# Snowmass White Paper: the Double Copy and its Applications

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ABSTRACT: The double copy is, in essence, a map between scattering amplitudes in a broad variety of familiar field and string theories. In addition to the mathematically rich intrinsic structure, it underlies a multitude of active research directions and has a range of interesting applications in quantum, classical and effective field theories, including broad topics such as string theory, particle physics, astrophysics, and cosmology. This Snowmass white paper provides a brief introduction to the double copy, its applications, current research and future challenges.

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## 1 Introduction

Scattering amplitudes provide a powerful arena for advancing our understanding of relativistic quantum field theory and its applications in particle physics and beyond. Traditional Feynman diagrammatics grow rapidly in complexity and computational difficulty with increasing number of external particles and loop order. In contrast, modern amplitudes methods have provided not only computational power to calculate otherwise intractable processes, but they have also shed new light on field theories, both classical and quantum.

The *amplitudes program* takes the physical observables, the scattering amplitudes, as the central objects of interest and constructs them directly from knowledge about their analytical structure and symmetries, often without direct reference to Lagrangians. Crucially, the amplitudes program rests on two pillars: (1) its connection to and direct relevance for particle phenomenology and experiment and (2) its strong potential to further our fundamental understanding of quantum field theories. Moreover, the program makes significant and lasting connections to adjacent fields: pure mathematics, string theory, and gravitational wave physics.<sup>1</sup>

In this Snowmass 2021 white paper, we describe recent progress, challenges, and opportunities in research on *the double copy*. Conceptually, the double copy [2–6] is a way to calculate amplitudes in one theory using, as input, amplitudes from one, or two different, simpler theories. It is a multiplicative and bi-linear operation, hence the name "double copy" is appropriate.

As an example, the double copy expresses graviton tree amplitudes in terms of sums of products of gluon tree amplitudes: this relation is obscured at the level of off-shell Lagrangians and Feynman rules, but takes a surprisingly simple form in terms of on-shell amplitudes. For example, the 4-graviton tree amplitude of Einstein gravity is simply

$$M_4 = -\frac{s_{12} s_{14}}{s_{13}} A_4 [1234]^2, \qquad (1.1)$$

where  $A_4[1234]$  is the color-stripped four-gluon<sup>2</sup> amplitude of Yang-Mills (YM) theory and  $s_{ij} = (p_i + p_j)^2$  are the Mandelstam variables. Explicit relations like (1.1) are known for any multiplicity of external states. This relationship is often summarized compactly as

$$gravity = (Yang-Mills)^2.$$
(1.2)

The double copy was first observed in string theory. In the mid-1980s it was found [2] that closed string tree amplitudes could be written as sums of products of open string amplitudes. This is a non-trivial manifestation, at the level of physical observables, of the factorization of closed string states into products of left- and rightmoving open string states. The low-energy limit of the string-theory double copy is the field-theory double copy exemplified in (1.1).

The existence of a multiplicative structure on the space of certain classes of field theories raises many interesting questions of fundamental interest in field and string

<sup>&</sup>lt;sup>1</sup>See e.g. ref. [1] for a recent edited compilation of reviews on scattering amplitudes.

<sup>&</sup>lt;sup>2</sup>We use "gluon" to denote the on-shell excitation of a massless non-abelian adjoint vector field, independent of whether the gauge group is the QCD SU(3) or a more general gauge group.

theory, and has sprouted several directions of research. At the same time, the double copy is also relevant for more practical applications where it becomes a tool to make otherwise intractable calculations, such as the evaluation of higher-loop corrections, accessible.

In the remainder of the Introduction we provide some background to motivate the interest in graviton scattering. Also, to give examples of the power of the double copy, we preview (i) its application to the binary inspiral problem relevant for gravitational wave physics, (ii) its ability to simplify complex multiloop calculation like determining the UV behavior of supergravity theories, (iii) its utility to organize and classify a web of theories.

#### 1.1 What is Graviton Scattering?

Before getting too technical, let us clarify what is meant by perturbative gravity and graviton amplitudes. Expanding the Einstein-Hilbert action around flat space as  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$  gives an action for a fluctuating spin-two field  $h_{\mu\nu}$ , with two-derivative interactions involving any number of fields controlled by the powers of the coupling  $\kappa = \sqrt{8\pi G}$ , where  $G = 6.67 \times 10^{-11}$  N m<sup>2</sup>/kg<sup>2</sup> is Newton's constant. Couplings to matter are introduced by expanding the covariant derivative and metric and are likewise controlled by powers of  $\kappa$ . The Feynman rules are exceedingly complicated in standard gauges, but nonetheless the sums of diagrams simplify to give relatively compact expressions for the on-shell amplitudes, such as the double-copy formula in (1.1).

Let us now address the physical relevance of perturbative graviton scattering. In natural units,  $c = \hbar = 1$ , the scale of G is set by the Planck mass as

$$\kappa \sim \sqrt{G} \sim 1/M_{\text{Planck}} \sim 10^{-19} \,\text{GeV}^{-1}$$
. (1.3)

Since  $\kappa$  is dimensionful, the gravitational interactions are controlled by the dimensionless effective coupling  $E\kappa$ , where E is the typical energy of the scattering process. This means that even for a 100 TeV scale process, the effective gravitational coupling  $E\kappa \sim 10^{-14}$  is very small. This may make perturbative scattering look like a particularly exotic exercise, however, let us now show how perturbative graviton scattering connects directly to familiar gravitational physics.

We know that gravity is an important force in our daily lives: it helps us keep our feet on the ground and makes our planet orbit the Sun. So suppose we consider massive (scalar) particles canonically coupled to gravity<sup>3</sup>. The leading order diagram

<sup>&</sup>lt;sup>3</sup>At the level of the Lagrangian, the canonical coupling of massive scalars  $\phi$  to gravity arises through the kinetic term  $\sim g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$ , where the inverse metric  $g^{\mu\nu}$  is an infinite series in the spin-two field.

for two massive particles interacting via the exchange of a graviton is then

The equivalence principle tells us that gravity couples universally, so the coupling of each vertex is  $\kappa$  and hence the diagram is proportional to G. In the non-relativistic limit,  $v \to 0$ , a Fourier transformation of this scattering amplitude results in the familiar Newton potential as indicated in (1.4). Thus we see how a familiar result from classical physics can arise from a perturbative scattering process. Let us now describe how gravity self-interactions are also relevant to study.

#### **1.2** Preview: Application to Gravitational Wave Physics

Newton's potential (1.4) describes the leading order gravitational interaction between two static masses and to obtain it from the scattering amplitudes, we set the relative speed v between the two particles to zero. Corrections to this potential can depend on the dimensionless quantities  $v^2/c^2$  and on  $GM/(rc^2)$  and their relative importance depends on the nature of the motion (while time-reversal-non-invariant contributions to observables may depend on v/c).

For motion on bound trajectories, the virial theorem implies that  $v^2/c^2$  and  $GM/(rc^2)$  are of the same order, where G is Newton's constant and r is the separation of the two masses. Thus, corrections to Newton's potential are organized in powers of inverse speed of light, and the n-th order corrections contains  $(v^2/c^2)^k (GM/(rc^2))^{n+1-k}$  for all k between zero and n.

The virial theorem does not constrain unbound motion in Newton's potential. Requiring relativistic invariance leads to a resummation of the complete dependence in  $v^2/c^2$  at fixed order in Newton's constant, presenting the corrections as a series in  $GM/(rc^2)$ .

While the origin of the velocity-dependent corrections is intuitively clear, the origin of those depending on Newton's constant is revealed by carefully enforcing the requirements of Bohr's correspondence principle that all charges are large. This implies that loop corrections can contain classical contributions and that they are given by a specific dependence on the momentum transfer [7], which is Fourier-conjugate to the relative distance. For instance,  $O(G^2)$  corrections include diagrams with two gravitons

exchanged as well as diagrams with graviton self-interactions,

$$m_1 \longrightarrow m_2 \longrightarrow m_2$$

while diagrams such as

$$\begin{array}{c} m_1 \\ \hline m_2 \\ \hline m_2 \\ \hline m_2 \\ \hline m_3 \\ \hline m_3 \\ \hline m_2 \\ \hline m_3 \\ \hline m_2 \\ \hline m_3 \\ \hline m_2 \\ \hline m_3 \\$$

contribute to  $O(G^3)$ .

Newton's potential (1.4) can be determined from the tree-level scattering of massive particles. That same Newton potential also describes the bound state physics of, say, orbiting planets. This continues to hold true for corrections to this potential. While scattering amplitudes naturally yield corrections at fixed order in G and all orders in velocity, they can be suitably expanded so that terms of the same order in accordance to the virial theorem are kept. Orbiting bodies radiate gravitationally towards the eventual merger. Close to the merger the complete velocity dependence is relevant. For example, the complete  $O(G^3)$  and  $O(G^4)$  conservative Hamiltonians obtained from scattering data were analyzed by theorists of the LIGO-Virgo-KAGRA collaboration in [8] and [9], respectively. While the results at  $O(G^4)$  are in an early stage with assumptions that require further investigation, the key messages are that they can be included in the effective one-body theory framework [10, 11] in a way that improves comparison with numerical general relativity towards merger, and that further developments are strongly encourages.

Three-graviton Feynman rules in generic gauges can contain over 100 terms [12], with comparable size expressions for contact terms associated with every additional multiplicity. The complicated expressions that appear in Feynman-graph-based calculations at higher orders in G are avoided using a combination of unitarity cuts that both reduce the input for the loop integrand to products of graviton and graviton & massive-particle tree diagrams and weed out terms with no classical contributions. The gravitational tree diagrams can be computed with little effort via double-copy expressions such as (1.1). Calculations using the double copy were the first to push the state of the art first through  $O(G^3)$  and then through  $O(G^4)$  [13, 14].

Recent progress on the application of the double copy to gravitational wave physics is described in further detail in Section 3.2. See also the dedicated Snowmass White

Loop order	$\mathcal{N} = 8$ supergravity	$\mathcal{N} = 5$ supergravity	$\mathcal{N} = 4$ supergravity
2	$D_c = 7 \ [17]$	$D_c^{(*)} > 6$ [18]	$D_c = 6$ [19]
3	$D_c = 6$ [20]	$D_c > 14/3$ [21]	$D_c = 14/3$ [22]
4	$D_c = 5\frac{1}{2}$ [23]	$D_c > 4$ [21]	$D_c = 4 \ [24]$
5	$D_c = 2\tilde{4}/5 \ [25]$		

Table 1. List of direct computations that lead to bounds on the lowest spacetime dimension in which supergravity theories diverge (critical dimension) at various loop orders realized through explicit calculation using double-copy and unitarity methods. <sup>(\*)</sup> The two-loop critical dimension for  $\mathcal{N} = 5$  SG has not been reported in the literature, to our knowledge, but it is reasonable to expect improved behavior relative to  $\mathcal{N} = 4$  SG [19], much like at three and four loops.

Paper, ref. [15].

#### **1.3** Simplifying Calculations

On-shell methods, such as recursion relations and unitarity, help simplify calculation of scattering amplitudes enormously. The double copy adds to this calculation power. We saw an example of this above. Another example is furnished by loop-calculations in perturbative supergravity (SG) with the goal of understanding potential cancellations of ultraviolet (UV) divergences, for example as depicted in Table 1. Unitarity and doublecopy constructions have allowed the first direct access to multiloop order calculations in SG theories that would be otherwise inaccessible through traditional Feynman-diagram based methods. Here the double copy is not only useful for computing cuts in terms of much simpler tree-level super-Yang Mills (sYM) amplitudes, but manifesting color-dual  $\mathcal{N} = 4$  sYM integrands through four-loops [4, 16] has allowed direct access to  $\mathcal{N} \geq 4$ SG integrands through the same loop order. This is discussed more in Section 3.8.

The double copy has impact even on simplifying calculations in gauge theories. This is because internal self-consistency of the double copy implies certain relations among color-ordered amplitudes. These relations reduce the number of independent color-ordered amplitudes in the theory: in (super)Yang-Mills theory for example, they reduce the number of independent color-orderings at n points from (n-2)! to (n-3)!. This matters in particular when on-shell tree amplitudes are used as input in higher-loop calculations. For instance, for the 4-gluon 2-loop amplitude in Yang-Mills theory there are naively 48 color-ordered cuts involving a six-point tree sewn with a four-point tree, but these are not independent. Indeed, because of self-consistency of the double copy we need only consider 6 color-ordered cuts. These ideas, described further in the

$\mathrm{FT}\otimes\mathrm{FT}$	YM	$\mathcal{N} = 4 \text{ sYM}$	$\chi \mathrm{PT}$	BAS
YM	gravity <sup>+</sup>	$\mathcal{N} = 4 \text{ SG}$	BI	YM
$\mathcal{N} = 4 \text{ sYM}$	$\mathcal{N} = 4 \text{ SG}$	$\mathcal{N}=8~\mathrm{SG}$	$\mathcal{N} = 4 \text{ sDBI}$	$\mathcal{N} = 4 \text{ sYM}$
$\chi \mathrm{PT}$	BI	$\mathcal{N} = 4 \text{ sDBI}$	sGalileon	$\chi \mathrm{PT}$
BAS	YM	$\mathcal{N} = 4 \text{ sYM}$	$\chi \mathrm{PT}$	BAS

Figure 1. Example of the web of theories connected by the double copy. As discussed in the main text, the double copy of Yang-Mills theory (YM) with itself gives gravity; the "+" indicates that in addition to gravity one also gets the dilaton and anti-symmetric 2-form. Supersymmetry is inherited additively by the double copy; as an example  $\mathcal{N} = 4$  super Yang-Mills theory (sYM) with its total of 16 states — gluons, gluinos, and scalars — double-copies with itself to give the  $16^2 = 256$  states of  $\mathcal{N} = 8$  supergravity (SG). Leading order 2-derivative chiral perturbation theory  $\chi$ PT can also act as input for the double copy and as shown in the table when it is double-copied with YM,  $\mathcal{N} = 4$  sYM, or itself it produces the amplitudes of a set of models relevant in various aspects of high-energy theory: Born-Infeld theory (BI) of non-linear electrodynamics,  $\mathcal{N} = 4$  super-Dirac Born-Infeld (sDBI), and the special Galileon (sGal) which has appeared in the contexts of modifications of gravity [26–28] and as proposed subleading terms in brane actions [29]. Finally, the cubic bi-adjoint scalar model (BAS) plays the role of an identity element for the double copy; we discuss this further in Section 3.6.

Section 3.9, have not only helped with higher-loop formal studies, but hint at future applications to amplitudes calculations of relevance to particle phenomenology.

#### 1.4 A Map on the Space of Field Theories

The double copy goes beyond a relation between gravity and YM theory (1.2). In fact, there is a web of theories connected via the double copy (see Figure 4 below). This includes, but is not limited to, example field theories in Figure 1 that illustrates the double copy as a "multiplication table".

There is much variety among these models. Yang-Mills theory (YM) is a renormalizable model, gravity is not. Chiral perturbation theory ( $\chi$ PT) is a low-energy effective field theory of pions whereas its double copy, the special Galileon, by itself is not a UV completable model. The Born-Infeld model (BI) of nonlinear electrodynamics has electromagnetic duality in 4d or, equivalently, has only helicity-conserving amplitudes, but the input theories YM and  $\chi$ PT do not. On top of that,  $\mathcal{N} = 4$  super Yang-Mills theory (sYM) is a conformal theory in 4d, while  $\mathcal{N} = 4$  super Dirac-Born-Infeld (sDBI) is the low-energy effective action on a flat D3-brane in Minkowski space. Finally, with its cubic interaction the bi-adjoint scalar model (BAS) has a potential unbounded from below. It is clear that the field-theory double copy connects models with wide range of properties and applications.

#### 1.5 Outline

We begin with a review in Section 2. There are a variety of different formulations of the double copy. Each representation has its own advantages and highlights different properties of the double-copy map. We present three formulations — known in shorthand as KLT, BCJ, and CHY — for tree amplitudes in Section 2.1 and then discuss progress on adapting them at loop-level in Section 2.2. We conclude our review with a brief reminder of the importance of choice of kinematic variables in Section 2.3.

In Section 3 we describe applications of the double copy. Classical double copy and its applications may be the most accessible to interested readers outside of the amplitudes community so we will start there. Besides, despite its discovery in and relevance to quantum mechanical scattering amplitudes, one of the most compelling and remarkable aspects of the double-copy construction is that it offers fresh insight at classically relevant scales. Section 3.1 presents a discussion of classical double copy, beginning with the Kerr-Schild and Weyl double copy understanding of black holes and related solutions. We continue with the sharpest empirical predictions of double copy in the precision gravitational wave predictions arising from black hole collision in Section 3.2. We continue the theme by considering the use of double-copy structure to lift flat space-time scattering amplitude insight to the interaction of quantum fields evolving on non-trivial classical backgrounds in Section 3.3. This naturally leads into a discussion of the potential opportunities and challenges for double copy constructions in both early and late-stage cosmology in Section 3.4.

Double-copy insight has its origins in the KLT relations originally recognized in tree-level perturbative string theory, and the dynamic interchange between QFT and string theory is strong and fruitful. We discuss continuing prospects for the rich interplay between QFT and string theory amplitudes in Section 3.5. From the local QFT perspective, string theory represents an effective UV completion in the form of a tower of higher-derivative operators. An important generalization of double-copy insight is to incorporate phenomenological higher-derivative operators for which string theory amplitudes represent an inspirational and clarifying goal post. Progress in generalizing the double copy in the KLT formalism is discussed in Section 3.6 and in terms of the duality between color and kinematics in Section 3.7. A related sharp question — driving higher-loop progress in non-planar theories — is whether all supergravity theories with a finite number of counterterms must require perturbative UV completions in four-dimensions, or if they could possibly be finite. We discuss current progress and open questions with a focus on double-copy constructions in Section 3.8.

Finally, one of the most attractive aspects of the double copy is how it simplifies calculations while exposing new ways to understand what makes individual theories unique and how they can be related to other theories in sometimes surprising ways. We explore this theme in three sections. The first discusses the reduction in complexity of independent building blocks relevant to phenomenological and formal integrands at the multiloop level in Section 3.9. We continue in Section 3.10 with an overview of the quest towards a sharp understanding of the underlying off-shell kinematic algebra behind double-copy constructions as well as the simplifications we expect such a recognition to bestow. We close by confronting the ultimate in recycling opportunities in Section 3.11: as a small number of primary constituents can combine to build a vast web of doublecopy theories, advances in these primary theories propagate through the web, and appreciating double-copy structures offers a new approach towards classifying theories by identifying shared constituents.

## 2 Review: Formulations of the Double Copy

There are various formulations of the double copy. Here we introduce three complementary formulations of the double copy for tree-level amplitudes in theories of massless particles in the adjoint <sup>4</sup> of a color (or flavor) group. These form the basis for the applications and generalizations. We then briefly discuss extensions to loop-level and choices of kinematic variables for the different types of building blocks.

#### 2.1 Tree-Level: KLT, BCJ, CHY

The three tree-level formulations of the double copy are:

• **KLT double copy**: named after Kawai, Lewellen, and Tye [2], the KLT formula is a manifestly gauge-invariant incarnation of the double copy. At *n*-point, the KLT formula for the tree amplitudes  $M_n^{L\otimes R}$  in the double-copy theory  $L\otimes R$  takes the form

$$M_n^{\mathrm{L}\otimes\mathrm{R}} = \sum_{a,b} A_n^{\mathrm{L}}[a] S_n[a|b] A_n^{\mathrm{R}}[b] , \qquad (2.1)$$

where  $A_n^{\rm L}$  and  $A_n^{\rm R}$  are color-stripped *n*-point tree amplitudes of the "left" and "right" theories L and R, such as the ones in Figure 1. The labels a, b run over two choices of (n-3)! of the n! possible single-trace color-orderings of nparticles. Finally,  $S_n[a|b]$  is the KLT kernel which is a function of the *n*-point Mandelstam variables [30, 31], and can be understood as the inverse of a matrix of bi-colored scalar amplitudes [32]. Connecting to the 4-point example (1.1), we

<sup>&</sup>lt;sup>4</sup>As we discuss later, much progress has been made on double-copy constructions involving arbitrary representations, including the fundamental, and as well as for theories with both massive and massless particles.

see that  $S_4[1234|1234] = -s_{12}s_{14}/s_{13}$ . Note that the same kernel is used in all the double-copies in Figure 1.

In string theory,  $A_n^{\rm L}$  and  $A_n^{\rm R}$  are color-stripped open-string disk amplitudes and  $M_n^{\rm L\otimes R}$  is the closed-string sphere amplitude. The string-theory version of the KLT kernel depends explicitly on the string tension via  $\alpha'$  and simplifies to the field-theory kernel  $S_n[a|b]$  in the low-energy limit. Both variants of the KLT kernel can be mathematically understood in terms of intersection numbers of twisted (co-)cycles [33, 34].

• BCJ double copy: Bern, Carrasco, and Johansson [3] discovered a form of the double copy that relies on the proposal (proven at tree level [35] and tested impressively at loop-level [4, 16]) of color-kinematics duality. The idea is to write the color-dressed amplitudes of the L and R theories as

$$A_{n}^{\rm L} = \sum_{I} \frac{n_{I}^{\rm L} c_{I}^{\rm L}}{\prod_{j \in I} P_{j}^{2}}, \qquad A_{n}^{\rm R} = \sum_{I} \frac{n_{I}^{\rm R} c_{I}^{\rm R}}{\prod_{j \in I} P_{j}^{2}}, \qquad (2.2)$$

where the sum is over trivalent graphs I, the  $c_I$ 's are color factors (e.g.  $c_{ab,cd} = f^{abx} f^{cdx}$  at four points in terms of the anti-symmetric structure constants), and the  $P_j$  are the momenta on the internal lines j of the trivalent diagrams summed over. The  $n_I$  are so-called "numerator factors" (also called "kinematic weights") that are made from Lorentz-invariant contractions of the external momenta and polarizations / fermion wavefunctions.

This way of writing the amplitude is highly non-unique. BCJ proposed [3] that if numerator factors can be found which obey the same Jacobi-like identities as the color-factors<sup>5</sup>, e.g.

$$c_{12,34} + c_{13,42} + c_{14,23} = 0 \quad \iff \quad n_{12,34} + n_{13,42} + n_{14,23} = 0, \quad (2.3)$$

where, e.g.,



<sup>&</sup>lt;sup>5</sup>It is worth pointing out that Jacobi satisfying numerators were recognized at four-point tree-level for pure Yang-Mills in the context of motivating certain radiation zeros [36, 37]. See e.g. refs. [38, 39] for recent developments.

then by replacing the color factors by these numerator factors, one finds the tree amplitude of another local field theory:

$$M_n^{\mathrm{L}\otimes\mathrm{R}} = \sum_I \frac{n_I^{\mathrm{L}} n_I^{\mathrm{R}}}{\prod_{j \in I} P_j^2} \,. \tag{2.5}$$

The existence of numerators  $n_I$  subject to kinematic Jacobi identities as in (2.3) is the key prediction of the *color-kinematics duality*.

Note that if instead the numerator factors  $n_I^{\rm L}$  had been replaced by another set of color factors  $c_I^{\rm R}$  in (2.2), one obtains the tree amplitudes of the cubic bi-adjoint model BAS discussed above and in Figure 1.

• CHY double copy: The scattering equation approach to scattering amplitudes by Cachazo, He, and Yuan [5, 6] gives rise to yet another formulation of the double copy. In this form, the color-stripped L and R amplitudes are formulated in terms of integrals over n auxiliary variables  $\sigma_i$  (punctures on the Riemann sphere) as

$$A_n^{\rm L}[12\dots n] = \int \mathrm{d}\mu_n(k,\sigma) \, \frac{\mathcal{I}^{\rm L}(k,\epsilon,\sigma)}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} \tag{2.6}$$

and similarly for  $A^{\mathrm{R}}$ . Here  $d\mu_n(k,\sigma)$  is an integration measure that localizes the integral over  $\sigma_i$  on the solutions to the scattering equations.<sup>6</sup> The function  $\mathcal{I}^{\mathrm{L}}(k,\epsilon,\sigma)$  encodes the on-shell external momenta  $k_i$ , the polarizations and fermion wavefunctions collectively denoted by  $\epsilon_i$ , and the  $\sigma_i$ . The product of  $\sigma_{ij} = \sigma_i - \sigma_j$  is similar to the denominator of the Parke-Taylor expression for the *n*-gluon tree amplitude. (See [6, 41] for examples of the polarization-dependent part  $\mathcal{I}^{\mathrm{L}}(k,\epsilon,\sigma)$  of the integrand.)

The CHY statement of the double copy is then

$$M_n^{\mathrm{L}\otimes\mathrm{R}} = \int \mathrm{d}\mu_n(k,\sigma) \,\mathcal{I}^{\mathrm{L}}(k,\epsilon,\sigma) \mathcal{I}^{\mathrm{R}}(k,\epsilon,\sigma) \,. \tag{2.7}$$

The soft behavior of amplitudes is more manifest in the CHY formulation. A consequence is also the now widespread recognition<sup>7</sup> that the double copy is not

<sup>&</sup>lt;sup>6</sup>The scattering equations firstly seen in [40] read  $\sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \quad \forall i = 1, 2, ..., n$  in the momentum phase space of massless *n*-point amplitudes.

<sup>&</sup>lt;sup>7</sup>Ref. [42] was the first to recognize that NLSM tree-level amplitudes satisfy the duality between color and kinematics and indeed represented a double copy. Subsequently ref. [41] clarified the NLSM's double-copy relationship with DBI and special Galileon theories. Different approaches to explicit NLSM numerators subject to kinematic Jacobi relations can for instance be found in ref. [43–45].

limited to Yang-Mills theory and gravity, but connects a large variety of field theories such as illustrated in Figure 1.

These formulations of the double copy are equivalent at tree-level. We have presented the three versions of double-copy constructions with just enough detail to facilitate the discussion of these formulations in later sections. An important point about all three formulations is that they either rely upon, or predict, additional relationships between gauge invariant color-ordered amplitudes in theories that can participate in standard adjoint-type double-copy constructions, as we now discuss.

Consider tree-level *n*-particle gluon scattering. Factoring out color factors in a trace basis leaves (n-1)! possible arrangements of the external states due to the cyclicity of color-traces. One could, instead, keep the color-weights in adjoint  $f^{abc}$  form but express them in terms of a minimal color basis due to anti-symmetry and Jacobi relations. This is less than in the trace basis because color-traces span inclusion of the symmetric  $d^{abc}$ color weights as well which are irrelevant to gluons at tree-level in pure Yang-Mills. Looking at the coefficients of these basis color-weights yields an (n-2)! element basis amongst color-ordered tree-amplitudes. Reduction from (n-1)! ordered amplitudes to (n-2)! ordered amplitudes yields the Kleiss-Kuijf relations [46, 47] which include reflection relations such as  $A_n[123...n] = (-1)^n A_n[n...321]$ .

The existence of the double copy suggests a further reduction. Why? Consider the KLT formulation (2.1). In the sum, one gets to make a choice of (n-3)! colororderings out of the in principle n! possible color-orderings. The result of the double copy cannot depend on this choice, so there have to be relations among the L and R amplitudes that ensure this "basis-independence". These relations are, in addition to the the Kleiss-Kuijf relations, an additional set of relations known as "BCJ relations" [3]. In the BCJ formulation, these relations arise from the color-kinematics duality (2.3). Complementary derivations have been based on string-theory methods [48–50], on-shell recursion relations [51, 52] or BRST cohomology techniques [53]. The punchline is that the BCJ relations reduce the number of independent single-trace amplitudes from (n-2)! to (n-3)!.

As a result of the combined cyclicity, Kleiss-Kuijf, and BCJ relations, rather than having 6! = 720 independent color-arrangements to handle for the 6-point gauge theory amplitudes, there are only 3! = 6. Similar reductions in complexity occur in other more phenomenologically relevant contexts in applications both at tree- and loop-level.

#### 2.2 Loop-Level

A major motivation for the study of the double copy stems from loop-level applications. In each of the three formulations of the double copy in Section 2.1, finding the optimal generalizations from tree-level to loop-level amplitudes is on-going research.

So far, the BCJ formulation has led to the most impressive tests and applications at the multiloop level<sup>8</sup>. The basic idea is to promote the trivalent-graph expansion of treelevel amplitudes (2.2) and (2.5) to the integrands with respect to the loop momenta  $\ell_j$ . Once the numerators  $n_I^{\rm L}$ ,  $n_I^{\rm R}$  in a loop-integrand version of (2.2) obey kinematic Jacobi identities at fixed  $\ell_j$ , then the double-copy formula (2.5) is claimed to yield the loop integrand of the L  $\otimes$  R theory.

It follows from unitarity considerations that the BCJ double-copy construction is going to hold for loop integrands to all orders if color-dual representations can be found. While there is evidence (at least in particular limits) that double-copy relationships hold to all orders between Yang-Mills and gravity [56, 57], it is a central open question whether BCJ numerators exist for all loop orders in theories that satisfy the duality between color and kinematics at tree-level. For recent attempts to prove all order relationships see refs. [58, 59], although there is some evidence that if there are global color-dual numerators they may have to be non-local, at least in external momenta, or may only satisfy color-dual equations cut by cut [60].

Color-dual integrands have been found through four loops in the maximally supersymmetric theory [4, 16], and have shaped the state-of-the-art in multiloop computations in  $\mathcal{N} = 4,5$  supergravity [21, 24, 61]. The five-loop amplitude in  $\mathcal{N} = 8$  supergravity was constructed [62] through a generalization of the BCJ double copy beyond kinematic Jacobi identities and trivalent graphs [63], see Section 3.8 for the far-reaching implications on the UV properties.

For the CHY double copy, the quest for loop-level generalizations is guided by the underlying ambitwistor string theories [64–68] (see [69] for a review). Similar to the integral formulae (2.6) and (2.7) at tree level, the ambitwistor-prescription for loop amplitudes derives loop integrands from integrating over auxiliary variables  $\sigma_i$ on a (multi-)nodal sphere [70, 71]. The loop-level generalizations of the polarizationdependent parts  $\mathcal{I}^{\mathrm{L}}\mathcal{I}^{\mathrm{R}}$  in (2.7) again take a factorized form as familiar from chiral splitting for conventional strings [72, 73].

At tree level [32, 74–76] and one loop [77, 78], the connection between the CHY and BCJ double copies is well understood at all multiplicities, also in the absence of supersymmetry [79]. Also at two loops, the BCJ double copy has been related to the ambitwistor-framework [80–82], and one of the future goals is to simplify the two-loop instances of the function  $\mathcal{I}^{L}$  to make the polarization dependence fully explicit at all multiplicities. At three loops, the BCJ double copy was used to propose expressions for the kinematic building blocks in both ambitwistor and conventional string theories

<sup>&</sup>lt;sup>8</sup>See refs. [54, 55] for recent reviews

[83], see Section 3.5 for further details.

At the time of writing, the multiloop systematics of the KLT double copy is least explored. The one-loop KLT formula [84] for field-theory amplitudes derived from ambitwistor methods [77] is tied to linearized variants of the Feynman propagators and calls for a reformulation in terms of traditional quadratic propagators.<sup>9</sup> Loop-level KLT formulae for closed-string amplitudes are uncharted terrain, though monodromy relations among open-string loop amplitudes and closely related methods from twisted de Rham theory [91–95] may shed light on their existence or construction.

#### 2.3 Kinematic Variables

In order to advance our understanding of the double copy formulations in Section 2.1, it is essential to have explicit testing grounds at different loop and leg orders. This calls for compact representations of the polarization-dependent building blocks such as the color-stripped amplitudes  $A_n^{\rm L}[a]$  in the KLT formula (2.1), the kinematic numerators  $n_I^{\rm L}$  in the BCJ double copy (2.5) and the integrands  $\mathcal{I}^{\rm L}(k, \epsilon, \sigma)$  in the CHY formula (2.7). With a growing number of loops and legs, the compactness of explicit  $A_n^{\rm L}[a]$ ,  $n_I^{\rm L}$  or  $\mathcal{I}^{\rm L}(k, \epsilon, \sigma)$  crucially depends on the kinematic variables parametrizing the external polarizations. Suitable choices of kinematic variables have repeatedly been driving forces for structural progress – in both the amplitudes program in general and in implementations of the double copy in particular.

As a shining example from the 80s, four-dimensional spinor-helicity variables for massless states of various spins [96–100] paved the way towards the Parke-Taylor formula, a one-liner expression for maximally helicity violating *n*-gluon tree amplitudes [101]. Generalizations of spinor-helicity variables to massive external legs [102–107] and to different spacetime dimensions [108–111] are still under active development and continue to facilitate both the construction and compact representation of amplitudebuilding blocks in a vast bandwidth of theories. In planar  $\mathcal{N} = 4$  sYM, momentum twistors [112] provided an unconstrained way to specify on-shell kinematics while manifesting dual superconformal invariance (cf. for instance [113–119]).

Spinor helicity and twistor variables have a rich interaction with the CHY formulae. One way in which this arises is through the ambitwistor strings that underpin these formulae. Ambitwistor space is fundamentally a dimension-agnostic concept, but for fixed spacetime dimension, one can find spinorial representations of the ambitwistor geometry. Formulating ambitwistor strings in these dimension-specific representations leads to notions of 'polarized' scattering equations [120–125], where the usual set of

<sup>&</sup>lt;sup>9</sup>The conversion between linearized and quadratic propagators has been discussed from a variety of perspectives [85–90].

scattering equations is graded by the polarization data. These polarized scattering equations can lead to amplitude formulae with dramatically fewer explicit moduli integrals (at least in certain polarization configurations, like MHV in 4-dimensions), and have links to formulae for massive scattering [126].

Since the color-kinematics duality and the double copy are universally observed in any number of spacetime dimensions, it is important to construct kinematic numerators subject to Jacobi identities in terms of dimension-agnostic polarization vectors of gauge bosons. This has been a key motivation in the construction of the CHY formulae, and recent studies of multiparticle amplitudes in (ambitwistor and conventional) string theories at tree and loop level led to striking compactifications of their polarization dependence in arbitrary dimensions.

In maximally supersymmetric settings, pure-spinor superspace [127, 128] led to breakthroughs in encoding the dependence of multileg amplitudes on  $D \leq 10$ -dimensional gauge- and supergravity-multiplet polarizations (see [129] for an automated extraction of bosonic components). Together with recursive Berends-Giele techniques [130] and perturbiner methods [131–133], pure-spinor superspace turned out to be crucial both for the simultaneous manifestations of locality and color-kinematics duality beyond four points [53, 134–136] and the construction of multileg and multiloop string amplitudes [137–139].

From these developments, we expect that structural discoveries of new symmetries and relations between physical theories will keep on going hand in hand with the quest for convenient variables to manifest different simplifying features of scattering amplitudes. And at the very least, innovative choices of variables can change the load of analytic data by numerous orders of magnitude in advancing the multiloop and multileg frontiers of gauge theories, gravity or string theories.

## 3 Applications

## 3.1 From Coulomb Charges to Schwarzschild Black Holes and Generalizations

The perturbative construction of solutions to classical field equations is closely related to the construction of semi-classical tree-level scattering amplitudes [54, 131–133, 140–145]. It is perhaps unsurprising then that the double-copy structure of scattering amplitudes is reflected in properties of classical solutions, albeit in a gauge dependent<sup>10</sup> manner.

 $<sup>^{10}\</sup>mathrm{Much}$  like how manifesting the duality between color and kinematics requires a particular generalized choice.

Many classical metric solutions admit Kerr-Schild coordinates,

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi \, k_\mu \, \tilde{k}_\nu \,, \tag{3.1}$$

which, due to properties of  $\{\varphi(x), k(x), k(x)\}$ , have the remarkable effect of linearizing Einstein's equations. Such gravitational solutions therefore have a natural double-copy interpretation in terms of linearized solutions to gauge theory equations of motion [146– 148]. A particularly striking example of a classical double-copy construction is that the gauge field of a Coulomb charge builds the metric of the Schwarzschild black hole [146]. The *exact* Schwarzschild metric in Kerr-Schild coordinates is

$$\mathrm{d}s^2 = \left(\eta_{\mu\nu} + \frac{2GM}{r}k_{\mu}k_{\nu}\right)\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}\,,\tag{3.2}$$

where  $k_{\mu} dx^{\mu} = dt - dr$ . The gauge theory "single-copy" is

$$A^a_\mu = \frac{Q}{r} k_\mu c^a \,. \tag{3.3}$$

This is nothing but the familiar gauge field of an abelianized point charge written in a particular gauge [146], which can be sourced e.g. by massive static charges. Any non-abelian structure in the YM case is irrelevant in the static limit. The exact departure from the flat space metric, encoded by the Schwarzschild solution, therefore arises from the single one-propagator diagram that generates the Coulomb field, but where the abelianized color factor  $c^a$  is replaced by another copy of the "kinematic" object  $k_{\mu}$ .

Obtaining the Schwarzschild metric required the Coulomb field to be written in a particular gauge; this can be avoided by exploiting a lesson from amplitudes and focusing on gauge-invariant observables. Motivated by gravitational-wave physics, a formalism for computing classical observables from amplitudes has been developed [149]. (We will discuss this formalism in more detail in the next section.) The field strength  $F_{\mu\nu}(x)$  is an example of such an observable [150]; it is a gauge-invariant quantity in electrodynamics. Turning to gravity, we can instead compute the linearized Riemann curvature, which is also gauge-invariant (in the linear theory). In the static case, the field strength and the linearized curvature are [151] on-shell Fourier integrals of three-point amplitudes and therefore they can be written manifestly as double copies of one another.

An alternative perturbative approach to this double copy is the following. Suppose we take our massive sources to be very heavy and probe them by scattering very light objects off them, gathering information in analogy to Rutherford scattering. These probes will travel on geodesics (in the gravitational case) or follow the solution of the Lorentz-force equation of motion (in electrodynamics). The scattering angle the probes experience can be computed from amplitudes; therefore the double copy of amplitudes is inherited by the angles. To the extent that the scattering angle gives us information about the fields, we again see that there must be a relation between charges in gauge theory and black holes in gravity [152, 153]. Remarkably, these statements generalize from Coulomb charges to magnetically-charged dyons; the double copy is a gravitational solution with both mass and NUT charge [147, 152]. It is also possible to include classical spin [146, 153, 154], making contact with the Kerr black hole (which is the double copy of a spinning disk of charge). More general exact classical double copies are available in gravitational theories with dilaton and axion matter [155, 156], as well as for curved backgrounds like (A)dS using generalized Kerr-Schild metrics [157, 158].

A convenient presentation of the curvature of the Schwarzschild solution which also reveals a path to generalizing its double-copy properties to other space-times is the Weyl spinor, which is simply a spinorial formulation of the Weyl curvature tensor. The Weyl curvature is essentially the Riemann curvature minus its traces, which are the Ricci tensor and Ricci scalar. Both traces vanish in vacuum solutions such as Schwarzschild. Therefore the Weyl spinor encodes the complete information about its curvature. One advantage of the spinorial curvature is that it is a scalar under coordinate transformations, though it nevertheless transforms under local change of the spinor basis. The surprise is that there is a basis in which the Weyl spinor is precisely linear in the mass of the black hole. In this basis, then, the Weyl spinor equals its linear approximation, which is nothing but an integral of the three-point amplitude. This is suggestive of a non-perturbative double copy.

The Weyl double copy [159] is a non-perturbative relation between the Weyl spinor of a wide class of spacetimes and the corresponding spinorial curvature, known as the Maxwell spinor, in electrodynamics. Writing the Weyl spinor in terms of spinorial indices as  $\Psi_{\alpha\beta\gamma\delta}$ , and the Maxwell spinor as  $\Phi_{\alpha\beta}$ , the Weyl double copy is the relationship

$$\Psi_{\alpha\beta\gamma\delta} = \frac{1}{S} \Phi_{(\alpha\beta} \Phi_{\gamma\delta)} \,. \tag{3.4}$$

In this four-dimensional equation, the object S is a scalar function, and the parentheses indicate symmetrisation over spinorial indices. The relationship is known to hold for large classes of Petrov type D and type N solutions [159, 160], as well as asymptotically in algebraically general cases [161]. We have also seen above that the Weyl double copy follows from the double copy of amplitudes, at least at low perturbative orders. The conjecture is that the Weyl double copy *is* the double copy of scattering amplitudes non-perturbatively in four dimensions. A similar relationship to (3.4) holds [162, 163] in three-dimensions in topologically massive theories between the Cotton spinor and the dual field strength spinor, and indeed can be derived via a dimensional reduction of the Weyl double copy under certain circumstances [163]. The symmetries of gravity have a double copy interpretation. In fact, symmetry generators in linearized gravitational theories have been explicitly reconstructed from gauge-theory ingredients. This is based on interpreting the double copy as a convolution of position-space gauge theory fields, both physical and ghosts [164–172].Since in Kerr-Schild coordinates Einstein's equations linearize, the convolutional double copy should reproduce the Schwarzchild and other solution with such a description. It can also be used to construct more general linearized solutions of Einstein's equations. For example, ref. [173] used it to construct the linearization of the Janis-Newman-Winicour solution, which apart form the metric also contains a nontrivial dilaton.

Classical aspects of the double copy have received intense attention in the literature in recent years [173–191], including non-abelian classical solutions [174, 192] and quantum double-copy predictions against a classically double-copied background [193, 194]. They offer the prospect of a more non-perturbative understanding of the double copy, and provide a glimpse of the double copy at work far from its original Minkowskian home. For example, there is no problem formulating a classical Kerr-Schild double copy for the Coulomb charge in AdS: the result is simply the AdS-Schwarzschild solution [147]. There are rich connections to other topics, including twistor theory [190, 195– 200], dualities [152, 201–205], three-dimensional physics [162, 163, 206–209] and — as we will now discuss — classical gravitational wave physics.

#### 3.2 Precision Gravitational Wave Science

Next generation gravitational-wave observatories will surpass the incredible precision of the LIGO and Virgo detectors and demand commensurately-accurate theoretical predictions for the waveform templates utilized in the detection and extraction of source parameters. The required calculations, requiring contributions up to seventh order in Newton's constant [210, 211], rival in complexity the exploration of high energy properties of supergravity theories [20, 21, 24, 25, 61]. In conjunction with unitarity methods, the double copy has been key to deriving new state-of-the-art results of interest<sup>11</sup> to the gravitational wave community. These methods naturally complement the application of Effective Field Theory approaches to gravitational wave physics, already a mature<sup>12</sup> and thriving field.

<sup>&</sup>lt;sup>11</sup>See the dedicated Snowmass White Paper [15] for a broad discussion of QFT scattering approaches applied to gravitational wave science.

 $<sup>^{12}</sup>$ See, e.g., refs. [212, 213] for recent reviews.

The double copy and classical gravitational waves were first linked at the level of the classical double copy [148]. The situation involved a particularly simple gravitational Bremsstrahlung process, which could be understood using Kerr-Schild coordinates. An important breakthrough was made by Goldberger and Ridgway [175], who found a direct form of the double copy in the leading-order gravitational radiation generated by scattering two masses. This work was performed in a strictly classical context, involving an iterative solution of the classical equations of motion. This double copy, intuited by Goldberger and Ridgway, was connected to scattering amplitudes, specifically at five points [214]. Advancing the Goldberger-Ridgway method to next-to-leading order required understanding how to disentangle classical numerator factors (which should be double-copied) from kinematic propagators (which should not). This was achieved by Shen through consideration of classical bi-adjoint propagation [215]. This tour-deforce result demonstrated the first appearance of the kinematic Jacobi identities beyond field-theory scattering. The same formalism has provided evidence of the double copy in the radiation emitted by bound systems [216].

The connection between scattering amplitudes and observables was developed into an all-orders "KMOC" formalism [149]. This formalism is designed to enable the extraction of observables, which are well-defined in both classical and quantum theory, from amplitudes. An initial state  $|\psi\rangle$  is chosen which is classically meaningful (involving, for example, point-like particles with positions and momenta which have uncertainty negligible compared to other scales relevant to the process). Scattering amplitudes arise through time evolution to the final state  $S|\psi\rangle$ . Classical observables are obtained from expectation values of quantum operators in the future. Since all the dynamics is captured by amplitudes, the double copy and unitarity methods are guaranteed to be available to all orders. Spinning particles can be included in this method [217], and the gravitational waveform itself can be extracted from the expectation value of the Riemann tensor [150]. The double copy is the main source for the gravitational amplitudes required in this formalism.

Complementary recent worldline quantum field theory [218] approaches have produced state-ot-the-art results for the Bremstrahlung waveform [219] including spin [220] (see also [221]), as well as spinning  $G^3$  observables in the scattering scenario [222, 223]. In fact, the underlining double copy structure of the worldline approach has been investigated in [224] connecting to the classical equation of motion solutions of [214, 215].

Precision observables relevant to gravitational-wave physics have been found by direct applying double copy methods in [13, 224–236]. Indeed, the determination of the highest precision post-Minkowskian  $\mathcal{O}(G^4)$  corrections to the scattering of classical non-rotating black holes involved a synthesis of effective field theory techniques, advanced multiloop integration, and the double copy applied to tree-level scattering amplitudes [13, 14]. This calculation determines the classically-relevant part of a three-loop four-point scattering amplitude between two massive scalars. The method exploited the double copy on the cuts, building the gravitational integrand using unitarity methods from double-copied gravitational trees. A similar strategy was used in [237] to obtain Compton amplitudes for arbitrary-spin particles and the  $\mathcal{O}(G^2S_1S_2)$  correction to the leading order amplitude for the scattering of two Kerr black holes [238].

In massive scalar QCD, at one loop, it is possible to find a color-dual expression for a classically-relevant five-point amplitude. The relevant amplitude involves two massive particle scattering while emitting gluonic radiation [239]. The existence of a color-dual expression opens a direct path to gravitational predictions [240] without having to build up the calculation from gravitational trees, potentially offering greatly improved scalability at higher loops. This example is the first loop-level result involving external massive matter and manifesting the double copy at the level of the integrand. It has direct relevance to observables using, for example, the KMOC formalism.

There is by now also an increasing number of classical double-copy constructions, see e.g. refs. [152, 153, 158, 159, 202, 241–247] which describe the emission of gravitational waves from various systems and in some instances can be interpreted from the perspective of scattering amplitudes.

The program to apply scattering amplitude methods to gravitational wave physics has produced results sought after by the gravitational wave community [248], uncovered new structures and developed new tools that will enable future progress towards addressing the precision needs of future facilities, such as Einstein Telescope, the Cosmic Explorer, and LISA. The double copy has been an integral part in this young and dynamic field, and will be instrumental in meeting remaining challenges, such as the higher-order calculations for binaries of generic mass ratio.

Of particular importance are extreme mass-ratio inspirals, which can be observed over long times, expected to complete  $10^4 - 10^5$  cycles in band for LISA [249], and may appear in different frequency windows in different detectors. With waveforms of sufficient precision, this much data could be used [250], for example: (1) Precision tests [251] of the no-hair theorem<sup>13</sup> of general relativity. (2) Measurement of the masses of both black holes to better than 10% precision [253]. (3) Reconstruction [254] of the mass distribution of massive black holes out to redshift  $z \gtrsim 4$ . (4) Precision measurement of the Hubble constant [255]. Current methods adapted for such systems, known as the self-force approach [256–258], involve determining the motion of the lighter body in the exact classical field of the heavier body. Future observatories will

<sup>&</sup>lt;sup>13</sup>For an example of amplitude techniques applied to derive classical observables related to binary systems involving helicity-0 modes see ref. [252].

be sensitive to contributions to the dynamics which are of second-order in self-force, i.e.  $\mathcal{O}(m_{\text{light}}^2/m_{\text{heavy}}^2)$ . Such contributions have not yet been completely calculated. Approaching this problem with scattering methods motivates us to study the double copy in the context of scattering amplitudes in background fields. Further motivation comes from describing the last phase of a binary's evolution – the ringdown – through double-copy methods, as it may open another window on the binary's geometry prior to merger, cf. e.g. [259, 260].

#### 3.3 Double Copy in Nontrivial Backgrounds

It is usually an unspoken assumption that the computation of scattering amplitudes is performed perturbatively around a trivial field configuration: one ordinarily considers gluon scattering in vacuum, or gravitational scattering in Minkowski spacetime. This assumption of a trivial background is sufficient for many scenarios of physical interest, but there are myriad experimental, phenomenological and theoretical contexts for which non-trivial backgrounds are better motivated. These range from the Schwinger effect in QED (see [261–263]) to QCD in strong magnetic fields (see [264]) to heavy ion and high-energy hadron collisions in QCD (see [265–267]) to neutron star atmospheres in astrophysics [268] to gravitational wave memory [269] and cosmology [270] in GR.

In such scenarios, the background fields are treated as *exact*, *non-linear*, *non-perturbative* solutions to the classical equations of motion, sometimes called "strong" backgrounds. (It is often a reasonable approximation to neglect backreaction effects on the background, or to deal with them perturbatively.) The traditional framework for this in QFT is the background field formalism (cf., [271–274]). This is a rich playground, where perturbative and non-perturbative physics meet in many interesting ways.

While the background field formalism is as old as QFT itself, performing explicit computations in even the simplest strong backgrounds is substantially more involved than in trivial backgrounds. The absence of translation invariance requires the use of a position-space formulation of Feynman rules, leading to tree-level amplitudes and loop-integrands which are no longer rational functions of kinematic data. For instance, even in highly symmetric gravitational backgrounds like (anti-)de Sitter space, rational functions are replaced by complicated transcendental functions.

This is reflected by the precision frontier for scattering in gauge theory and gravity with strong background fields. For instance, in constant-curvature symmetric spacetimes the state-of-the-art for gluon scattering<sup>14</sup> is only 4-points [275–281] or 5points [282, 283] at tree-level, while for tree-level graviton scattering it is four points

 $<sup>^{14}\</sup>mathrm{We}$  abuse terminology by referring to such objects as "scattering amplitudes"; in asymptotically (A)dS spacetimes, the S-matrix is replaced by boundary correlation functions or coefficients in the wavefunction of the universe, which are the analogous observables.

in  $AdS_5$  [284–286] and four points in  $AdS_4$  [287]. In AdS-momentum space algorithms have been developed for *n*-point scalar [288], gluon [289] graviton [290] tree-level correlators, and at loop level [291]. Complete explicit expressions beyond five-points are yet to be found, however. Certain AdS gauge theory calculations rely on supersymmetry to relate gluon amplitudes to amplitudes of scalars in the same multiplet, which allows the use of powerful tools such as Mellin space. By contrast, in Minkowski space the tree-level S-matrix of gluons or gravitons is known to *arbitrary* multiplicity [2, 6, 292–294]<sup>15</sup>. Likewise, recent progress for direct loop-level calculations in AdS (cf., [298–307]) is largely restricted to 4-points<sup>16</sup>. For calculations of direct experimental relevance, the situation is similar. The tree-level precision frontier is essentially at 4-points in strong-field QED (a subject that is over 75 years old) for on-going and future high-energy laser and electron/positron beam experiments, such as E320 at FACET-II, ELI, CoReLS, and LUXE (e.g., trident pair production or double Compton scattering, see [261, 263, 310, 311]).

Going beyond these low-multiplicity results is an opportunity to develop novel tools adapted to position-space problems and to not rely on special symmetry properties. It is also an opportunity to revisit the validity of perturbation theory in background fields and the need for resummation beyond that for trivial backgrounds. Highermultiplicity in strong backgrounds is also of practical importance: while higher-point amplitudes come with additional factors of the "small" coupling constant, they are also accompanied by additional background-dressed propagators (cf., [261, 263, 312, 313])<sup>17</sup>. These propagators contain powers of the background itself, which can compensate for the additional coupling factors. The net effect is that low-multiplicity processes do not necessarily dominate high-multiplicity ones. Furthermore, high-multiplicity scattering in curved backgrounds is related to the emission of radiation; this can be seen in terms of classical observables (e.g. [314–316]), and for certain backgrounds is linked with resummations in high-energy scattering on a flat background (e.g. [317–323]).

With such a large knowledge gap compared to Minkowski space, it is clear that double copy can be a powerful tool for the study of gravitational scattering in curved spacetimes, but there are several important challenges which must be overcome. Firstly, does a notion of the double copy even exist for scattering amplitudes in background fields, and, if so, what is its general structure? This is not just a practical question: if dou-

<sup>&</sup>lt;sup>15</sup>We should note that there are all-multiplicity formulae for "maximal U(1)-violating" correlation functions in  $\mathcal{N} = 4$  sYM [295–297], which can be interpreted as certain amplitudes of type IIB supergravity in AdS<sub>5</sub> × S<sup>5</sup>.

<sup>&</sup>lt;sup>16</sup>Features of four-point correlation functions to all loop orders have also been discussed, with input from the dual gauge theory (e.g. [308, 309]).

<sup>&</sup>lt;sup>17</sup>We would like to thank Anton Ilderton for emphasizing this point to us.

ble copy is really an intrinsic property of perturbative gravity, it should hold on any perturbative background (not just Minkowski space).

One then requires novel methods to generate suitable gauge theoretic "data" to seed the double copy. In background fields, it is a challenge to calculate amplitudes even at tree-level, much less arrange them into whatever the strong-field version of a color-kinematics representation might be. Clearly, novel techniques (i.e., beyond the background field expansion on space-time) are required. Finally, on both sides of any curved double copy map, new tools are needed to check and constrain results. Most modern unitarity methods break down in the presence of background fields, meaning that the usual ways of verifying whether a formula for some scattering amplitude is correct are no longer available.

While these are substantial challenges, exciting progress in recent years makes it clear that they are not insurmountable.

**Double copy in curved spacetimes:** As it stands, there is no definitive prescription for what are the building blocks that lead to gravitational scattering amplitudes in curved spacetime. Do we start with gluon amplitudes (in a color-kinematics representation) in the *same* spacetime? Or should we instead start with gluon amplitudes in flat space but with non-trivial *gauge* background fields? If so, then which gauge background do we select to obtain graviton scattering in the desired curved spacetime<sup>18</sup>?

In the first case, a substantial amount of work has focused on gravitational "scattering" (i.e., boundary correlation functions) in AdS, where CFT methods on the boundary can be brought to bear. Here, the proposed double copy map is constructed by "squaring" gluon amplitudes in the *same* AdS space-time, based on observations of a double copy structure between the gluon and gravity boundary correlators in AdS momentum space [324–327] and Mellin space [328, 329]. Additionally, there has been progress in realizing that gluon, NLSM and biadjoint scalar scattering in AdS manifests colorkinematics duality, with amplitudes obeying the BCJ relations [279–281, 330–332]. These observations build on a variety of methods, ranging from Mellin representations of the AdS amplitudes [333, 334] to geometric formulations of the kinematics in terms of isometry generators [145, 330, 331]. A key test of these methods in the coming years will be whether or not they can be used to compute gravitational scattering amplitudes in AdS beyond the state-of-the-art set with AdS/CFT techniques.

Arguments that color-kinematics duality should persist in any curved spacetime have been put forth in [145, 335], and it was shown in [330] that it does hold for NLSMs and the biadjoint scalar theory on symmetric spaces. Some explicit calculations *have* 

 $<sup>^{18}</sup>$ Note that this question is distinct from, but related to, the question of *classical* double copy for exact solutions discussed in Section 3.1

been possible for plane wave spacetimes [193, 336]; these admit an S-matrix [337], extend to solutions of string theory [338–341] and can be viewed as local approximations to any spacetime [342]. Here, the proposed double copy map does *not* involve gluon scattering on the same plane wave spacetime, but rather gluon scattering in a gauge theoretic plane wave background *in Minkowski space*. This gluonic plane wave scattering has been shown to obey a strong-background version of the color-kinematics duality [194], and is of interest in its own right for various topics in strong-field QED/QCD (see [263, 322, 343]).

A definitive verdict on which sort of prescription is correct (indeed, it may be both – or neither) will rely on their ability to extend calculations beyond the reach of standard background perturbation theory methods (including Witten diagrams). Recently discovered all-multiplicity formulae for gluon and graviton scattering in chiral/self-dual gauge-theoretic and gravitational plane waves [344–346] should provide an important data set on which proposals can be tested in the future.

Generating gauge theory data: To improve the precision frontier of perturbative gravity in curved spacetime one must have gauge theoretic scattering data to feed into the double copy map. This means improving on the state-of-the-art for gluon scattering in strong background gauge fields as well as curved space-times. One promising toolkit for doing this is ambitwistor string theory [64–66], which underpins the CHY version of double copy [6, 32, 41]. Ambitwistor strings can be exactly coupled to non-trivial gauge and gravitational backgrounds [347–349], and have the remarkable property that the worldsheet CFT remains solvable (i.e., local anomalies and worldsheet OPEs can be computed exactly). Ambitwistor strings can compute 3-point scattering amplitudes of gluons and gravitons in plane wave backgrounds [350], n-point gluon/graviton scattering in AdS<sub>3</sub> [351], and n-point scalar scattering in (A)dS spacetimes of arbitrary dimension [304, 352, 353]. Developing a formalism to compute the worldsheet correlators of ambitwistor strings in generic backgrounds and target space dimensions would provide a mechanism to generate large amounts of data on both sides of any double copy correspondence for curved backgrounds.

Another promising avenue is provided by twistor theory. This trivializes the selfdual sectors of gauge theory and gravity in four-dimensions [354, 355], making the computation of scattering amplitudes on self-dual backgrounds tractable. For instance, twistor theory provides formulae for the complete tree-level S-matrix of Yang-Mills theory on any self-dual radiative gauge field background [345], and there is scope to describe gauge and gravitational scattering on *any* self-dual background, including dyons, instantons and self-dual black holes. How to study non-chiral background fields with twistor theory remains an important open question. Finally, there has been progress in the use of worldline methods for strong background fields (cf., [356, 357]). An impressive example is a generating functional for n-point, one-loop photon scattering amplitudes in electromagnetic plane wave backgrounds [358]. Further developments in this area (including generalizing the formalism to non-abelian gauge theories) could provide a rich source of data to seed any doublecopy map for curved spacetime.

Unitarity methods in background fields: While some on-shell methods, like spinor-helicity variables in (A)dS [359–361] and beyond [336, 344, 346, 362], still work in curved spacetimes, the familiar toolkit of generalized unitarity fails as soon as generic background fields are introduced, since tree-amplitudes and loop-integrands are no longer rational functions. To make progress with double copy in curved spacetimes, new perspectives on constraining candidate amplitude formulae are required. In the context of (A)dS scattering, versions of BCFW recursion, "cutting" rules and generalized unitarity have been developed [363–382]; while promising, these have yet to deliver results for gluon or graviton scattering that surpass the state-of-the-art from standard perturbation theory or boundary CFT methods. However, it is hopefully only a matter of time before these techniques become as powerful as their flat space counterparts.

More general backgrounds are much less studied. One tantalizing avenue was recently explored in the context of plane wave backgrounds in QED, where it was shown that the simple requirement of gauge invariance imposes a factorization-like decomposition of strong field amplitudes [383]. While the study of such structures is still in its infancy (and yet to be generalized to non-abelian gauge theories), it shows the potential for factorization-based arguments to be lifted to background field settings. A similarly enticing idea would be to use worldsheet descriptions of curved space-time scattering (based on strings or ambitwistor strings) to study factorization properties. Here, one would simply focus on the cutting properties of the underlying worldsheet CFT [384–387]; on general grounds this should then imply a "worldsheet factorization" argument for the amplitudes themselves, regardless of background fields in space-time.

Even without unitarity-based tools, there are still basic consistency checks like the flat or perturbative limit (i.e., where the background is treated as a single gluon or graviton) which can be deployed. These are well-studied at tree-level and even at loops; for instance, multi-perturbative limits of loop-level amplitudes in strong field QED/QCD must match multi-collinear limits of higher-multiplicity loop amplitudes in trivial backgrounds [388]. Finally, studying flat spacetime scattering in alternatives to the usual momentum eigenstate basis could provide new insights into how to constrain curved spacetime scattering amplitudes. For example, in the conformal primary basis [389, 390] there is no longer momentum conservation and tree amplitudes are not rational functions of the kinematic data. Thus, the arena of celestial holography [391] may have important lessons to teach us about double copy in curved spacetimes [392–395].

#### 3.4 Double Copy and Cosmology

An intriguing application of the double copy consists of providing new insights and tools for cosmology. While some initial progress has been made [158, 279–281, 324–332, 335, 396], there are still many challenges to overcome. There are two primary directions that one can consider, namely early and late universe cosmology. In the following, we discuss how the double copy can contribute towards progress in both areas.

It is widely believed that the structure of the Universe that we observe today has its origin in quantum fluctuations that arose during an inflationary period (quasi-de Sitter phase). By looking at the spatial correlation functions of these fluctuations living in the future boundary of the inflationary spacetime, one can learn about the particles present in this early epoch and their dynamics, thus giving an opportunity to observe beyond the Standard Model physics. These fluctuations can be observed by looking at the statistics of the cosmic microwave background and large-scale structure. Currently, we only have measurements of the scalar 2-point function, but ongoing and future experiments will be able to provide measurements or tight constraints on tensor 2-point functions and non-gaussianities encoded in three-point functions [397–403].

While higher-point statistics will be hard to observe in the near future, one can learn valuable lessons regarding the structure of QFT in curved-spacetimes from them. The traditional computations of primordial correlation functions can become involved beyond the 2-point and tree-level cases<sup>19</sup> [270, 404, 405]. Therefore, a double copy relation between primordial correlators could give a new simpler method for performing calculations at loop order and for higher points in addition to providing insights on the symmetries of QFT in curved backgrounds.

As discussed also in Section 3.3, some progress in this direction<sup>20</sup> has been made in refs. [324–326, 329], but a better understanding of the double copy on curved spacetimes is still required. In refs. [279, 280, 324–327] it was shown that the residue of the total energy pole of certain tensor (A)dS correlators is given as the square of Yang-Mills amplitudes (as expected since this residue is given by the flat space gravitational

<sup>&</sup>lt;sup>19</sup>Note that a complementary approach dubbed the Cosmological Bootstrap approaches this problem from a different perspective using unitarity, locality, and symmetries; see the dedicated White Paper [382] for a thorough discussion on this topic.

<sup>&</sup>lt;sup>20</sup>Progress made in an AdS/CFT context [279–281, 327, 328, 331, 332] and maximally symmetric spacetimes [330, 335] can provide lessons for a dS double copy.

amplitude [359, 365]). These papers also give hints that the double copy relation can be extended beyond this special kinematic limit; thus, it would be interesting to understand how to build an explicit double copy and whether it can be solely expressed in terms of correlators.

One should also note that the most progress has been made for conformally flat spacetimes, such as de Sitter, but inflationary spacetimes break de Sitter symmetries. Fortunately, in the slow-roll regime the symmetries are softly broken and one can compute inflationary correlators from de Sitter ones by taking soft limits [368, 406–408]. To go beyond the slow-roll regime, one would need to be more ambitious and construct a double-copy relation in a background with broken time translations. In fact, this is an intricate challenge for constructing a double-copy relation that holds in more generic cosmological contexts.

Concerning late universe cosmology, we will focus on theories of dark energy, i.e., those that drive the accelerated expansion at late times. Whether these theories are within the web of theories related through double-copy relations is an open question. Some initial insights come from the appearance of the special galileon as a double copy. Galileon theories are higher-derivative theories with spacetime-dependent shiftsymmetries that can behave as dark energy [26]. Nevertheless, the special galileon corresponds to a branch that cannot drive accelerated expansion. Besides galileons, there are more general scalar and vector effective field theories that can describe dark energy and generically involve higher derivative operators, see for example refs. [409-411] and references therein. Explorations of effective field theories within the context of the double copy, see Section 3.6 and Section 3.7, can shed light on whether dark energy could be obtained as a double copy. A perturbative double copy could be useful in some restricted scenarios where the theory is weakly coupled. Nevertheless, most of these theories rely on screening mechanisms that render their effects negligible within the solar system to be consistent with observations [409, 412]. These screening mechanisms depend on classical non-linearities becoming strong, thus only a non-perturbative double copy could shed light on these regimes.

Another intriguing class of theories that could give rise to a late accelerated expansion are those that involve massive gravitons [413–416]. As an aside, we also note that massive spin-2 fields can be relevant in cosmological scenarios as possible dark matter states [417–419]. Well-behaved gravitational theories involving massive gravitons are reviewed in refs. [413–415], a new construction with a higher strong coupling scale can be found in [420], and an exactly solvable two-dimensional version, which is equivalent to  $T\bar{T}$  deformations, is analyzed in ref. [421]. Calculations in these theories are highly intricate due to the complicated tensor structures appearing in their interactions. Thus, the possibility of performing computations for massive gravity theories from simpler massive gauge theories is largely appealing. It has been shown that the standard double copy of massive gauge fields does not always correspond to a well-defined local gravitational theory [422, 423]. Some constraints have been formulated so that the double copy is a healthy theory, but these constraints have several assumptions on the spectrum, interactions, or spacetime dimensions of the theory [422–424]. Most of these cases consider formulations that resemble the massless case and thus lead to theories arising from simple patterns of spontaneous symmetry breaking of massless theories. The special three-dimensional case [162, 163, 208, 209, 424–428] deviates from this paradigm<sup>21</sup> and shows that there could be different methods for obtaining healthy double-copies that do not resemble the structure of the massless one. Extending this to four-dimensional theories would be an important breakthrough in describing massive gravity theories. Beyond the cosmological implications, theories of massive gravity could also be relevant for studying black hole physics in theories beyond general relativity [429–431].

#### 3.5 Interplay Between QFT and String Theory

String theories provide a natural framework to understand the double copy structure of gravitational interactions since closed-string degrees of freedom are built from tensor products of color-stripped open-string degrees of freedom. Gravitational states due to massless vibration modes of closed strings are therefore organized into double copies of gauge multiplets from open-string excitations. As will be detailed below, this double copy structure of the spectrum propagates to closed-string scattering amplitudes such that the point-particle limit<sup>22</sup>  $\alpha' \rightarrow 0$  relates gravitational interactions to bilinears in gauge-theory building blocks. Recent developments in string perturbation theory – see the White Paper [432] for an overview – led to concrete multi-loop and multi-leg manifestations of the gravitational double copy and the closely related color-kinematics duality of gauge theories.

By the KLT relations between closed- and open-string tree-level amplitudes [2], classical gravity predictions are known to reduce to gauge-theory input. At loop level in turn, chiral splitting [72, 73] expresses the integrand of closed-string amplitudes at fixed loop momenta as a square of chiral halves which individually integrate to open-string amplitudes upon specifying the boundary conditions for their endpoints. Accordingly, the double-copy construction of quantum-gravity interactions and the color-kinematics duality of gauge theories manifest themselves at the level of the loop integrand [4].

<sup>&</sup>lt;sup>21</sup>While the case of topologically massive theories is gauge invariant just as the massless theories, a healthy double copy can also be constructed from the square of massive Yang-Mills restricted to a special set of polarizations [424].

<sup>&</sup>lt;sup>22</sup>More precisely, the low-energy expansion of string amplitudes around the point-particle limit is performed in the dimensionless combinations  $\alpha' k_i \cdot k_j$  formed by the inverse string tension  $\alpha'$  and light-like external momenta  $k_i$ .

Manifestly gauge-invariant incarnations of the color-kinematics duality [3, 433] descend from monodromy relations among color-stripped open-string amplitudes, both at tree level [48–50] and in loop integrands [91–95]. The manifestly local formulation of the duality via kinematic Jacobi relations of gauge-theory numerators was realized via point-particle limits of open strings at tree level [53, 134], one loop [135, 434, 435] and two loops [136, 139]. In field-theoretic terms, kinematic Jacobi relations within tree-level subdiagrams originate from a non-linear gauge transformation of perturbiner solutions to the classical equations of motion [131–133]. The construction of perturbiners in so-called BCJ gauge [141, 142, 144] is crucially inspired by BRST cohomology methods of the pure-spinor superstring [127].<sup>23</sup> Also for the NLSM, string-theory methods led to compact all-multiplicity expressions for color-kinematics dual tree-level numerators [44, 438]. More general string-motivated perspectives on kinematic Jacobi identities can be obtained from vertex operator algebra constructions [439, 440], the pure-spinor master action of ten-dimensional sYM [441] or residue theorems for moduli spaces of Riemann surfaces with marked points [442].

The CHY formalism [5, 6] and the underlying ambitwistor strings [64–66] generalize the Witten-RSV [292, 293] and Cachazo-Skinner [294] formulae beyond four spacetime dimensions and share several facets of the worldsheet description of conventional string theories. However, ambitwistor strings do not involve any analogue of  $\alpha'$  and directly compute field-theory amplitudes, see [69] for a review. The tree-level realization of the color-kinematics duality via CHY and ambitwistor methods [32, 74–76] follows the lines of conventional-string computations [443, 444], also see [445, 446] for a closely related geometric construction in kinematic space. At loop level, the ambitwistor string yields modified Feynman propagators linearized in loop momentum [70, 71, 447] which introduces additional flexibility to find color-kinematics dual gauge-theory numerators and KLT-formulae for loop integrands in supergravity [77, 84].

Conversely, the color-kinematics duality and double-copy structure of field theories provided striking insights into string amplitudes including their full-fledged  $\alpha'$ dependence! For instance, the recent proposal for three-loop four-point superstring amplitudes [83] grew out of the color-kinematics dual representation of their sYM counterpart [4] and the higher-loop framework for ambitwistor-string amplitudes [80–82]. The three-loop four-point expression in [83] borne out of field-theory considerations conjecturally uplifts the low-energy limit [448] to all orders in  $\alpha'$ . Similarly, the one-loop double-copy formulae due to ambitwistor strings inspired the construction of one-loop matrix elements with insertions of the tr $(D^{2k}F^n)$  and  $D^{2k}R^n$  operators seen in superstring tree-level effective actions [449]. These matrix elements carry key information

<sup>&</sup>lt;sup>23</sup>Also see [436, 437] for recent worldline approaches to BCJ gauge.

on the discontinuity structure in the  $\alpha'$ -expansion of one-loop superstring amplitudes.

String tree-level amplitudes benefit from a particularly rich interplay with fieldtheory structures: disk amplitudes of open superstrings line up with the field-theory version of the KLT formula that double copies sYM with a basis of  $\alpha'$ -dependent modulispace integrals [137, 450]. The latter can be interpreted as (doubly-ordered) tree amplitudes of a putative effective theory of bi-colored scalars dubbed Z-theory [44, 438, 451]. This generalizes the appearance of bi-adjoint  $\phi^3$  [32] and NLSM amplitudes [44] in the low-energy limit of disk integrals to infinite families of scalar higher-derivative interactions subject to the color-kinematics duality. It is a two-fold surprise to encounter open strings as the *output* rather than the *input* of a double-copy construction and to find all orders in  $\alpha'$  resonating with a field-theory double copy!

Also, closed strings turn out to furnish field-theory double copies beyond the scope of the traditional string-theory KLT formula [2]: tree-level amplitudes of type-II superstrings were expressed in terms of sYM double copied with the single-valued version of open superstrings [452, 453]. The underlying single-valued map [454, 455] acts on the multiple zeta values (MZVs) in the  $\alpha'$ -expansion of the disk integrals and thereby reproduces the sphere integrals of closed-string tree amplitudes [456–459]. All the  $\alpha'$ -corrections to the open-string monodromy relations are washed out by the singlevalued map such that single-valued open superstrings obey field-theory BCJ amplitude relations at all orders in  $\alpha'$  [456].

Similar field-theory double copy constructions apply to heterotic and bosonic strings [456, 460, 461]. Open bosonic strings for instance arise from Z-theory double copied with a massive gauge theory known as  $(DF)^2$ +YM [461] that was constructed in the context of (mass-deformed) conformal supergravity [462]. Tree-level amplitudes of heterotic strings with any combination of external gauge and gravity multiplets are double copies of single-valued open superstrings with the  $(DF)^2$ +YM+ $\phi^3$  theory [461, 462] which reproduces the double-copy structure of Yang–Mills–Einstein theories [463] at  $\alpha' \rightarrow 0$ . Hence, massless tree amplitudes in a variety of perturbative string theories fit into the array of double copies in Figure 2 below, where both the KLT kernel and one of the double-copy components refer to field theories. Figure 2 may be viewed as a string-theory analogue of the field-theory multiplication table in Figure 1.

The  $\alpha'$ -expansions of open-superstring [451, 452, 464] and  $(DF)^2$ +YM tree amplitudes pinpoint infinite families of higher-derivative operators that respect the colorkinematics duality and participate in field-theory double copies [456, 460, 465]. The matrix elements of color-kinematics dual tr $(D^{2k}F^n)$  operators can be made fully explicit at any multiplicity by isolating the coefficients of different MZVs in open-string amplitudes: since all the  $\alpha'$ -corrections in the monodromy relations [48–50] occur in the rigid combination of  $(\pi \alpha')^2$ , any operator without powers of  $\pi^2$  in its coefficient is bound to

$\mathrm{string}\otimes\mathrm{QFT}$	sYM	$(DF)^2 + YM$	$(DF)^2$ +YM+ $\phi^3$
Z-theory	open superstring	open bos. string	compactified open bosonic string
$\operatorname{sv}\left( \begin{array}{c} \operatorname{open} \\ \operatorname{superstring} \end{array} \right)$	closed superstring	heterotic (gravity)	heterotic (gauge/gravity)
$\operatorname{sv}\left( \begin{array}{c} \operatorname{open} \\ \operatorname{bos. \ string} \end{array} \right)$	heterotic (gravity)	closed bos. string	compactified closed bosonic string

Figure 2. Double copy constructions of string amplitudes as presented in [461].

satisfy BCJ relations.<sup>24</sup> The simplest color-kinematics dual examples derived from this logic are matrix elements with single-insertions of  $\alpha' \text{tr} F^3$  from the open bosonic string and  $\alpha'^3 \zeta_3 \text{tr}(D^2 F^4 + F^5)$  universal to open bosonic strings and open superstrings [465]. Higher-order examples of color-kinematics dual operators include the coefficients of all single-valued MZVs (whose double copy is realized in closed-string tree amplitudes) but also coefficients of non-single-valued MZVs such as  $\alpha'^{\geq 8} \zeta_{3,5} \text{tr}(D^{\geq 10} F^5)$  that do not enter any closed-string amplitude.

While the string-theory KLT and monodromy relations apply universally to external states at arbitrary mass levels, the field-theory double copy structures in Figure 2 are specific to massless string excitations. As a first echo for massive external states, open-superstring amplitudes with a single mass-level-one state and any number of gauge multiplets have been brought into a field-theory KLT form [467]: Z-theory amplitudes are double copied with field-theory kinematics of a colorless massive spin-two multiplet coupled to sYM and supergravity as realized by the heterotic version of chiral strings [468, 469]. It would be interesting to test this double copy for multiple external massive spin-two multiplets and to further explore their role in conformal supergravity [470] and bimetric gravity [471]. Another important question (with potential input from the asymmetrically twisted variants of chiral strings [472]) is whether such massive double-copies generalize to higher mass levels of open-string spectra.

Future targets: An immediate goal for the future is to investigate the loop-level systematics of (i) generating color-kinematics dual gauge-theory numerators from the  $\alpha' \rightarrow 0$  limit of string amplitudes and (ii) identifying field-theory double-copy structures that apply to the full  $\alpha'$ -dependence.

<sup>&</sup>lt;sup>24</sup>In the first place, powers of  $\pi^2$  arise from the coefficients of even zeta values  $\zeta_{2k} \in \mathbb{Q}\pi^{2k}$ . A more precise identification of open-string amplitude contributions that obey BCJ relations relies on the *f*-alphabet description of MZVs [452, 466].

- (i) The current frontier of deriving BCJ numerators from open superstrings is at the two-loop five-point level [139]. Higher-loop and -leg orders are a central research target which will require refined methods to handle the spin-structure sums of RNS superstrings or b-ghost correlators in the pure-spinor formalism, see the dedicated White Paper [432]. At the same time, one-loop six-point supergravity amplitudes in general dimensions  $D \leq 10$  have not yet been derived from sYM numerators (see e.g. [135, 435]) through the BCJ double copy without resorting to the linearized propagators of the ambitwistor-string prescription.
- (ii) For open superstrings at one loop, a double-copy structure based on building blocks in the chiral-splitting procedure [72, 73] was proposed in [138, 473], and it remains to explore their connections with loop amplitudes of Z-theory. More generally, double-copy prescriptions for loop amplitudes of open and closed strings are likely to hinge on suitable bases of moduli-space integrals that admit interpretations in terms of scalar effective field theory. A parallel goal pioneered in [474–477] is to generalize the relations between closed strings and single-valued open strings beyond tree level, see [478] for a one-loop proposal covering a conjectural basis of open-string integrals.

**Big Picture:** In summary, double copy and color-kinematics duality were seen as surprisingly universal phenomena interweaving string- and field-theory structures to a growing web of theories, see Figure 4 below. On the one hand, the field-theory limit  $\alpha' \rightarrow 0$  of string amplitudes keeps on sharpening our understanding of the elegant interplay between gauge theories and gravity. On the other hand, field theories start to return the favor and shed new light on string perturbation theory. It will be rewarding to target string and point-particle amplitudes in parallel in future research and to search for drastic re-formulations of their S-matrices which manifest their synergies.

### 3.6 Generalizing the Double Copy and the KLT Bootstrap

We described in the Introduction how the double copy links a variety of field theories to each other; see Figure 1. We can think of this as a multiplicative map on a subspace of field theories. It is interesting to understand better what it takes for such a map to work and how the double copy may be generalized to include a larger subspace of field theories. Examples:

• Different representations of the color group. Gluons are in the adjoint representation, but it is of interest too to include fundamental matter (quarks, leptons) [239, 479–482] or to study which representations and group-theory structures are compatible with the double copy, not only for the particles themselves but also for their interactions.

- Massive matter and mediators. When masses are associated with matter particles in generic representations of the gauge group, e.g. quarks in QCD, the generalization is relatively straightforward [480]; different masses can be bundled along with flavor and one simply associates distinct graph edges for different masses and follows group theory commutation identities to relate their weights. This color-dual perspective generalizes nicely to relations between graphs contributing to integrands at loop level [239]. At tree level having different graphs with constrained algebraic relations does result in modified amplitude relations, KLT kernels, and CHY representations [480, 482–485]. The situation with more general massive states is more subtle with additional constraints beyond colorkinematics duality required for a physically consistent double copy [422, 486]. These constraints are automatically satisfied for spontaneous symmetry breaking in the adjoint [423, 487, 488]. Consistent double-copy constraints for massive mediators is a subject of ongoing study [489], including generalizations to topological massive theories in three dimensions, see e.g. [208, 209, 424, 425, 427, 489, 490] and references therein.
- **Higher-derivative operators**. This can be done with an eye on how to generate specific operators, as needed for example as counterterm input for loop-calculation, or systematically in the sense of what are the most general higher-derivative terms allowed [491–493].

In this section we focus on recent progress on how to generalize the KLT form of the double copy. We consider input theories with with adjoint massless particles, such as for example Yang-Mills theory. One class of applications is the double copy in the context of effective field theory, such as

$$(YM + h.d.) \otimes (YM + h.d.) = gravity + h.d.$$
(3.5)

where h.d. stands for higher-derivative operators.

It turns out that not all gauge theory operators can participate in the standard field theory double-copy construction. Recall from Section 2.1 that in order to be input for the double-copy relation, the color-ordered tree amplitudes have to satisfy the Kleiss-Kuijf (KK) and BCJ relations. For YM theory with higher-derivative operators,  $trF^3$ satisfies the KK and BCJ relations, but  $trF^4$ , for example, does not [465]. From a bottom-up effective field theory approach, it is interesting to understand better the selection-principle of which higher-derivative operators are allowed and which ones are not. And, to the point of this section, it is relevant to ask if the double copy can be generalized to admit a larger class of operators. To describe such generalizations, let us first take a closer look at the double copy to appreciate how non-trivial its existence really is.

Consider for example the color-stripped gluon tree amplitude  $A_4[1234]$ . It has simple poles in the  $s_{12} = (p_1 + p_2)^2$  and  $s_{14} = (p_1 + p_4)^2$  channels, but none in the  $s_{13} = (p_1 + p_3)^2$  channel. If we naively square it, we get an unphysical object  $A_4[1234]^2$ which has double poles at  $s_{12} = 0$  and  $s_{14} = 0$ . In contrast, the 4-graviton tree amplitude  $M_4(1234)$  has simple poles in all three channels,  $s_{12}$ ,  $s_{13}$ , and  $s_{14}$ . In light of this, how can a double copy ever work? In the KLT formulation, the key ingredient is the *KLT kernel*  $S_n$ : the field theory KLT formula for the 4-point amplitude can be written as

$$M_4(1234) = A_4[1234] S_4[1234|1234] A_4[1234], \quad \text{with} \quad S_4[1234|1234] = -\frac{s_{12}s_{14}}{s_{13}}.$$
(3.6)

The kernel serves two crucial purposes:

- (1) it has zeroes that precisely cancel double poles in the product of gauge theory amplitudes, and
- (2) it provides the missing poles, in this case in the  $s_{13}$ -channel.

One thing is getting the pole structure right, another is getting the residues of those poles correct so that the resulting amplitude is indeed that of gravitons. This is all non-trivial; yet, it works. And not just at 4-point, but for any n points.

The non-triviality of the double copy is important to keep in mind when we consider its possible generalizations. In the KLT formulation of the double copy, it is natural to consider modifications of the KLT kernel. The discussion above makes it clear that changes to the KLT kernel can easily wreck properties (1) and (2) and render the construction unphysical. Moreover, spurious poles in the kernel could result in an unphysical expression for  $M_4$  that is not a tree amplitude in any local theory. To address these issues, we must understand what the rules are for generalizations of the double copy kernel.

A recent proposal [493] for generalizing the KLT double copy was based on the KLT algebra: the idea that the double copy has an associated identity element. Given a KLT product  $\otimes$ , defined by a given kernel  $S_n$ , the proposition is that there exists an associated identity element such that

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$$
 and  $\mathbf{1} \otimes \mathbf{R} = \mathbf{R}$ ,  $\mathbf{L} \otimes \mathbf{1} = \mathbf{L}$ . (3.7)

For the field theory double copy, the identity element (sometimes called the zeroth copy) is the field theory known as cubic bi-adjoint scalar theory (BAS), as can be seen from

the double multiplication table in Figure 1. This was first noticed in the CHY formalism [32], though the BAS model also naturally arises in the BCJ language, namely when the numerator kinematic factors of the Yang-Mills amplitudes are replaced by color-factors,  $n_I \rightarrow c_I$  in (2.2), one finds the tree amplitudes of the BAS model. For string theory, the amplitudes of the identity element model [494] can be understood to arise in the  $\alpha'$ -expansion from the BAS model with a tower of very particular higher-derivative operators with fixed coefficients.

Written out, the meaning of  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$  at 4-point is that for any choices of color orderings a, b, c, d, the doubly color-ordered 4-point tree amplitudes  $m_4$  of the identity model must satisfy

$$m_4[a|b] = m_4[a|c] S_4[c|d] m_4[d|b].$$
(3.8)

It is quite remarkable that such an identity model exists! To understand it better, it is useful to unpack the (3.8). First, note that the relation uniquely fixes all components of the kernel  $S_4$  in terms of the  $m_4$ 's. In particular, choosing c = b, we see that  $S_4[b|d] = (m_4[d|b])^{-1}$ . This illustrates the unique link between the kernel and the identity model which also extends to higher points. Second, plugging this result for the kernel back into (3.8) and rearranging the equation it reads:

$$0 = m_4[a|b] m_4[d|c] - m_4[a|c] m_4[d|b] = \begin{vmatrix} m_4[a|b] m_4[a|c] \\ m_4[d|c] m_4[d|b] \end{vmatrix} .$$
(3.9)

That is, any  $2 \times 2$  minor of the  $4! \times 4!$  matrix of all doubly color-ordered 4-point tree amplitudes of the 1-model must vanish: the matrix must have rank 1. At *n*-point the statement is that the  $n! \times n!$  matrix of doubly color-ordered *n*-point tree amplitudes must have rank (n-3)!. This property holds for the BAS model, but adding a generic (higher-derivative) operator to BAS increases the rank above (n-3)!. The particular higher-derivative operators of the string-kernel in the  $\alpha'$  expansion do however preserve rank (n-3)!. That turns out not to be the only solution.

The KLT algebra (3.7) gives a well-motivated pathway for generalizing the double copy: if we modify the identity element, this will — via the equation  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$  — result in a unique new double-copy kernel, thus giving a *bootstrap for the KLT kernel*. Next, the requirements  $\mathbf{L} \otimes \mathbf{1} = \mathbf{L}$  and  $\mathbf{1} \otimes \mathbf{R} = \mathbf{R}$  on the amplitudes of the single-copy amplitudes become generalizations of the Kleiss-Kuijf relations and the BCJ (Bern-Carrasco-Johansson) relations (and likewise generalize the string monodromy relations). Thus the KLT double-copy bootstrap of [493] can be summarized as

$$\underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{1}}_{\text{KLT kernel bootstrap eq}} \quad \text{and} \quad \underbrace{\mathbf{1} \otimes \mathbf{R} = \mathbf{R}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{L} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{R} = \mathbf{R}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{L}}_{\text{generalized KK\&BCJ / monodromy relations}} \cdot \underbrace{\mathbf{1} \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes$$
These relations ensure that the result of the double copy is independent of the representation chosen for the KLT double copy. One must additionally impose locality.

As discussed, the generalized double copy can be applied in the context of EFTs. It was shown in [493] how  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$  uniquely fixes the KLT kernel in terms of the amplitudes of the identity model. Thus, adding all possible local higher-derivative single-trace operators to the cubic bi-adjoint theory (BAS) and plugging it into the bootstrap equation  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$  selects a certain set of admissible operators that can be added to the BAS model and it imposes certain relations among their Wilson coefficients. The result, as tested at 4- and 5-point in [493] and at 6-point in [495] indicates that the bottom-up approach to the generalized double copy gives a kernel that is more general than the strings kernel. In particular, unlike the strings kernel in which everything is fixed in terms of  $\alpha'$ , the generalized kernel involves a growing number of unfixed parameters at each order in the derivative expansion.

The generalized KLT double copy has been applied to to Yang-Mills theory with higher derivative operators and chiral-perturbation theory ( $\chi$ PT) with higher-derivative operators to get models of gravity, axion-dilaton-gravity, special Galileons, and Born-Infeld with higher-derivative corrections [493]. For example, the generalized KLT double copy allows for tr $F^4$  with any Wilson coefficient as a higher-derivative correction to YM theory; but note that coefficient of tr $F^4$  is linked by the generalized KK and BCJ relations to the coefficients of a  $\partial^2 \phi^4$  higher-derivative correction to BAS.

There are several motivations for studying generalizations of the double copy, including the need in higher-loop calculations to include counterterms in the double-copy construction. But there are also various puzzles that a better understand of the doublecopy framework may help address:

The Galileon scalar models have been proposed as higher-derivative corrections to the Dirac-Born-Infeld (DBI) action, however, the DBI-Galileon does not appear to be produced by the double copy. Since N = 4 super-DBI is the result of χPT ⊗ (N = 4 sYM), one might have thought that simply including higher-derivative terms would produce the Galileon corrections. However, this is impossible because the 4d Galileons interactions are not compatible with N = 4 supersymmetry [496]; nor are they compatible with N = 2 supersymmetry in the presence of the DBI leading interaction [496]. There is evidence in favor of an N = 1 supersymmetric quartic Galileon [497, 498], but there is no known way to construct it using the double copy: its scalar-part, the special Galileon, simply arises from χPT ⊗ χPT, as can be seen from the table in Figure 1, but χPT is not compatible with supersymmetry. Thus, at this time, it remains a mystery how an N = 1 quartic Galileon could possibly arise from a double copy, either directly or as a

higher-derivative correction to DBI.

- At 5-points, there is also a double-copy puzzle. The 4d quintic Galileon is a 5-point 8-derivative interaction. Yet there are no higher-derivative 5-point operators in χPT that can be double-copied to the quintic Galileon. The closest candidate is the Wess-Zumino-Witten term, but it is not compatible with the KK&BCJ relations [499]. An open question is if therefore if there exists a different version of the double copy that can produce the quintic Galileon.
- 4d Born-Infeld theory has electromagnetic (EM) duality. It is rather remarkable that the tree amplitudes in this theory can be produced as a double copy of  $\chi$ PT and Yang-Mills theory, which does not have EM duality. As symmetry of the equations of motion, but not of the Lagrangian, EM duality is only expected to be a tree-level symmetry of the amplitudes. However, it turns out that at 1-loop order there is evidence [500, 501] that EM duality can be restored via finite local counterterms. As it turns out, even the simplest counterterm needed to restore EM duality in the 4-point 1-loop amplitudes cannot be produced by any known consistent form of the double copy. This is then challenging for higher-loop calculations that may use the double copy and need to take such counterterms into account. This example from Born-Infeld theory serves as a simpler toy example of a similar issue that arises in  $\mathcal{N} = 4$  supergravity [502].

These bullet points outline examples of three puzzles with the standard field theory double copy. The generalizations of the double copy with higher-derivative terms added to BAS do not resolve the issues. However, the double-copy bootstrap is a more general framework and perhaps it admits a more general double copy that provide new insights and resolutions.

There is an alternative version of incorporating higher-derivative corrections into the double copy, namely by including kinematics into the color-factors [491, 492]. Recently, the connection between that approach and the KLT bootstrap [493] was examined in [503]. Combining the approaches may lead to new fruitful avenues of exploring the double copy.

Finally, let us point out that the EFT generalization of the KLT kernel described here is a bottom-up approach that does not assume anything about existence of a UV model. A particularly interesting question is then to understand what constraints UVcompletability brings in terms of bounds on the Wilson coefficients of the identity model associated with the KLT kernel. The modern powerful S-matrix bootstrap program intersects the double-copy program in a very interesting way that in the future will shed light on what makes string theory so special.

### 3.7 Color-Dual Effective Field Theory

From an effective field theory (EFT) perspective, any new higher-energy physics will be encoded in the low-energy theory as Wilson coefficients of higher-derivative operators, motivating an understanding of what we can definitively clarify about such predictions to all orders in mass dimension. Double-copy construction has already been used in attempts to understand potential constraints on the Wilson coefficients of higherderivative operators in gravity theories [504–506].

Adding increasingly higher-dimension operators to encode novel UV physics traditionally means working with increasingly finicky and difficult operators in gauge and gravity theories. Color-kinematics, double copy, and the success of Z-theory amplitudes (cf. Section 3.5) suggest a new approach for higher-derivative operator predictions in gauge theories: decomposing such predictions into much simpler color-dependent higher-derivative building blocks double-copied with a minimal basis of bare gaugetheory building blocks. The idea is to attempt to reduce all higher-derivative operators for gauge theories with and without massive matter to their most atomic components. By virtue of being color-dual these higher-derivative blocks will never interfere with the gauge invariance of the complete scattering amplitudes. If one is interested in going on to also construct higher-derivative gravity predictions, gravitational amplitudes then follow by replacing color with known color-dual gauge-theory weights.

How should one construct (or identify) all distinct amplitudes associated with higher derivative operators of a given mass-dimension relevant to a particular n-field contact? A natural S-matrix approach invokes ansatze of the appropriate kinematic mass-dimension, external fields, then fixes on relevant symmetries. Color-kinematics duality means only needing to dress a small number of cubic basis graphs at tree-level, but even this becomes expensive as mass-dimension grows. Familiarity with the strict constraints of color-dual kinematics combined with the type of functional symmetry required to dress loop graphs suggests a more constructive route.

This perspective has recently born fruit at tree-level [491, 492, 507, 508]. There exists a simple pattern to all orders in higher-derivative structure and an amazingly small number of primary building blocks for color-dual gauge and gravity higher-derivative operators. A new verb has entered the color-dual story: *composition*. Rather than using ansatze in Lorentz invariants and polarizations to satisfy functional color-kinematics constraints, color-dual functions can be *composed* [491] to generate new color-dual functions of arbitrarily high mass dimension.

This has profound implications for building and classifying the predictions of EFT operators as we can, at each multiplicity, find a pure-scalar color-dual function that is linear in Mandelstam invariants. This means there exists a unit step ladder to

access higher-dimension operator predictions. Rather than needing to confront an ever more excruciating ansatz at higher order in mass dimension, these building blocks can be simply composed with themselves and a scalar unitary weight. It is like building higher-derivative corrections by putting together tinker-toys.

At four points [491] it is possible to entirely climb the ladder of color-dual singletrace UV corrections to gauge theory and associated higher-derivative operators to gravity theories with a handful of building blocks. The same four-point composition rules have been shown to also hold in the multi-trace sector [507, 508]. The spectacular confirmation of the success of this approach involves exposing this very structure in the Z-theory description of the open super and bosonic strings at four-points. The reduction in complexity at five points [492] is even more profound, involving the classification of a number of relevant double-copy structures complementary to the adjoint. A surprising primary building block at five-points – similar to adjoint but with central vertex antisymmetry condition relaxed, has lead to the identification of a candidate allmultiplicity linear building block: obeying adjoint relations at even multiplicity, and relaxed-adjoint relations at odd [492].

The higher-derivative interactions at string tree level realize some of the colorkinematics dual operators of [491, 492] with particular (rational) combinations of multiple zeta values as their Wilson coefficients [451, 452, 464]. It would be interesting to compare the entirety of loop-level effective interactions in different perturbative string theories with the operators constructed from the method in [491, 492].

Color-dual building blocks provide [491, 492, 507, 508] a powerful tool that can be simply composed to all mass-dimension without having to resort to ansatz to construct the physical predictions of effective operators in gauge and gravity theories. Next steps involve addressing the following sharp questions:

- Can these ideas be brought front and center in the description of the higherderivative operators themselves at the heart of the action?
- Do the tree-level composition rules change for matter with mass and in arbitrary representations of the gauge group? Do color-dual bootstraps hold for fermionic matter and massive gauge theories?
- What are the relevant scalar integrands at the multiloop level required to encode the predictions of higher-dimension operators of the compositional building blocks? Four-points through four loops should be completely feasible at least at relatively low mass-dimension. Do there exist similar composition rules at loop level that allow all-order in mass dimension building-blocks?

**Big Picture:** Even being able to articulate these questions demonstrates the power of exploiting novel S-matrix double-copy structure to attain all-order in mass-dimension predictive control for effective gauge and gravity theories through a small number of simple building blocks, constructively generated. Leveraging double-copy structure offers a new way of thinking about higher-derivative operators in EFT—applicable across many fields of physical inquiry, one that may lead to much easier higher-loop predictions. The long view is that having these types of structures manifest in our predictions could ultimately lead to new descriptions of our physical theories, placing these atoms of prediction front-and-center.

### 3.8 UV Behavior of Supergravity Theories

Simple power-counting arguments make it clear that individual gravitational Feynman diagrams must manifest unruly UV behavior at some loop order in four dimensions relative to gauge theory. Ultimately this is due to the dimensionality of the coupling constant, and the consequent need for additional momentum upstairs in the integrand for every gravitational interaction. Supersymmetry is known to only provide a finite amount of protection in the UV. There are for example constraints from supersymmetry together with linearly and nonlinearly realized symmetries up to 7-loop order, but not in any known way beyond, in 4d  $\mathcal{N} = 8$  supergravity [509–512]. This suggests at most a delay for which loop order UV divergences must appear. Are all pointlike (local) quantum field-theories of gravity therefore doomed to be *effective*, requiring perturbative completion in the UV? Arguments based on individual Feynman diagrams only hold if there is no additional symmetry or structure enforcing cancellation between graphs. The only known way to show that there are no missing or unrecognized symmetries in a theory is to do the explicit calculation and find a divergence.

From both symmetry considerations and technical accessibility, the theory to test with the best hope of perturbative finiteness in four dimensions would appear to be maximally supersymmetric supergravity ( $\mathcal{N} = 8$  SG). While counterterms have been found which would be relevant starting at seven loops [511, 513], their coefficients have not yet been calculated and could indeed vanish in four dimensions. Indeed, if one wants to celebrate optimism, there are a number of hints that suggest that four dimensions may indeed be special [514–517].

A recent milestone was the explicit calculation of the UV behavior of  $\mathcal{N} = 8$  SG at the five-loop order. Using the method of maximal cuts in combination with discovering [63] a crucial generalization of double-copy integrand construction, obviating the need for finding a manifest color-kinematics gauge-theory representation, the integrand of the five-loop correction to the two-to-two graviton scattering in the  $\mathcal{N} = 8$  SG theory was constructed [62]. Shortly thereafter, a representation tailored to integration

in the UV confirmed [25] a critical dimension of 24/5. This is the dimension in which the theory diverges due to insufficiently soft UV behavior. Notably this is the first demonstration that complete amplitudes in the maximal supergravity theory can possess a worse UV behavior than its double-copy progenitor (maximally supersymmetric gauge theory) whose formal five-loop critical dimension is 26/5. This may herald a potential higher-loop divergence in four-dimensions – a topic of much speculation that itself awaits explicit calculation, now potentially within reach.

Perhaps more important than the UV behavior of this particular theory, performing the five-loop calculation exposed [25] entirely surprising and potentially universal consistency relations between loop-orders (*in different dimensions!*) for the vacuum *integrals* that dominate the UV. The newly discovered consistency conditions are exactly the same in both the maximally supersymmetric gauge theory and the gravity theory. Having UV relations between loop orders suggests a path towards a UV bootstrap program which could allow the direct probing of higher-loop behavior – something to be explicitly tested in the near future.

Six-loops in the  $\mathcal{N} = 8$  theory is the obvious and critical next step to be calculated both via the new UV bootstrap approach as well as unitarity methods towards verification of this new approach. Within the past year the six-loop integrand for the color-dressed maximally supersymmetric gauge theory has finally been calculated [518]. Using traditional methods this task is enormous. Completing the task required the invention of new and efficient means of extracting cut information—methods that ultimately have color-dual representations at their heart but exploit various soft-limits to relate information at a given loop level to more accessible information at lower loops, or simpler structures at higher loops. A clear next step will be to turn this gaugetheory result into a supergravity calculation and to test the range of the UV bootstrap developed at five-loops.

Complementarily, the first UV divergence in four-dimensions ever calculated in a pure supergravity theory was at four-loops in the half-maximally supersymmetric theory [24]. This divergence has been associated with the presence of the so-called Marcus anomaly whose behavior at one-loop can be explained in terms of a double copy between maximally supersymmetric gauge theory and pure Yang-Mills [519]. A local counterterm has been proposed whose effects have been calculated through two loops [502, 505, 520]. Higher-loop calculations with this higher-derivative counterterm prove prohibitive with traditional approaches, especially if it requires higher-derivative operators for consistency. However this counterterm may be easily understood in terms of the higher-derivative building blocks described in Section 3.7 – involving a double copy seeded by one vector weight proportional to  $F^3$ . Ultimately we are interested in whether the anomaly tempering counterterm provides for finiteness to all orders, or whether the theory requires [521] an infinite number of counterterms.

**Big Picture:** The question is to understand definitively whether or not QFT in fourdimensions can admit a perturbatively finite QFT of gravity without the need for infinitely many counterterms — or extended structure as they are understood in the context of string theory. These are "big-theory"-type projects: multi-year calculations that push current technology to its limit, forcing innovation and recognition of previously unappreciated structure in order to arrive at an answer. More loops is different, and considering supersymmetric theories offers a simpler playground to build tools and identify potentially helpful structure. Sometimes this structure is generic, and sometimes it is very special. The ability to only consider graphs without one-loop triangle and bubble subdiagrams is special to gravity theories whose single-copy involve maximally supersymmetric gauge theory. On the other hand, the color-kinematics duality and double copy appear to be properties of field theories not only beyond supersymmetry, but indeed beyond gauge and gravity theories. It is an open question as to how general the UV bootstrap of [25] might be. It is worth noting that pushing the envelope of finite-field related methods (cf. for example [522] and references therein) applied to simplify the integral-level book-keeping will be a nontrivial concrete outcome of this line of investigation.

## 3.9 Simplifying Calculations in Gauge Theories with Matter

One research thrust has been to consider applying the duality between color and kinematics towards simplifying Standard-Model type integrands. The following is a summary of current progress:

- QCD with massive matter in arbitrary representations coupled to massless gauge theories admits [480–482, 523] color-dual representations at tree-level, with associated reduction in number of independent ordered (stripped) tree-amplitudes.
- Massive scalar QCD with matter in arbitrary representations is known to be color-dual through one-loop five-points [239, 480, 524].
- Supersymmetric generalizations, (S)QCD, with external glue and matter in the fundamental are color-dual through two-loop four-points [525–528].
- While one must be careful with the duality between color and kinematics for generic massive gauge theories [422, 486], spontaneous symmetry breaking ("Higgsing") in the adjoint is color-dual at tree-level, and indeed massive gauge theories are understandable at tree-level as a consistent dimensional reduction [487, 488].

It is absolutely fair to ask what this program could buy us in terms of precision QCD and Standard Model calculations more generally.

The relevant quantitative predictions for hard processes in QCD are obtained [529] using perturbation theory, after accounting for the non-perturbative bound state dynamics of partons and hadron fragmentation. The most immediate and omnipresent challenge when performing multiloop calculations towards IR safe observables is integration [530], especially over virtual loop momenta involving multiple internal mass scales. While there has been a tremendous amount of progress, new ideas and approaches to integration remains an active area of pressing interest [530–534]. When applicable, the duality between color and kinematics appears to hold at the integrand level and so does not confront, at least in any direct sense, the primary challenge of loop integration. Nevertheless, perturbative calculation relies on organization into local Feynman integrals, and therefore even at the integrand level, as loop level and multiplicity increase, we have a problem: the number of contributing diagrams grows factorially. At the integrand construction level, the duality between color and kinematics has the potential to simplify the situation as we will describe.

Unitarity, or on-shell, methods [535-540] have become standard in the current era of precision calculation. The idea is to avoid calculating with unwieldy arbitrary-gauge off-shell Feynman rules that carry around unphysical information that must cancel in final gauge invariant observables. Instead we employ compact on-shell gauge-invariant quantities like tree-level scattering amplitudes. It is sufficient to verify an integrand if it satisfies all unitarity cuts to tree-level results. Since cuts satisfy a spanning relationship, only a relatively small number of cuts need to actually be performed to verify an integrand. Furthermore, because gauge theory tree-amplitudes can be expressed in terms of a color-basis, we only need to consider simpler color-ordered (or color-stripped) cuts. Having an easy approach to verification leads to natural construction techniques, whereby one builds the integrand so as to simply satisfy all cuts. Furthermore when a minimal basis of integrals is known, such as one-loop, it is possible to directly target the coefficients of integrals themselves via cut techniques. It is important to be able to capture *D*-dimensional information when using variants of dimensional regularization, so often cuts are performed in D-dimensions. Furthermore there are subtleties [537] around massive tadpoles that are in principle only directly extractable via on-shell methods via forward-limit cuts, but could in principle receive cut information propagated from more accessible graphs by color-dual kinematic relations.

Consider a four-point two-loop gluon QCD calculation with quarks in the fundamental. At higher multiplicity or loop level the gain can be even more impressive, but NNLO processes in many ways still represent the state of the art. One of the most complicated cuts is between two 5-point trees. The most complicated part of this calculation will involve gluons crossing the cut. Prior to the recognition that there are in fact only (m-3)! distinct color-ordered amplitudes at *m*-points under the BCJ relations as described in Section 2.1, 36 different color-ordered cuts would in principle need to be performed as per the Kleiss-Kuijf basis. Now only four distinct cuts need be considered. A similar reduction holds for the cut between a six-point tree and a four-point tree. For example, if both the external and the cut lines are gluons, before the duality between color and kinematics we would have had to worry about 48 color-order cuts, while now we only need to consider six.<sup>25</sup> These gains arise simply from the theory being color-dual at tree-level.

What if we can make the duality between color-and-kinematics manifest directly at the level of the integrand? The scattering amplitude for supersymmetric QCD with massive matter in the fundamental for the two-loop correction has been constructed [525] and integrated [528], including the maximally helicity violating case (MHV). MHV, at four-points, refers to the situation when two of the external gluons are positive helicity and two external gluons are negative helicity. As depicted in Figure 3, there are 24 distinct graph topologies that could contribute to the amplitude, appearing with all distinct permutations of labels. In order to build the amplitude, each graph must be mapped to its relevant color-weights, kinematic weights, and propagators for all relevant labelings. This procedure of mapping a graph to its specific weights is called dressing the graph, and the weights are often called dressings. An advantage of graph-based unitarity approaches is that one only needs to constrain dressings for individual topologies, allowing automorphism invariance to account for all distinct labeling permutations. The goal of cut construction is to find the appropriate dressings for each topology such that all cuts are satisfied.

For the two-loop MHV  $\mathcal{N} = 2$  SQCD amplitude it is possible to write a spanning ansatz for kinematic dressings of each graph in terms of a minimal basis of Lorentz invariants involving momenta and external polarizations. Any parameter not constrained by physical cuts will inevitably turn out to be a generalized gauge choice that will never contribute upon integration. There are about 4.5k parameters per graph. If it had been impossible to make the duality between color and kinematics manifest at the level of the integrand this would mean a total ansatz size for the amplitude on the order of 105k parameters. The duality between color and kinematics however relates the kinematic dressings of graphs to each other. For example, the double-triangle kinematic weights

 $<sup>^{25}</sup>$ For a cut between a six-point tree and a four-point tree of a four-fermion amplitude with one cut fermion and two cut gluon lines, the duality reduces the number of cuts from 24 to 6, cf. Tables 2 and 4 of [480].



Figure 3. A complete list of non-vanishing graphs contributing to four-point 2-loop  $\mathcal{N} = 2$  SQCD taken from ref. [525]. The eight basis graphs whose kinematic weights combine to dress all graphs are (1)–(5), (13), (19) and (22).

are given as differences between the kinematic weights of double boxes,

$$n_{11}(1234; \ell_1, \ell_2) = n_1(1234; \ell_1, \ell_2) - n_1(1243; \ell_1, p_3 + p_4 - \ell_2),$$

$$n\binom{4}{3} \underbrace{\ell_2 \ell_1 \atop \ell_2}{\ell_1} = n\binom{4}{3} \underbrace{\ell_2 \ell_1 \atop \ell_2 \ell_1 \atop \ell_2 \ell_1}{\ell_2 \ell_1 \atop \ell_2 \ell_2} - n\binom{3}{4} \underbrace{\ell_2 \ell_1 \atop \ell_2 \ell_2 \ell_2}{\ell_2}.$$
(3.11)

The authors of ref. [525] used such linear relations to reduce all kinematic weights to those of just eight basis graphs. This reduces the entire size of the kinematic ansatz for the integrand to 35k parameters, of which many are drastically constrained due to symmetries of the graphs before even admitting cut data. The resulting integrand was subsequently integrated in ref. [528].

Even more dramatic reductions occur when all particles transform in the same representation of the gauge group. The  $\mathcal{N} = 4$  super Yang-Mills amplitude requires only ~ 80 non-vanishing topologies at the four-loop correction to four-point scattering. The reason the number of topologies is so small is because maximal supersymmetry removes a need to consider any graphs with internal one-loop bubble or triangle sub-diagrams. All of the kinematic dressings can be expressed in terms of linear functions of a single [16] nonplanar basis graph by exploiting color-dual Jacobi-like relations.

There is often residual generalized gauge freedom upon finding color-dual numerators for a given theory at the integrand level. This means that certain parameters cancel for any physical cut or upon integration. This freedom can be exploited to make manifest UV behavior graph by graph as per three and four loops in  $\mathcal{N} = 4$  super Yang-Mills [4, 16]. Intriguingly this freedom can be used to expose IR behavior as well, as in the example of MHV two-loop SQCD, cf. the original integrands of ref. [525] vs the relatively IR safe integrand of ref. [527].

It is worth noting that the act of discovering simplicity is often driven by the needs of encountering tremendous complexity. We are well past the days where the boundary of perturbative calculation proceeded without automation. Pushing the boundary of perturbative calculation often means the development and application of massively parallel computational techniques, analytic as well as numeric. This has been critical for a number of projects from event generation [541], to supergravity calculations [25], to precision gravitational waves [13, 14], to cosmological large scale structure [542]. The amount of analytic data generated in integrand construction can be enormous. The size of the six-loop integrand of the friendliest gauge theory, the  $\mathcal{N} = 4$  super Yang-Mills theory, is about a gigabyte [518]. Constructing it meant evaluating and considering orders of magnitude more data in the form of unitarity cuts. It will not be long before analytic calculations approach the terabyte scale.

Success in these calculations yields not only new understanding of the language of perturbative field theory but also the development of tools to handle large-scale analytic data, aiding researchers in identifying meaningful patterns. This expertise pays dividends. The ultimate goal, of course, is to extract from this data, when possible, the correct reformulation minimizing the need for such large intermediary stages. Indeed historically, this is how the duality between color and kinematics was first identified [3] in the midst of performing a four-loop calculation in the  $\mathcal{N} = 8$  maximal supergravity theory [23].

#### 3.10 Towards Understanding the Kinematic Algebra

The celebrated double copy slogan, eqn. (1.2), could be expressed as: "understanding gravity is no more complicated than understanding gauge theory." A corollary, emerging from attempts to understand the duality between color and kinematics at the level of equations of motion [140–142, 144, 145, 192, 543–545] and the action [45, 428, 441], could be phrased "gauge theory need not be more complicated than a scalar theory." To fully realize the idea, it is necessary to write the non-trivial kinematic details of gauge theory as a Lie-algebraic structure that is dual to the color Lie algebra.

A detailed understanding of the kinematic algebra would be transformative from many possible perspectives. Given a diagram with any number of loops or external legs, we could in principle write down its numerator by simply contracting appropriate kinematic "structure constants". The diagrams themselves would be convenient devices for specifying invariant tensors associated with the kinematic algebra. Gravity would be realized as a peculiar Yang-Mills theory: one in which the gauge algebra is equal to the kinematic algebra. Loop integrations could potentially be reinterpreted as kinematic traces. The study of the space of functions emerging from loop integration may take on a wholly new algebraic character. Finally, the mathematical structure of the kinematic algebra could reveal new hidden symmetries or clarify details about the presumed emergence of spacetime.

In special situations it has been possible to identify a simple kinematic algebra. For example, it is useful to restrict attention to the "self-dual" sector of pure gauge theory and gravity [140, 546, 547]. This restriction can be interpreted as choosing to scatter only gluons or gravitons of a particular helicity, say positive helicity. At tree level this implies the vanishing of all amplitudes beyond three points. However, a closely-related quantity is non-vanishing: the solution of the classical equations of motion sourced by an ensemble of plane waves. In general, this solution is a generating function of tree amplitudes, and can be thought of as a sum over tree graphs with exactly one external off-shell leg.

The expansion of the self-dual classical solution in powers of couplings can be organized in terms of trivalent diagrams. In the Yang-Mills case, each diagram corresponds to a particular kinematic and color weights. Remarkably, it is possible to choose a gauge in which the kinematic weights, computed from the equations of motion, manifestly satisfy the same algebraic relations as the Jacobi relations enjoyed by the corresponding color factors. A closely related gauge choice is available in gravity, revealing that the gravitational solution is indeed a double copy of the gauge solution to all perturbative orders in the self-dual theories. It is possible to understand the details of the self-dual kinematic algebra: it is an algebra of area-preserving diffeomorphisms of a two-dimensional plane embedded in four-dimensional spacetime. The kinematic structure constants appear as two copies in the three-point vertex of gravity, and thus in the gravitational equations of motion. While it is no great mystery that that a diffeomorphism algebra plays a central role in self-dual gravity, it is more surprising that the same algebra controls self-dual Yang-Mills theory. There are some indications that the origin of this fact may be traced back to the Sugawara construction of a related CFT in two dimensions [548]. The self-dual sector has recently been generalized to other theories via Moyal deformations and relaxing the symmetry [549]. An interpretation of the kinematic algebra based on the Drinfeld double of the Lie algebra of vector fields was discussed in reference [550].

In more recent work, it has been shown [428] that the kinematic algebra includes volume-preserving diffeomorphisms in the special case of three-dimensional Chern-Simons theory. In contrast to the above self-dual case, which does not constitute a consistent quantum theory, the complete action of pure Chern-Simons theory gives rise to off-shell Feynman rules that manifest the duality between color and kinematics. This holds both at tree level and any loop order. However, similar to the self-dual case, the Feynman diagrams in pure Chern-Simons theory non-trivially conspire to give vanishing amplitudes when taken on shell. The off-shell correlation functions are non-vanishing in Lorenz gauge and provides the first fully off-shell realization of the duality.

In the self-dual case, since we choose to study only one helicity, it follows that we can describe the theory in terms of a scalar field (at the expense of manifest Lorentz invariance). The self-dual kinematic algebra is therefore relevant to a kind of scalar theory. Closely related constructions are available in other scalar theories, notably the non-linear sigma model [45, 143, 551].

Another lesson from the self-dual theory is that it can be useful to think about colorkinematics duality, and the double copy, at the level of the classical equations of motion. Indeed, using traditional field theory methods in a first-order formalism, ref. [145] not only identified the relevant algebraic relations for Yang-Mills covariantly, but also produced associated explicit realizations of all multiplicity color-dual numerators at tree-level. Explicit color-dual solutions have found an off-shell realization through classical perturbiner solutions [131–133] in so-called BCJ gauge [141, 142, 144]. Just this year there has additionally been new insight into double-copying off shell currents [552] and the kinematic algebra [553] from the perspective of double-field theory.

If we zoom into any differentiable manifold associated with a continuous group we will see its defining local generators obeying a Lie algebra. Indeed, scattering amplitudes in the NLSM can be shown to encode such a field-space geometry [554, 555]. Could we from this perspective understand the origin of the kinematic numerators describing various theories by relating them to scalar interactions in a theory-specific field-space geometry? This idea was recently explored [556] by writing massless bosonic tree-amplitudes for general theories in terms of NLSM amplitudes and replacing the field-space geometry with a notion of kinematic-space geometry.

Despite significant recent progress, the kinematic algebra of Yang-Mills theory remains an enigmatic problem. A central obstacle to overcome is the vast generalized gauge freedom that is associated to kinematic numerators, and additional principles are needed for narrowing down possible constructions. Stratifying the kinematic numerators according to the MHV sectors of Yang-Mills theory was shown to be a powerful organization principle in refs. [544, 545], which permitted a formulation of the kinematic algebra up to the next-to-MHV level. This formulation was motivated by current algebras and produced all-multiplicity numerators using a handful of "fusion product" rules for the generators of the algebra. The approach later expanded in a combinatorial direction in refs. [233, 557], where a well-known quasi-shuffle Hopf algebra was shown to be isomorphic to the fusion product of generators that give gauge-invariant treelevel kinematic numerators, to all multiplicity and all MHV sectors. These kinematic numerators correspond to a heavy-mass effective theory coupled to Yang-Mills theory, and include poles of the heavy-mass particles in the numerators. This mild non-locality is what makes it possible to give a manifestly gauge-invariant formulation, and echoes the construction of gauge-invariant numerators in ref. [145]. Curiously, the number of terms in the numerators (as well as number of generators) are given by the Fubini numbers.

**Big picture**. While still in its infancy, the exploration of the kinematic algebra of Yang-Mills theory, and of other gauge and scalar theories, constitute a cornucopia of rich mathematical structures that deserve further attention. As briefly reviewed, there are many options for formulating the kinematic algebra: through differential operators, kinematic structure constants, geometric maps, field equations, and at the Lagrangian level. We hope these interlocking perspectives will eventually crack open the problem of finding an elegant, mathematically deep, and practical formulation of the kinematic algebra underlying color-kinematics duality. Once fully revealed, the algebra invites deep physical interpretations and possible consequences for new physics.

## 3.11 Unifying the Web of Theories

From its original formulation, it has been clear that the double-copy structure does not rely on the presence of supersymmetry. Much early work focused on using the double copy to facilitate calculations in maximal and half-maximal supergravity, but that was mostly a consequence of the increased simplicity of these theories, as well as of the interest in examining their UV properties. Indeed, from the very beginning,



Figure 4. Web of theories as presented in ref. [54]. Nodes represent the main double-copyconstructible theories, including gravitational theories (rectangular nodes), string theories (oval nodes) and non-gravitational theories (octagonal nodes). Undirected links connect theories with a common gauge-theory factor. Directed links are drawn between theories constructed by modifying both gauge-theory factors.

examples of non-supersymmetric theories admitting double-copy constructions have been analyzed. These include the double copy of pure Yang-Mills theory, which yields Einstein gravity with some additional states (in four dimensions, a complex scalar field, which can be seen as the double copy of gluons of opposite polarizations). Over the years, more complicated examples of the construction have emerged, leading to the realization that the double-copy structure might be considerably more general than originally envisioned, as depicted in Figure 4. However, an important question remains open on exactly how general the double-copy structure is.

The space of all possible gravitational theories is extremely difficult to chart and, in order to delve deeper into the above question, we need to use some classifying principle. Once again, supersymmetry comes to the rescue: the supergravity community has long taken on the task of charting the space of all possible supergravity theories, uncovering and exploiting some beautiful geometrical structures in the process [558]. This classification begins from theories with a large number of supersymmetries, including maximal  $\mathcal{N} = 8$  supergravity<sup>26</sup> [559] and half-maximal  $\mathcal{N} = 4$  supergravity [560, 561]. Because of the high amount of symmetry, the former theory is unique. The latter can be specified completely by a single parameter — the number of vector fields in the theory. Once the field content is known, the symmetries are so constraining that all interactions are fixed.

However, when supersymmetry is reduced to  $\mathcal{N} = 2$ , supergravity theories are much less constrained [562–565]. In particular, the spectrum alone no longer qualifies a theory completely, and additional information on the interactions is needed. The simplest theories with this property constitute a fundamental test for the application of the double copy to generic theories of gravity. These are the so-called homogeneous  $\mathcal{N}=2$ Maxwell-Einstein supergravities. After earlier progress in describing particular cases [41, 566, 567], a general double-copy construction for these theories was formulated in 2015 in ref. [568]. These theories were previously explicitly classified in the supergravity literature [569]. Remarkably, the construction of ref. [568] succeeds in realizing all theories in the supergravity classification [569] as double copies of suitably-chosen gauge-theory factors, demonstrating that the double copy is a property of very general classes of theories. In particular, the key ingredient in the construction for homogeneous theories is the fact that the gauge theories involved can admit extra non-adjoint representations in which some of the matter fields transform while still obeying the duality between color and kinematics. This idea has also been used to present constructions giving pure theories [479] and  $\mathcal{N}=2$  theories with hypermultiplets [168, 568]. In turn, the double copy has been used to conduct highly-nontrivial loop calculations in these theories along the lines of Refs. [525, 526, 570].

In the past years, more and more gravitational theories have been recognized to secretly admit a double-copy description. A particularly important class is given by Yang-Mills-Einstein theories, some simple supergravities which include non-abelian gauge interactions. Aside from the double-copy method [463, 487, 571–577], a variety of approaches have been employed in studying the amplitudes in these theories, facilitating contact between closely-related subfields [41, 578–589].

Another very important class of constructions arises in case of the so-called gauged supergravities. These are theories obtained from the more conventional (ungauged) supergravity theories by promoting part of the R-symmetry to a local symmetry under

 $<sup>^{26}</sup>$ Here we specify the amount of supersymmetry in four spacetime dimensions. The reader should be aware that most of these theories can also be specified in higher dimensions.

which some of the supergravity fields are charged. This operation results in an array of interesting physical features, including the possibility of non-trivial scalar potentials, spontaneously-broken supersymmetry (in case the theory admits Minkowski vacua) and the possibility of anti-de Sitter vacua [590]. Most importantly, gauged supergravities are a subject of current investigation from the supergravity community, with the recent discovery of various families of novel theories [591–596]. While the study of gauged supergravities from the vantage point of the double copy is still in the early stages [488, 597], this is one of the instances in which amplitude methods in general and the double-copy technique in particular can provide new insights to solve challenging problems that have long remained open, as well as potential for cross fertilization between different fields in theoretical physics. Another notable set of constructions is the one for conformal supergravities [462, 598].

The net result of a significant body of work over the last few years is a wide web of theories connected by a double-copy construction (see Refs. [54, 55] for recent reviews). This web illustrates the power of the double copy as a classifying principle. In the coming years, the double copy will be applied to theories with minimal or no supersymmetry. Much more information is required to specify these theories in comparison with the instances of the double copy known so far. The open challenge is to map this abundance of supergravity information into gauge-theory data, as it was done in the case of homogeneous supergravities.

**Big picture**. While the results so far have raised the prospect of the double copy being a general feature of gravitational interactions, extension of the web of doublecopy-constructible theories to include generic theories with minimal or no supersymmetry will be the fundamental test to determine whether this is the case. At a conceptual level, this is a question with deep implications in terms of gravitational interactions admitting simpler descriptions using gauge-theory building blocks. Additionally, the near future will reveal whether the double copy can be used as an organizing principle to chart the space of all possible theories. As previously emphasized, this organizing principle is independent from supersymmetry — we started from analyzing supersymmetric theories simply because they are usually simpler and more under control than their non-supersymmetric relatives. More work also needs to be done in understanding the physical significance of the connections between theories revealed by the double copy. The double copy organization of the web of theories connects theories sharing a common gauge-theory factor, but the physics behind this connection often remains elusive and necessitates further study. Finally, the double copy remains above all a formidable computational tool. As more and more instances of this construction are identified, the doors open to the possibility of conducting highly non-trivial precision calculations in these theories.

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# Solicited Feedback

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