Lie symmetry.

2. Conditional symmetry

Consider a system of nonlinear PDE of order n

$$\begin{gathered} L(x, u_1, u_2, \ldots, u_n) = 0, \qquad x \in R(1,n), \\ u_1 = \left(\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \qquad u_2 = \left(\frac{\partial^2 u}{\partial x_0^2}, \frac{\partial^2 u}{\partial x_0 \partial x_1}, \ldots\right). \end{gathered} \tag{18}$$

Let some operator Q do not belong to the invariance algebra of equation (18) and its prolongation satisfy the relations

$$\begin{gathered} \widetilde{Q}L = \lambda_0 L + \lambda_1 L_1, \\ \widetilde{Q}L_1 = \lambda_2 L + \lambda_3 L_1 \end{gathered} \tag{19}$$

with some functions $\lambda_0, \lambda_1, \lambda_3, \lambda_3$.

We say that equation (18) is *conditionally invariant* if relations (19) hold. In this case we can impose an additional condition

$$L_1 \equiv L_1(x, u_1, u_2, \ldots)$$

and system (18), (19) is invariant under Q.

We say that equation (18) is Q-invariant provided

$$\widetilde{Q}L = \lambda_0 L + \lambda_1 (Qu).$$

The essence of this definition is that we can extend a symmetry of PDE by adding some addititonal conditions on its solutions. The conditional and Q-invariance approaches make it possible to find a lot of new exact solutions for great many of important nonlinear equations. Let us enumerate some of them:

1.The nonlinear Schrödinger equation

$$i\Psi_t + \Delta\Psi = F(x, \Psi, \Psi^*).$$

2. The nonlinear wave equation

$$\Box u = F(u).$$

3. The nonlinear eikonal equation

$$u_{x_0}^2 - (\vec{\nabla} u)^2 = \lambda.$$

4. The Hamilton-Jacobi equation

$$u_{x_0} - (\vec{\nabla} u)^2 = \lambda.$$

5.The nonlinear heat equation

$$u_t - \vec{\nabla}(f(u)\vec{\nabla} u) = g(u).$$

6. The Monge-Ampéré equation

$$\det\|u_{x_\mu x_\nu}\|_{\mu,\nu=0}^n = F(u).$$

7. The nonlinar Born-Infeld equation

$$(1 - u_{x_\mu} u_{x^\mu})\Box u + u_{x_\mu} u_{x_\nu} u_{x^\mu} u_{x^\nu} = 0.$$

8. The nonlinear Maxwell equations

$$\Box A_\mu - \partial_{x_\mu}\partial_{x_\nu} A_\nu = A_\mu F(A_\nu A^\nu).$$

9. The nonlinear Dirac Equations

$$i\gamma_\mu \Psi_{x_\mu} = F(\Psi^*, \Psi).$$

10. The nonlinear Lev-Leblond equations.

$$i(\gamma_0 + \gamma_4)\Psi_t + i\gamma_a \Psi_{x_a} = F(\Psi^*, \Psi).$$

11. Equations of the classical electrodynamics.

$$i\gamma_\mu \Psi_{x_\mu} + (e\gamma_\mu A^\mu - m)\Psi = 0,$$
$$\Box A_\mu - \partial_{x_\mu}\partial_{x_\nu} A_\nu = e\overline{\Psi}\gamma_\mu \Psi.$$

12. $SU(2)$ Yang-Mills Equations.

$$\partial_{x_\nu}\partial_{x^\nu}\vec{A}_\mu - \partial_{x^\mu}\partial_{x_\nu}\vec{A}_\nu + e\Big((\partial_{x_\nu}\vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_{x_\nu}\vec{A}_\mu) \times \vec{A}_\nu + +(\partial_{x^\mu}\vec{A}_\nu) \times \vec{A}^\nu\Big) + e^2\vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}.$$

Summary

In conclusion, I should like to say that the main heritage of Prof. Fushchych is a scientific school created by him. About 60 Philosophy Doctors whose theses he supervised work at many institutions of the Ukraine and abroad. And his former students will make their best to continue the ideas of Prof. Wilhelm Fushchych.

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