

Snowmass TF04 Report: Scattering Amplitudes and their Applications.

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ABSTRACT: The field of scattering amplitudes plays a central role in elementary-particle physics. This includes various problems of broader interest for collider physics, gravitational physics, and fundamental principles underlying quantum field theory. We describe various applications and theoretical advances pointing towards novel descriptions of quantum field theories. Comments on future prospects are included.

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Executive summary

Virtually everything we have learned about the behavior of elementary particles has been gleaned from experimental and theoretical studies of scattering processes. The past few decades have taught us that scattering amplitudes offer remarkable insights into the structure of quantum field theories, as well as efficient routes to precision theoretical results needed to interpret modern experiments. These insights, including those that follow from novel descriptions of scattering amplitudes suggest that some of our most cherished notions about quantum theories using the principles of locality need revision. These novel approaches include those using on-shell approaches, twistor-space and geometric approaches. It has also become abundantly clear that scattering amplitudes have led to important progress in issues of interest to the broader community.

Scattering amplitudes have a long history of applications to collider physics, string theory, supergravity, mathematical physics, and more recently to gravitational-wave physics, summarized in relevant Snowmass white papers [1–13]. The basic premise of the field is a virtuous cycle between explicitly calculating quantities of experimental or theoretical interest and identifying new structures that teach us basic facts about quantum field theories. These structures in turn lead to improved methods to carry out out new calculations that then lead to new insights. This positive feedback loop has continued to infuse the field with new ideas and energy informing and guiding new advances. This has been used to push the state of the art for collider physics, supergravity and

more recently for precision calculations of direct importance to gravitational-wave emission from binary black holes and neutron stars.

Scattering amplitudes also serve as a wonderful theoretical playground to test new ideas and connections. This has led to great advances in our understanding of gauge theories, connections to positive geometry and Amplituhedron, cluster algebras and produced efficient bootstrap methods for higher-loop amplitudes. We have seen also use of integrability techniques and an intriguing imprint of AdS/CFT correspondence in the structure of S-matrix at strong coupling. The color-kinematics duality and related double copy uncovered a deep connection between various quantum field theories, and allows us to construct scattering amplitudes using universal building blocks.

In the coming years we expect that scattering amplitudes will continue to address nontrivial problems in collider physics, gravitation, conformal field theories, gravitational-wave physics, effective field theories, and as well as open up new direction such as cosmology and completely unexpected ones. Tantalizing hints, such as from new geometric approaches to scattering, bootstraps and unexplained ultraviolet cancellations in extended supergravity theories, suggests that we need to rethink fundamental principles in quantum field theory.

1 Introduction

From Rutherford’s discovery of the atomic nucleus more than a century ago by scattering α particles from gold foil, to the much more recent discovery of the Higgs boson at the Large Hadron Collider (LHC) at CERN, the observation and interpretation of scattering events have been central to our understanding of elementary-particle interactions. In recent years the field of scattering amplitudes has taken on a renewed vitality, not only because of the continued importance to experimental and theoretical studies, but also because of the realization that scattering amplitudes offer deep insight into the very structure of quantum field theories. It has had a broad variety of state of the art applications to collider physics, supergravity, string theory, mathematical physics and gravitational-wave physics.

The past few decades of research have revealed remarkable new structures in scattering amplitudes that provide striking insights into the structure of modern quantum theories, as well as efficient routes to carry out theoretical results needed to interpret various experiments. These insights, including geometric structures in amplitudes, suggest that some of our most cherished principles about constructing quantum theories using locality need revision. It has also become abundantly clear in recent years that deep issues in quantum gravity, including its relation to gauge theories, can be understood through studies of scattering amplitudes.

As illustrated in Fig. 1, a virtuous cycle between new explicit calculations and new identified structures that then lead to improved methods is central to progress. Many examples abound, starting from the n -gluon Parke-Taylor MHV amplitudes [14, 15], which was originally studied in the context of jet physics at particle colliders. These amplitudes form the basis for many other advances including the construction of n -point one-loop MHV amplitudes [16], and the Cachazo–Svrcek–Witten diagrams [17] for obtaining all tree-level helicity amplitudes starting from the MHV ones, as motivated by twistor-space representations of amplitudes [18, 19]. The modern unitary method [16, 20, 21] was developed following computations using earlier methods that led to the simple form of the explicitly computed one-loop five-gluon amplitude of QCD. The Britto–Cachazo–Feng–Witten (BCFW) on-shell recursion relations [22] were in turn motivated by explicit forms of

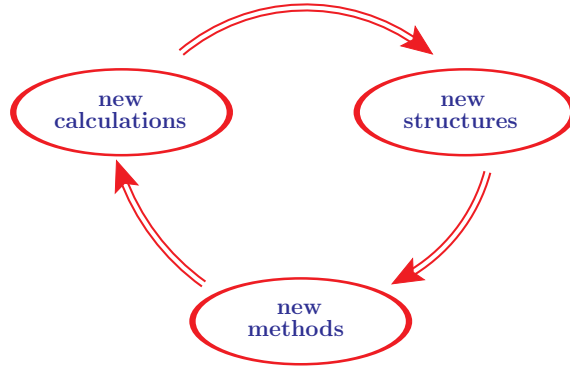


Figure 1. The virtuous cycle between explicit results from calculations, new theoretical structures, and new methods.

tree amplitudes appearing as coefficients of infrared singularities in one-loop amplitudes [23]. The pace of development of new methods has continued in recent years with examples being new methods for describing massive states using helicity methods [24], new methods for writing down multi-loop amplitudes bypassing integration [25] and improved methods to obtain results directly relevant to precision predictions of gravitational waves from astrophysical sources [26–28]. There are many other examples of the synergy between explicit results and the development of new methods that then lead to further new results, with the expectation that the cycle will continue well into the future.

1.1 Further Reading

Snowmass is a community planning exercise, and the present document aspires to represent the excitement and interests of the growing community of theorist who work in the area of scattering amplitudes and topics with direct overlap. We gratefully acknowledge the contributions from the authors of the white papers offering valuable insights and guidance for the future. The following people have contributed to white papers helpful as input for this summary: Tim Adamo, Nima Arkani-Hamed, Benjamin Basso, Daniel Baumann, Xavier Bekaert, Nathan Berkovits, Nicolas Boulanger, Jacob L. Bourjaily, Broedel Broedel, Alessandra Buonanno, Andrea Campoleoni, John Joseph Carrasco, Mariana Carrillo-González, Ekta Chaubey, Marco Chiodaroli, Claudia de Rham, Lance J. Dixon, Claude Duhr, Eric D’Hoker, Henriette Elvang, Fernando Febres Cordero, Dario Francia, Hjalte Frellesvig, Steven B. Giddings, Walter Goldberger, Daniel Green, Michael B. Green, Maxim Grigoriev, Martijn Hidding, Henrik Johansson, Austin Joyce, Mohammed Khalil, Martin Kruczenski, Sandipan Kundu, Robin Marzucca, Andrew J. McLeod, Tobias Neumann, Donal O’Connell, Enrico Pajer, Joao Penedones, Guilherme L. Pimentel, Radu Roiban, Oliver Schlotterer, Ergin Sezgin, William Shepherd, Evgeny Skvortsov, Mikhail P. Solon, Marcus Spradlin, Lorenzo Tancredi, Massimo Taronna, Andrew J. Tolley, Jaroslav Trnka, Matthew Reece, Balt C. van Rees, Charlotte Sleight, Cristian Vergu, Anastasia Volovich, Matthias Volk, Matt von Hippel, Andreas von Manteuffel, Stefan Weinzierl, Matthias Wilhelm, Mao Zeng, Chi Zhang, and Shuang-Yong Zhou.

In this summary, due to the large number of papers, we include citations only to a relatively small number of selected papers and refer readers to the relevant white papers [1–13] for a detailed list of references. Besides the white papers, readers may also consult various review articles [29–37]

and books [38–40] for further details and references. We also limit our discussion here to a few selected topics for the purpose of illustrating various principles as well as the vitality of the field. A list of pertinent white papers that present an overview of developments, challenges and new opportunities related to the field of scattering amplitudes are as follows:

Computational Challenges for Multi-loop Collider Phenomenology [1]. High-order precision computations needed to match the experimental precision at the LHC continue to motivate the field of scattering amplitudes to develop ever more efficient theoretical tools.

$\mathcal{N} = 4$ super-Yang Mills [2]. $\mathcal{N} = 4$ super-Yang–Mills theory is an important special case, not only because its relative simplicity compared to QCD makes it possible to obtain results at spectacularly high-orders, but it links to both to Maldacena’s AdS/CFT conjecture and to supergravity via the double copy [3].

The Double Copy and its Applications [3]. The double copy began as a means for obtaining gravitational scattering amplitudes directly from corresponding gauge-theory ones, and has since spread in various directions to a web of theories, impacting string theory, particle physics, general relativity, and more recently gravitational-wave physics, astrophysics, and cosmology.

Gravitational Waves and Scattering Amplitudes [4]. Powerful tools from scattering amplitudes and effective field theory (EFT) have pushed state-of-the-art perturbative calculations of direct interest to the problem of gravitational-wave signals from inspiraling binary black holes and other astrophysical objects.

Effective Field Theories of Gravity and Compact Binary Dynamics [5]. The methods of effective field theory make it possible to directly apply scattering amplitude methods to gravitational-wave physics, with a useful synergy between the two areas.

Standard Model Effective Field Theory (SMEFT) at the LHC and Beyond [6]. The impact of new physics on scattering of Standard Model particles can be systematically described by EFTs; the understanding of scattering amplitudes in such EFTs will continue to be an important direction in the coming years.

String Perturbation Theory [7]. String theory scattering amplitudes are closely tied to those of quantum field theory and are an essential part of studies of gravitational physics, dualities and mathematical structures. Such studies will continue to lead to new insights.

Higher Spin Gravity and Higher Spin Symmetry [8]. The problem of consistent descriptions of higher-spin particles and their scattering amplitudes continues to be important, and has applications to quantum gravity, cosmology, conformal field theory, AdS/CFT, string theory, and very recently to the problem of the coalescence of binary spinning (Kerr) black holes [4].

Functions Beyond Multiple Polylogarithms for Precision Collider Physics [9]. Our ability to push the frontiers of scattering amplitudes, whether for collider physics or more theoretical studies, rely crucially on the mathematics of special functions.

UV Constraints on IR Physics and the S-matrix Bootstrap [10, 11]. Recent years has seen a renewed vigor toward fundamental principles, such as unitarity, crossing and good Regge behavior to constrain low-energy EFTs, with the goal of identifying the regions where physically

sensible EFTs live.

The Deepest Problem: Some Perspectives on Quantum Gravity [12]. Scattering amplitudes will continue to be an important tool towards the goal of realizing a fully satisfactory description of quantum gravity.

The Cosmological Bootstrap [13]. A promising and exciting new direction is to connect the basic principles of scattering amplitudes—unitarity, locality and symmetry assumptions—to the study of fluctuations in the early universe.

2 New Structures from Amplitudes

Scattering amplitudes display structures with deep implications that are completely hidden using standard Feynman diagram methods. The earliest example of such structures are the maximally-helicity-violating (MHV) tree amplitudes of quantum chromodynamics (QCD) [14, 15]. At the lowest perturbative (tree-level) order in QCD the n -gluon color-ordered amplitude is given by

$$A(1^-, 2^-, 3^+, \dots, n^+) = i \frac{\langle 1 2 \rangle}{\langle 1 2 \rangle \langle 3 4 \rangle \dots \langle n 1 \rangle}, \quad (2.1)$$

where the plus and minus signs refer to the helicity of the gluons, and the notation $\langle ij \rangle$ denotes spinor-inner products. (In this amplitude the color-charges have been stripped away, giving a color-ordered amplitude from which complete amplitudes can be reconstructed [29, 30]). Perhaps the most remarkable aspect of this formula is its simplicity which may be contrasted to the complexity of high-multiplicity Feynman diagrams. The observed simplicity in Eq. (2.1) eventually led to the development of new methods to exploit it, including the unitarity method [16, 20, 21] and on-shell recursion [22].

Over the years the field has continued to identify new and novel presentations of amplitudes. Some examples of novel structures that have driven the development of new methods include,

- Descriptions of scattering amplitudes in terms of algebraic curves in twistor space, as motivated by twistor-string theory [18, 19].
- Iterative descriptions based on unitarity for loop-level scattering-amplitude integrands starting from tree-level ones [16, 20, 21, 41].
- Recursive descriptions of tree-level scattering amplitudes, inspired by the twistor-space description and generalized unitarity [17, 22, 23].
- A hidden “dual conformal” symmetry in planar $N = 4$ super-Yang-Mills theory [42, 43] that guides the construction of amplitudes in the theory.
- Geometric representations of amplitudes, such as the Amplituhedron [44], which seek to recast the basic premise of quantum field theory.
- A duality between color and kinematics [45], that greatly clarifies the double-copy relations between gravity and gauge theory, first identified in string theory [46].

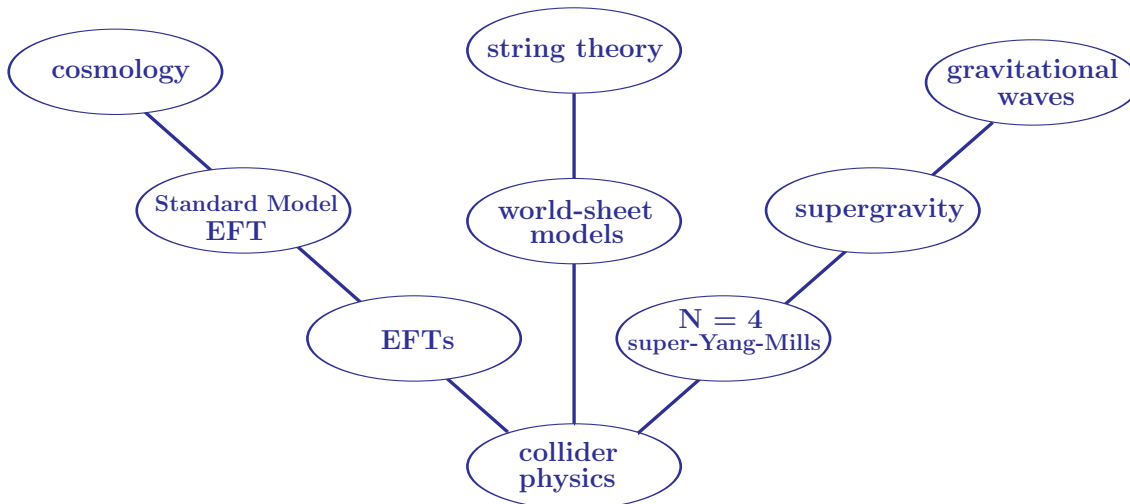


Figure 2. The modern field of scattering amplitudes began in collider physics and has since been applied to a wide variety of topics.

- New representations of tree amplitudes by an integral over the position of n points on a sphere restricted to satisfy a set of equations known as “scattering equations” [47].

Many of these novel descriptions and structures have led to greatly improved computational methods.

3 Highlights of Amplitudes

As illustrated in Fig. 2 the modern field of scattering amplitudes has its roots in collider physics, and has now spread to a large variety of topics.

3.1 Collider Physics

The past five years have been a golden era in pushing forward high-precision calculations in collider physics [1], driven by the unprecedented precision of the upcoming high-luminosity run at the Large Hadron Collider. Even rare processes such as Higgs boson production require theoretical control of cross sections at an unprecedented level of about one percent. For many processes, current theoretical uncertainties still do not match the anticipated experimental uncertainties. This continues to be a primary driver of the field of scattering amplitudes.

In the mid-1980’s as collider physics was progressing towards studies of ever increasing numbers of objects such as jets or vector bosons, as well as more precise measurements, a need arose for better methods which became apparent as more calculations were completed. This cycle of new collider physics calculations driving new methods continues today, except now this is occurring at the multi-loop level.

To fully exploit the new-physics discovery potential of on-going particle-collider experiments, it is essential to continue to reduce theory uncertainties in order to match experimental improvements. The current lack of new physics discoveries at the LHC beyond the landmark discovery of the Higgs boson, emphasizes the importance of the quest for ever more precise measurements that can tease

out long-awaited new physics. To reach the required level of precision requires control over all aspects of the collision including parton distributions functions, the final state parton shower, soft physics associated with hadronization, as well as the partonic high-energy interactions.

The field of scattering amplitudes is especially attuned to dealing with the problem of precision interactions at colliders. Scattering amplitudes play a central role in virtually all collider phenomenology predictions and contain the essential dynamical information associated with the underlying physics. In order to achieve percent-level control generally one needs at least two-loop calculations of scattering amplitudes. Despite numerous advances in recent years, such calculations remain challenging, in particular those that depend on multiple scales associated with either masses or kinematic invariants. In the past five years many such new calculations have been performed thanks in part to advances in our understanding of the analytic structure of scattering amplitudes in quantum field theory [9]. These calculations include, for example, two-loop amplitudes for $2 \rightarrow 2$ and $2 \rightarrow 3$ processes with multiple scales, three- and four-loop form factors. Important progress has been accomplished also for the calculation of related quantities, for example the complete five-loop beta function and first results for the four-loop splitting functions. (See the white paper [1] for references.) A central issue is to develop ever improved methods for reducing multiloop Feynman integrals to a small number of master integrals that can be evaluated using advanced methods. This progress is not only important in collider physics but the same advances carry over to other areas such as to conformal field theories, such as $N = 4$ super-Yang–Mills theory [2], and problems in gravitational-wave physics [4] which involve similar integrals.

Besides precision Standard Model calculations another crucial direction in collider physics is quantifying new physics models. In recent years effective field theories have risen in prominence as a means for systematically categorizing physics beyond the Standard Model [6, 48]. Scattering amplitudes can aid this in two directions, firstly being useful for organizing the independent interactions and also by providing tools for computing useful quantities such as anomalous dimensions and cross-sections. It has also proven useful for explaining new structures, such as nontrivial zeroes [49] in anomalous dimension matrices of the Standard Model Effective Field Theory [50].

3.2 Scattering Amplitudes and Gravitational Waves

Arguably the most exciting recent application of scattering amplitudes methods is to calculate new quantities of interest to the gravitational-wave community [4]. The experimental detection of gravitational waves has fundamentally transformed key areas in astronomy, cosmology, and particle physics, and will continue to do so for many decades to come given the anticipated advances in current and future detectors. This era of ever-increasing sensitivities holds the promise of dramatic new and unexpected discoveries, but relies crucially on complementary advances in our theoretical modeling of gravitational-wave sources. In recent years, a new program for understanding the nature of gravitational-wave sources based on tools from scattering amplitudes and effective field theory (EFT) has emerged.

At first sight the subject of scattering amplitudes in quantum field theory might seem to be rather distant from the problem of gravitational waves. Firstly the sources are purely classical, then all the measured events are for bound states not unbounded scattering processes, and finally the basic objects are kilometer scale not point-like elementary particles. On the other hand, compared to the even larger scale of the orbit and the wavelength of gravitational wave the black holes are

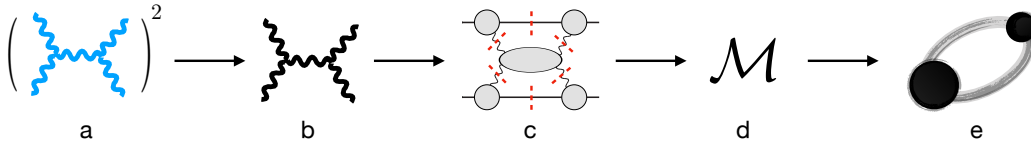


Figure 3. Amplitudes based methods start from the double copy (a) which constructs gravity scattering amplitudes (b) from gauge-theory ones. Generalized unitarity (c) then builds higher-order loop scattering amplitudes (d) from which the interactions (e) between black holes or neutron stars can be extracted. From Ref. [4].

effectively points making it possible to use effective field theory methods [5, 51]. Given the success of performing high-order calculations in quantum gravity, even through five loops, one might suspect that if powerful amplitude methods could be applied to the gravitational-wave problem one would be able to make progress (see e.g. Refs. [4, 26, 28, 52, 53]).

The scattering amplitudes community has taken on the challenge, pushing forward the state of the art in a variety of directions (summarized in Ref. [4]). A specific request from the gravitational-wave community [54] to obtain the conservative two-body Hamiltonian the third order in Newton’s constant was soon answered [28, 55] and more recently at the next order as well. New ideas have been flourishing that link scattering amplitudes to problems of direct importance (see, for example, Refs. [56, 57]) to theorists working on precision predictions for LIGO/Virgo/KAGRA and future detectors. This includes the development of new methods (see e.g. Refs. [26, 27, 55]) for linking scattering amplitudes to physical observables in the bound-state gravitational-wave problem. As illustrated in Fig. 3, amplitudes-based methods start with tree-level scattering which are amenable to double-copy methods, which then feed into unitarity methods, leading to a scattering amplitude from which the classical interactions between two black holes or other astrophysical objects can be extracted. New progress has also been accomplished on radiation effects, tidal effect and spin effects. Higher-spin theories are well studied in particle physics [8], further aiding recent progress. An interesting open problem is to apply scattering-amplitude methods to dissipative effects from the absorption of energy by black holes or neutron stars. Based on the recent advances it is clear that scattering amplitudes will continue to play a prominent role in pushing forward state-of-the-art perturbative gravitational-wave computations.

3.3 Planar $\mathcal{N} = 4$ super-Yang–Mills Amplitudes: from Weak to Strong Coupling

Maximally supersymmetric Yang–Mills (SYM) theory in the planar limit is from many perspectives the *simplest quantum field theory*, making it an ideal toy model for testing amplitude methods that can then be applied to other theories including QCD. One such success is the unitarity method which was first developed for maximally supersymmetric Yang–Mills theory [16, 21] and then later extended to QCD (see e.g. [58]). The perturbative expansion of the maximally supersymmetric theory has a number of simplifying features: it is convergent, its scattering amplitudes are ultra-violet finite and it exhibits a hidden infinite-dimensional Yangian symmetry [59]. The structure of the scattering amplitudes at weak coupling is surprisingly simple and follows various organizational principles such as maximal transcendentality, connections to cluster algebras, and allow for powerful bootstrap methods [2, 60]. These bootstrap methods offer hope that one day all multi-loop scattering amplitudes relevant for collider physics can be directly obtained bypassing the usual step

for first finding an integrand that must then be laboriously integrated. The leading IR divergence of the maximally supersymmetric Yang–Mills S-matrix, related to cusp anomalous dimension, is known to *all* loop orders in this case via integrability methods [61]. The integrability also plays a crucial role in the flux-tube methods [62]: the dual (to scattering amplitudes) null polygon Wilson loop can be calculated using the operator product expansion (OPE) and a new decomposition in terms of pentagon transitions related to the dynamics of flux tubes. Finally, the strong-coupling limit is controlled by Maldacena’s celebrated AdS/CFT correspondence [63, 64] and classical string configurations in AdS₅ [65]. In special cases, it is even possible to resum the entire perturbative series, linking weak and strong coupling [66, 67].

3.4 Gravity as a Double Copy of Gauge Theory

Perhaps one of the more surprising outcomes from studies of scattering in gravitational theories is the double copy [3, 36, 45–47]. The past few years have seen a burst of interest in this topic. At its core, the double provides a means to calculate amplitudes in one theory using, as input, amplitudes from two technically simpler theories. The most prominent case gives gravity scattering amplitudes in terms of two corresponding gauge-theory ones. Such relations were originally discovered in the context of string theory [7, 46] and have since been greatly clarified via a “duality between color and kinematics” [45], which allows gravity amplitudes to be generated by gauge-theory ones via a simple replacement of color factors by kinematic factors. Here color refers to the usual color charges of nonabelian gauge theories that describe either strong or electroweak forces. This duality is effectively a map between color and kinematic factors that extends to a broad variety of familiar field and string theories. It has its origins in perturbative scattering amplitudes but is currently being systematically extended to generic classical solutions. The simplest such example relates the Coulomb solution in electromagnetism to the Schwarzschild black hole in Einstein gravity [68], with many more sophisticated constructions available such as the Weyl double copy.

As the simplest example of the double copy, $2 \rightarrow 2$ graviton scattering amplitude at lowest order in Einstein gravity are simply related Yang–Mills (YM) theory,

$$\mathcal{M}(1, 2, 3, 4) = \left(\frac{\kappa}{2}\right)^2 \frac{st}{u} A(1, 2, 3, 4) \times A(1, 2, 3, 4), \quad (3.1)$$

where $A(1, 2, 3, 4)$ is a color-ordered gauge-theory four-gluon partial scattering amplitude (related to ordinary amplitudes by appropriately stripping off group-theory color factors), $\mathcal{M}(1, 2, 3, 4)$ is a four-graviton tree amplitude, κ is the gravitational coupling related to Newton’s constant via $\kappa^2 = 32\pi^2 G_N$, and s, t, u are the Mandelstam kinematic invariants. We can summarize the relation heuristically as

$$\text{gravity} \sim (\text{gauge theory}) \times (\text{gauge theory}). \quad (3.2)$$

In a precise sense the double copy gives us a “multiplication table” for converting pairs of gauge theories to gravitational theories.

In the coming years we can expect continued development of the double copy on its practical and theoretical sides. In supergravity besides the quest to understand whether *all* supergravity theories can be expressed as double copies of corresponding gauge theories, it will be important to carry out new higher-loop computations to finally understand whether all point-like supergravity theories must, as lore suggests, necessarily be ultraviolet divergent. The double copy has also been used in

state-of-the-art calculations of interest to the gravitational-wave community; it will be important see what further insights emerge from these studies. General questions about the set of all allowed higher-derivative interactions that permit a double copy also remain. An important outstanding puzzle is to fully understand the kinematic algebra behind the duality between color and kinematics. While there are many examples where double copies for classical solutions have been identified it would be important to find a coherent principle for identifying double-copy mappings for generic classical solutions.

3.5 String Scattering Amplitudes and World Sheet Models

The perturbative S-matrix is a central object not only in quantum field theory but also in string theory [7]. Superstring perturbation theory is a rich subject which reveals deep connections between QFT amplitudes, D-branes, gauge/gravity duality, gravitational waves, black holes, algebraic geometry, and modular forms. The central objects are superstring amplitudes, which are primarily considered for massless gravitons, gauge bosons and their respective supersymmetry partners. These amplitudes are integrals over the moduli spaces of Riemann surfaces, and exhibit some remarkable simplicity and unexpected properties, tightly connected to gauge theory in the infinite string-tension limit. In recent years, the calculation front of string perturbation theory was pushed to three loops with many non-trivial new results. New insights were obtained from string dualities and AdS/CFT correspondence, and string amplitude were shown to have some fascinating connections to transcendentality principle and multiple zeta values. Future targets include lifting technical obstacles to perform higher-loop calculations of string amplitudes, explore the rich mathematical structure of Feynman integrals associated with K3 or Calabi-Yau geometries in the context of higher-genus amplitudes.

There are also two related exciting directions that have benefited from the advances in the perturbative string theory [7]: ambitwistor string models [69] and Cachazo–He–Yuan (CHY) formalism [47]. The latter one expresses field-theory amplitudes as certain integrals over the worldsheet localized at the points which satisfy scattering equations. The integrals are built from simple building blocks and manifest connections between various field theories, including color-kinematics duality. The ambitwistor string express the same field-theory amplitudes as the worldsheet correlators of certain ambitwistor strings defined by a simple worldsheet action. The correlators reduce to the formulas in the CHY formalism, making a fascinating link between strings, worldsheets and field-theory amplitudes. The existence of such novel descriptions of field-theory scattering amplitudes points to new underlying principles in quantum field theories.

3.6 Web of Theories

Many scattering-amplitudes developments suggest that there are nontrivial fundamental links between amplitudes in different quantum field theories [3, 36]. Their tree-level amplitudes are uniquely fixed by simple physical conditions: locality and unitarity (pole structure and factorizations) together with additional constraints [16, 22]: gauge invariance in the context of gauge theory and gravity [70]; vanishing soft limits (of various degrees) for non-linear sigma model (NLSM), DBI action or special Galileons, Born-Infeld action, and combinations of those for Volkov-Akulov or Einstein–Yang–Mills actions, and others [71–74]. The uniqueness of these tree-level amplitudes, for example, allow for their reconstruction using recursion relations from elementary amplitudes [22].

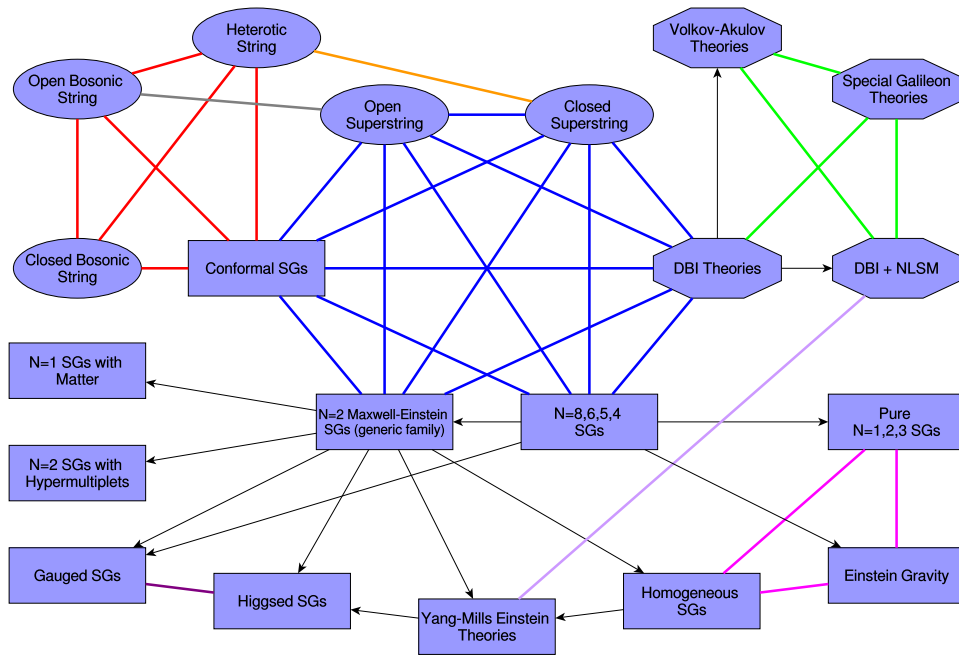


Figure 4. The web of theories exhibited by the double copy, as presented in Ref. [36]. The nodes represent double-copy-constructible theories, including gravitational theories (rectangular nodes), string theories (oval nodes) and non-gravitational theories (octagonal nodes). Undirected links connect theories with a common gauge-theory factor. Directed links are drawn between theories constructed by modifying both gauge-theory factors.

Infrared physics and soft limits are also under study from the novel perspective of transforming amplitudes to celestial sphere [75, 76]. In this picture the soft theorems are understood as symmetries of the celestial correlators.

The same theories also appear in the context of CHY world-sheet description of scattering amplitudes [3, 7, 47, 77], and are linked by double-copy relations: the CHY integrand of the special-Galileon theory contains two copies of the NLSM integrands, and the same combination appears in the double-copy construction. In fact, amplitudes in these special theories can be all built from elementary building blocks making the connections between theories and special kinematical behaviors manifest.

A specific example illustrating these ideas are relations between theories that can be expressed in a double-copy format as a product of two theories along the lines of Eq. (3.2). This type of multiplication table extends to a remarkably broad variety of theories beyond standard gravitational theories [3], illustrated in Fig. 4, which show links between theories that share a common theory in the product factor. A key goal is to extend these types of relations to a much larger class of theories and to find new building blocks in a unified description.

3.7 Constraints on Effective Field Theories

Physically sensible quantum field theories are constrained by basic assumptions of unitarity, causality, crossing, and good high-energy behavior. The conformal bootstrap program [78] has emphasized the power of such ideas. These constraints are natural to apply to scattering amplitudes in the

context of effective field theories that describe physics at scales lower than that of the underlying fundamental theories [79]. Such effective field theories are a basic tool for describing physics beyond the Standard Model [6], including also gravitational theories.

How strongly can we constrain generic EFTs using fundamental principles that all sensible theories must satisfy? Recent progress (see e.g. Refs. [80–82]) systematically constrains the Wilson coefficients of operators or equivalently the EFT coefficients appearing in amplitudes to bounded regions. EFTs describing weakly coupled gravity provide an an important test case, where we can use string theory, as well as various intermediate energy models to compare the known constraints to the regions where sensible models actually land [83]. Remarkably, sensible effective field theories seems to lie on tiny theory islands far smaller than anticipated from known constraints. This suggests that new principles constraining physically sensible effective field theories may very well exist. An example of such a principle is the concept of “low spin dominance”. Given the recent advances, in the coming years we can expect much more progress on understanding the islands where physically sensible EFTs live and identifying new constraints that all such theories satisfy.

3.8 Positive Geometry and the Amplituhedron

A completely new way to define and calculate perturbative scattering amplitudes has been developed for certain theories as volumes of positive geometries. The positive geometry encodes the combinatorics of singularities in the kinematical space and the geometry volume form reproduces the amplitudes. The prime example is the Amplituhedron picture [44] for planar $\mathcal{N} = 4$ SYM theory which defines all tree-level amplitudes and loop integrands in this theory. Recently the Associahedron geometry has been linked to amplitudes in scalar ϕ^3 theory [84]. These are new definitions of the perturbative S-matrix reformulating the physics problem of summing an infinite number of Feynman diagrams as the mathematical problem of triangulating geometric spaces. The central object is the positive geometry [85], a region in the kinematical space defined by certain inequalities, and the canonical differential form on this geometry reproduces the scattering amplitude. This picture has been used to provide some all-loop order calculations, not accessible using standard methods [86, 87]. There is also a fascinating connection pure mathematics: Amplituhedron provides a substantial generalization of the positive Grassmannian and is of great interests to combinatorists; the positive geometries are closely linked to cluster algebras and tropical geometries, which are both very active areas of research (see e.g. Refs. [88, 89]). The future goals include uncovering more mathematical connections and using them in the triangulations and explicit evaluations of differential forms, the discovery of positive geometries for other quantum field theories, and formulating a unified geometric picture for the perturbative S-matrix.

This approach stands out in the way it seeks to reformulate the usual principles of quantum field theories in terms of a completely different set of geometric principles from which the usual ones of unitarity, causality and locality follow. Further progress offers the promise of radical reinterpretations of quantum field theory.

3.9 S-Matrix Function Space

The perturbative scattering amplitudes are kinematical functions of many variables with special properties dictated by underlying physical constraints. The poles and branch cuts encode the basic principles of locality and unitarity, while scaling and various limits encode universal soft or

collinear properties of the S -matrix. While tree-level amplitudes are simple rational functions, the loop amplitudes are much more complicated and the universe of all functions that can appear in the loop amplitudes is not well understood even at next-to-next to leading order in the coupling [9], which is necessary to match experimental precision for many processes [1]. This important question has deep mathematical significance—mathematical properties of these functions can encode hidden physics—as well as practical use for finding a relatively small basis of objects we need to consider in any particular calculation. Over the last decade, it has become clear that a broad range of scattering amplitudes can be expressed in terms of functions called multiple polylogarithms. This realization had led to major computational advances for QCD amplitudes, as well as remarkably high-loop calculations in supersymmetric theories.

In planar $\mathcal{N} = 4$ SYM theory [2] knowledge of the function space, symbols and the connection to cluster algebras [90–92] was used to obtain results for six-point amplitudes up to seven loops (see e.g. Refs. [60, 93, 94]) for various helicity structures (more precisely IR finite objects called “remainder” and “ratio” functions), and uncover a remarkable new duality which relates the amplitudes to form factors [95].

Unfortunately, scattering amplitudes need further special functions beyond the polylogarithms, especially in processes with multiple kinematical variables. This includes multiple polylogarithms, extensions to elliptic polylogarithms and beyond. Similar functions beyond ordinary polylogarithms also enter into high-order calculations of gravitational-wave physics [4]. A summary of the state of the art, reviewing the “zoo” of non-polylogarithmic integrals and functions and providing future directions is given in Ref. [9].

3.10 Cosmological Bootstrap

An exciting new direction with important synergies with scattering amplitudes is the cosmological bootstrap [13]. The physics of primordial density fluctuations is a unique probe of an early universe. During inflation quantum fluctuations were stretched to very large distances, and these correlations provide today a rare insight of the early stages of our universe. All these are spatial correlations, and the time only appears in the scale dependence as the modes freeze at different times depending on the wavelength. The standard approach to calculate inflationary correlators is to evolve them from the origin of quantum fluctuations until reheating which requires the evaluations of complicated time integrals. The cosmological bootstrap is a new strategy to construct the cosmological correlation functions using basic physical principles like locality, unitarity and scale invariance as an approximate symmetry. The grand goal is to classify all possible patterns of primordial fluctuations based on these general principles, as well as uncover unexpected connections between fundamental principles and correlators, theory and data. Recent developments in various directions include the study of constraints imposed by unitarity on cosmological correlations, both in the perturbative and non-perturbative setup. These ideas naturally align with scattering amplitudes so we can expect new fruitful applications of scattering amplitude methods.

4 Outlook and Conclusions

In summary the modern amplitudes program continues to be a vibrant area significantly impacting various directions including collider physics, gravitational-wave physics, effective field theory, supersymmetric gauge and gravity theories, string theory and mathematical physics. At its core the

field of scattering amplitudes is about identifying new structure that then help us calculate difficult to obtain quantities of theoretical or experimental interest. Some of the identified structures, such as the geometric interpretation of scattering amplitudes or the double copy, suggest that our basic description of quantum field theory need revision. In the coming years we can expect continuing advances on both the applications and theoretical sides.

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