On the Importance of Rare Kaon Decays: A Snowmass 2021 White Paper

Jason Aebischer^a, Andrzej J. Buras^b and Jacky Kumar^b

^a Physik-Institut, Universität Zürich, CH-8057 Zürich, Switzerland ^bTUM Institute for Advanced Study, Lichtenbergstr. 2a, D-85747 Garching, Germany

Abstract

We stress the importance of precise measurements of rare decays $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi^0 \nu \bar{\nu}$, $K_{L,S} \to \mu^+ \mu^-$ and $K_{L,S} \to \pi^0 \ell^+ \ell^-$ for the search of new physics (NP). This includes both branching ratios and the distributions in q^2 , the invariant mass-squared of the neutrino system in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ and of the $\ell^+ \ell^-$ system in the case of the remaining decays. In particular the correlations between these observables and their correlations with the ratio $\varepsilon' / \varepsilon$ in $K_L \to \pi \pi$ decays, the CP-violating parameter ε_K and the $K^0 - \bar{K}^0$ mass difference ΔM_K , should help to disentangle the nature of possible NP. We stress the strong sensitivity of all observables with the exception of ΔM_K to the CKM parameter $|V_{cb}|$ and list a number of $|V_{cb}|$ -independent ratios within the SM which exhibit rather different dependences on the angles β and γ of the unitarity triangle. The particular role of these decays in probing very short distance scales far beyond the ones explored at the LHC is emphasized. In this context the role of the Standard Model Effective Field Theory (SMEFT) is very important. We also address briefly the issue of the footprints of Majorana neutrinos in $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$.

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1 Introduction

Rare decays of Kaons played already for decades a very important role in testing the Standard Model (SM) and in the search for new physics (NP). In this decade significant progress on these decays will be made, in particular through experiments at CERN (NA62, LHCb) [1–3], J-PARC (KOTO) [4] and later also KLEVER [5] at CERN. Among the rare Kaon decays considered by us $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \mu^+ \mu^-$ and $K_S \to \pi^0 \ell^+ \ell^-$ are CP conserving while $K_L \to \pi^0 \nu \bar{\nu}$, $K_S \to \mu^+ \mu^-$ and $K_L \to \pi^0 \ell^+ \ell^-$ proceed in the SM and in many of the beyond SM (BSM) scenarios governed by vector and axial-vector currents only in the presence of CP violation. The latter fact makes the search for these decays very important with the goal to find new sources of CP violation possibly responsible for the matter-antimatter asymmetry in the universe. A recent extensive review of these decays can be found in [6]. However, a less known fact should be emphasized here. In the presence of scalar currents $K_L \to \pi^0 \nu \bar{\nu}$, $K_S \to \mu^+ \mu^-$ and $K_L \to \pi^0 \ell^+ \ell^-$ can proceed also without any sources of CP violation [7].

Within the SM these decays are loop-induced semileptonic FCNC processes, receiving only contributions from Z^0 -penguin and box diagrams, in particular with W^{\pm} and top quark exchanges. A very important virtue of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ decays is their clean theoretical character. This is related to the fact that the low energy hadronic matrix elements required for the calculations of their branching ratios are just the matrix elements of quark currents between hadron states, which can be extracted assuming isospin symmetry from the leading (non-rare) semileptonic decay $K^+ \to \pi^0 e^+ \nu$ that is very well measured. Isospin breaking and electroweak corrections are also known [8].

The case of $K_{L,S} \to \mu^+ \mu^-$ and $K_{L,S} \to \pi^0 \ell^+ \ell^-$ is different as they are subject to long distance contributions. However, over the past years the understanding of the latter contributions has been improved by much [9–15]. In particular

- It has been demonstrated in [9, 10, 12] that with the help of the measurements of $K_S \rightarrow \pi^0 \ell^+ \ell^-$ and $K_L \rightarrow \pi^0 \gamma \gamma$ long distance (LD) contributions to both $K_L \rightarrow \pi^0 e^+ e^-$ and $K_L \rightarrow \pi^0 \mu^+ \mu^-$ can be determined. Consequently, the short distance (SD) contributions to these decays can be extracted from data as well. This is important because as stressed in [12] $K_L \rightarrow \pi^0 \mu^+ \mu^-$ and $K_L \rightarrow \pi^0 e^+ e^-$ considered simultaneously offer a powerful test of not only vector and axial-vector currents but in particular of scalar and pseudoscalar currents.
- It has been pointed out in [11] that the SD parameters of the decay $K \to \mu^+ \mu^-$ can be cleanly extracted from a measurement of the $K_L - K_S$ interference term in the time dependent rate and consequently to measure direct CP-violation in this decay.
- Subsequently it has been demonstrated in [15] that the SD contribution to $K_S \to \mu^+ \mu^-$ can be extracted from data, making it another precision observable.
- But also in the case of $K_L \to \mu^+ \mu^-$ significant progress has been made so that already for many years this decay served to bound the estimates for the $K^+ \to \pi^+ \nu \bar{\nu}$ rate in various NP scenarios [10,16–18], depending on whether NP contributions are dominated by left-handed or right-handed currents. Explicit models will be listed in the context of our paper.

The investigation of these low-energy rare decay processes in conjunction with their theoretical cleanliness allows to probe, albeit indirectly, high energy scales of the theory far beyond the reach of the LHC. They are also very sensitive to the values of the CKM parameters, in particular to V_{td} and $\text{Im}\lambda_t = \text{Im}V_{ts}^*V_{td}$ so that the latter could in principle be extracted from precise measurements of the decay rates for $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_{\rm L} \to \pi^0 \nu \bar{\nu}$, respectively. Moreover, the combination of these two decays offers one of the cleanest measurements of $\sin 2\beta$ [19] with β being one of the angles of the Unitarity Triangle. However, the very fact that these processes are based on higher order electroweak effects implies that their branching ratios are expected to be very small and not easy to access experimentally.

The large sensitivity of the decays in question to the values of the CKM parameters, in particular $|V_{cb}|$, is presently problematic in view of the tensions between the inclusive and exclusive determinations of this important CKM parameter [20–23]. Fortunately, as demonstrated recently in [24], constructing particular ratios of the branching ratios for $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ with the parameter ε_K allows to remove within the SM the dependence on $|V_{cb}|$ entirely and practically also the one due to γ leaving the dependence only on the angles β of the unitarity triangle. As the angle β is already well measured through the $S_{\psi K_S}$ asymmetry, this strategy allows to obtain the SM predictions for both decays that are most accurate to date. It can be applied to other decays [24] and we will summarize it in the context of our presentation. Yet, eventually, it will be very important to determine CKM parameters with the help of tree-level decays because taking ratios could in principle cancel out NP contributions. Moreover finding an anomaly in a ratio does not yet tell us in which of the two observables, taking part in the ratio, NP is present. In fact it could be present in both. We will also discuss this issue below.

As of 2022 one can look back at four decades of theoretical efforts to calculate the branching ratios for all these decays within the SM. Among early calculations are [25, 26] in which QCD corrections were neglected. The first LO QCD corrections have been calculated in [27, 28] and the NLO ones in the 1990s [29–33]. Already the NLO calculations reduced significantly various renormalization scale uncertainties present at LO. Yet, in the last twenty years further progress has been made through the following calculations:

- NNLO QCD corrections to the charm contributions in $K^+ \to \pi^+ \nu \bar{\nu}$: [34–36].
- Isospin breaking effects and non-perturbative effects: [8,37].
- Complete NLO electroweak corrections to the charm quark contribution to $K^+ \to \pi^+ \nu \bar{\nu}$: [38].
- Complete NLO electroweak corrections to the top quark contribution to $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$: [39].

As far as $K_{L,S} \to \mu^+ \mu^-$ and $K_L \to \pi^0 \ell^+ \ell^-$ are concerned

- The NLO QCD calculations have been performed in [30, 33] and [31], respectively and the NNLO ones for $K_L \to \mu^+ \mu^-$ in [40].
- The long-distance contributions have been investigated in [10–14].

This list of theoretical papers demonstrates very clearly the importance of these decays. Reviews on NLO and NNLO QCD corrections can be found in [41,42].

The main theoretical uncertainties in the SM predictions for the decays in question stem then from the CKM parameters, in particular $|V_{cb}|$ in the case of CP-conserving decays like $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and both $|V_{cb}|$ and $|V_{ub}|$ in the case of CP-violating ones like $K_L \rightarrow \pi^0 \nu \bar{\nu}$. But further improvements in theory would also be desirable. In particular in the case of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ an improved estimate of long distance effects in charm contributions would be welcome. Lattice QCD should be helpful in this respect and in fact first steps in this direction have been made by the RBC-UKQCD collaboration [43,44]. It is expected that in the second half of the 2020s the theoretical errors in present estimates of various branching ratios will be significantly reduced.

While, as far as the theory is concerned, the situation of the decays in question is satisfactory within the SM and actually also within a number of BSM scenarios, this is not the case on the experimental side. The main reason are the very low branching ratios which require years, even decades to be measured. In fact the first NLO QCD calculations [29–31] have been performed almost three decades ago and although the measurements of the branching ratio for $K^+ \to \pi^+ \nu \bar{\nu}$ with the error of 10% is expected in the coming years, it could still take the full decade to measure the remaining branching ratios with respectable precision.

It appears then that in this decade the main breakthrough will be made by experimentalists through the measurements of the six branching ratios in question and in particular through the measurements of the distributions in q^2 , the invariant mass-squared of the neutrino system in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ and of the $\ell^+ \ell^-$ system in the case of the remaining decays. These have already been discussed in particular in [9–15] and [45,46] but an improved analysis of them within the SM and various BSM scenarios would be very desirable so that also theorists will be able to make further advances in this field in the coming years. As we will see here also the insight from the Standard Model Effective Field Theory (SMEFT) will turn out to be useful.

Reviews of these decays can already be found in [6, 47–51] and the power of them in testing energy scales as high as several hundreds of TeV has been demonstrated in [52]. Yet, the presentation here includes also several recent insights which cannot be found in these papers.

Our paper is organized as follows. In Section 2 we list the relevant formulae for the branching ratios in question within the SM and BSM scenarios with left-handed and right-handed quark currents. We give also their estimates within the SM, stressing the issue of parametric CKM uncertainties. Furthermore, we summarize the experimental status of these decays. In Section 3 the results of [24] are reviewed, where the strong $|V_{cb}|$ dependence of the branching ratios considered by us has been eliminated in favour of ε_K . In Section 4 we discuss the differential q^2 distributions for all decays. As stressed two years ago in [45, 46] and analyzed in detail very recently in [53] these distributions in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ allow to distinguish between Dirac and Majorana neutrinos which is not possible on the basis of the branching ratios alone. In Section 5 we generalize the discussion of the previous sections beyond the SM, including right-handed currents, scalar/pseudoscalar and tensor operators. This is followed in Section 6 by the results obtained in the BSM scenarios stressing the correlations between these decays, ε'/ε , ε_K and the $K^0 - \bar{K}^0$ mass difference ΔM_K . An outlook is given in Section 7.

In the present paper when discussing BSM physics we concentrate on heavy particles with masses significantly larger than the electroweak scale. A systematic analysis for the case of light BSM physics, i.e. for NP models with new degrees of freedom lighter than the Kaon mass, can be found in [54].

2 Basic Formulae

The general formulae for the branching ratios in the presence of left-handed and right-handed vector quark currents within and beyond the SM listed below are sufficient to get an idea of the structure of various effects and the numerics. Further details, in particular the derivations of these formulae, can be found in [6] and in the original papers listed there and below.

2.1 $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$

$2.1.1 \quad K^+ o \pi^+ u ar{ u}$

Including isospin breaking corrections, summing over three neutrino flavours and generalizing the SM formulae in [8,33] to include both left-handed and right-handed quark currents one finds

$$\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu}) = \kappa_+ (1 + \Delta_{\rm EM}) \cdot \left[\left(\frac{{\rm Im} X_{\rm eff}}{\lambda^5} \right)^2 + \left(\frac{{\rm Re} \lambda_c}{\lambda} P_c(X) + \frac{{\rm Re} X_{\rm eff}}{\lambda^5} \right)^2 \right], \qquad (1)$$

where

$$X_{\rm eff} = V_{ts}^* V_{td} \left(X_L(K) + X_R(K) \right) \equiv V_{ts}^* V_{td} X_L^{\rm SM}(K) (1 + \xi e^{i\theta}).$$
(2)

Here

$$X_L(K) = X(x_t) + \Delta X_L(K) \tag{3}$$

represents contributions of left-handed currents where $\Delta X_L(K)$ denotes BSM contributions and $X(x_t)$ is the SM contribution. $X_R(K)$ represents the contributions of right-handed currents. Only *vector* parts of these currents contribute to these decays. The SM contribution is given by

$$X(x_t) = 1.462 \pm 0.017, \qquad P_c(X) = 0.405 \pm 0.024,$$
(4)

with the latter calculated for $\lambda = 0.225$. The value of $X(x_t)$ is the most recent one from [55] corresponding to the most recent value for $m_t(m_t)$ in Table 1. Examples of the BSM contributions for Z' scenarios can be found in [56].

Next

$$\lambda = |V_{us}|, \qquad \kappa_{+} = (5.173 \pm 0.025) \cdot 10^{-11} \left[\frac{\lambda}{0.225}\right]^{8}, \qquad \Delta_{\rm EM} = -0.003. \tag{5}$$

Here $x_t = m_t^2/M_W^2$, $\lambda_i = V_{is}^*V_{id}$ are the CKM factors and κ_+ summarizes all the remaining factors, in particular the relevant hadronic matrix element that can be extracted from leading semi-leptonic decays of K^+ , K_L and K_S mesons [8]. In obtaining the numerical value in (5)

$$\sin^2 \vartheta_W \equiv s_W^2 = 0.23116, \qquad \alpha(M_Z) = \frac{1}{127.9},$$
(6)

given in the $\overline{\text{MS}}$ scheme, have been used. Their errors are below 0.1% and can be neglected. Further details can be found in Section 9.5.3 of [6]. The SM prediction for the $K^+ \to \pi^+ \nu \bar{\nu}$ decay as usually quoted in the literature reads as follows [57,58]

$$\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})_{\rm SM} = (8.5^{+1.0}_{-1.2}) \times 10^{-11}, \qquad (2016). \tag{7}$$

However, as stressed in [24], this result corresponds to the values of $|V_{cb}|$ in the ballpark of inclusive determinations of this parameter and would be significantly lower if the value from exclusive determinations was used. We will return to this important issue in Section 3.

On the experimental side the NA62 experiment at CERN is presently running and is expected to measure the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ branching ratio with the precision of 10% by 2024, as described in [1,59], that would improve the accuracy of the most recent measurement by a factor of five.

This measurement from NA62 [60, 61] reads

$$\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{exp}} = (11.0^{+4.0}_{-3.5} \pm 0.3) \times 10^{-11}, \qquad (8)$$

fully consistent with the SM estimates but still leaving room for significant NP contributions.

$m_{B_s} = 5366.8(2) \mathrm{MeV}$	[62]	$m_{B_d} = 5279.58(17) \mathrm{MeV}$	[62]	
$\Delta M_s = 17.749(20) \mathrm{ps}^{-1}$	[62]	$\Delta M_d = 0.5065(19) \mathrm{ps}^{-1}$	[62]	
$\Delta M_K = 0.005292(9) \mathrm{ps}^{-1}$	[62]	$m_{K^0} = 497.61(1) \mathrm{MeV}$	[62]	
$S_{\psi K_S} = 0.699(17)$	[62]	$F_K = 155.7(3) \mathrm{MeV}$	[23]	
$ V_{us} = 0.2253(8)$	[62]	$ \epsilon_K = 2.228(11) \cdot 10^{-3}$	[62]	
$F_{B_s} = 230.3(1.3) \mathrm{MeV}$	[63]	$F_{B_d} = 190.0(1.3) \mathrm{MeV}$	[63]	
$F_{B_s}\sqrt{\hat{B}_s} = 256.1(5.7)\mathrm{MeV}$	V[64]	$F_{B_d}\sqrt{\hat{B}_d} = 210.6(5.5)\mathrm{Me}$	V[64]	
$\hat{B}_s = 1.232(53)$	[64]	$\hat{B}_d = 1.222(61)$	[64]	
$m_t(m_t) = 162.83(67) \mathrm{GeV}$	[55]	$m_c(m_c) = 1.279(13) \mathrm{GeV}$		
$S_{tt}(x_t) = 2.303$		$S_{ut}(x_c, x_t) = -1.983 \times 10$	$^{-3}$	
$\eta_{tt} = 0.55(2)$	[65]	$\eta_{ut} = 0.402(5)$	[65]	
$\kappa_{\varepsilon} = 0.94(2)$	[66]	$\eta_B = 0.55(1) \qquad [6]$	57, 68]	
$\tau_{B_s} = 1.515(4) \mathrm{ps}$	[62]	$\tau_{B_d} = 1.519(4) \mathrm{ps}$	[62]	

Table 1: Values of the experimental and theoretical quantities used as input parameters. For future updates see FLAG [63], PDG [62] and HFLAV [23].

2.1.2 $K_L ightarrow \pi^0 u ar{ u}$

Including isospin breaking corrections in relating $K_L \to \pi^0 \nu \bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$ and summing over three neutrino flavours and generalizing the SM formulae in [41,69] to include contributions from both left-handed and right-handed quark currents one finds

$$\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu}) = \kappa_L \cdot \left(\frac{\mathrm{Im} X_{\mathrm{eff}}}{\lambda^5}\right)^2,\tag{9}$$

where [8]

$$\kappa_L = (2.231 \pm 0.013) \cdot 10^{-10} \left[\frac{\lambda}{0.225}\right]^8 .$$
(10)

Due to the absence of $P_c(X)$ in (9), $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})$ has essentially no theoretical uncertainties. It is only affected by parametric uncertainties coming from m_t , $\text{Im}\lambda_t$ and κ_L of which only the one due to $\text{Im}\lambda_t$ is important.

The SM prediction for $K_L \to \pi^0 \nu \bar{\nu}$ decay usually quoted in the literature is given as follows [57, 58]

$$\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})_{\rm SM} = (3.2^{+1.1}_{-0.7}) \times 10^{-11}, \qquad (2016)$$
(11)

accompanied by the same remarks on $|V_{cb}|$ as made after (7). The most recent 90% confidence level (CL) upper bound on $K_L \to \pi^0 \nu \bar{\nu}$ from KOTO [4] reads

$$\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})_{\text{exp}} \le 3.0 \times 10^{-9} \,. \tag{12}$$

The expected measurement of $K_L \to \pi^0 \nu \bar{\nu}$ by KOTO at J-PARC [48,70] should reach the SM level by 2024. Moreover, the KLEVER experiment at CERN SPS [5,59] is expected to measure this decay in this decade.

The KOTO collaboration presented also data on four candidate events in the signal region, finding

$$\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})_{\text{KOTO}} = 2.1^{+2.0(+4.1)}_{-1.1(-1.7)} \times 10^{-9}, \qquad (13)$$

at the 68 (95) % CL. The central value is by a factor of 65 above the central SM prediction and in fact violates the Grossman-Nir bound [71] which at the 90% CL together with the present NA62 result for $K^+ \to \pi^+ \nu \bar{\nu}$ amounts to 0.8×10^{-9} . Theoretical analyses of this interesting data can be found in [72–75].

2.1.3 Interplay of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$

Beyond the SM the only unknown in (1) and (9) is the complex function X_{eff} . Once both branching ratios will be measured one day, X_{eff} will be determined model independently as follows [76]

$$\operatorname{Re} X_{\text{eff}} = -\lambda^5 \left[\frac{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})}{\kappa_+ (1 + \Delta_{\text{EM}})} - \frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{\kappa_L} \right]^{1/2} - \lambda^4 \operatorname{Re} \lambda_c P_c(X) , \qquad (14)$$

$$\operatorname{Im} X_{\text{eff}} = \lambda^5 \left[\frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{\kappa_L} \right]^{1/2} \,. \tag{15}$$

In choosing the signs in these formulae it has been assumed that NP contributions do not reverse the sign of the SM functions. For more general expressions admitting such a possibility see [77]. At the Grossman-Nir bound [71] the square root in (14) vanishes.

In Fig. 1 we show correlation between the branching ratios for $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$ in the SM for fixed values of β [24]. The SM values depend on $|V_{cb}|$ but the positions of the straight lines depend basically only on β and have negligible dependence on γ , m_t and $|V_{cb}|$ so that they have practically a universal slope. This observation made already in 1994 in [19] allows one day to determine the angle β from these two branching ratios alone and compare it with the value extracted from the mixing induced asymmetry $S_{\psi K_s}$.

In this respect an important comment is in order. A given line in Fig. 1 reminds us at first sight of the correlation between $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ branching ratios in models



Figure 1: The correlation between branching ratios for $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$ for different values of β within the SM. The ranges of branching ratios correspond to $38 \leq |V_{cb}| \times 10^3 \leq 43$ and $60^\circ \leq \gamma \leq 75^\circ$. From [24].

with MFV [77] for $X(x_t) > 0$ and in the plots showing this correlations in different models, like in [76,78], the SM value is represented by a point. But one should realize that in those papers the lines are obtained by varying X while keeping $|V_{cb}|$ and β fixed. On the other hand in Fig. 1 while X is kept at its SM value both $|V_{cb}|$ and β are varied. In other words the SM point in the plots in [76,78] and similar plots found in the literature is rather uncertain and Fig. 1 signals this uncertainty. Inspecting formulae (1) and (9) one finds that for fixed β the position on a given straight line in Fig. 1 is determined by the combination $|V_{cb}|^2 X(x_t)$. The solution to these large uncertainties has been found recently in [24] and we will report on it in Section 3.

Postponing the issue of strong $|V_{cb}|$ dependence to the next section, let us summarize what is known about the correlation between $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})$ in BSM models. In view of very small hadronic uncertainties this correlation depends fully on the short distance dynamics, represented by the two real parameters ξ and θ (2) that vanish in the SM. Measuring then these branching ratios one day will allow to determine those parameters and, comparing them with their expectations in concrete models, to obtain insight into the flavour structure of the NP contributions. Those can be dominated by left-handed currents, by right-handed currents, or by both with similar magnitudes and phases. In general one can distinguish between three classes of models [78]:

- 1. Models with a CKM-like structure of flavour interactions. If based on flavour symmetries only, they include MFV and $U(2)^3$ models [79]. In this case the function $X_L(K)$ is real and $X_R(K) = 0$. There is then only one variable to our disposal, the value of $X_L(K)$, and the only allowed values of both branching ratios are on the green branches in figure 2. But due to stringent correlations with other observables present in this class of models, only certain ranges for $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})$ are still allowed.
- 2. Models with new flavour and CP-violating interactions in which either left-handed cur-

rents or right-handed currents fully dominate, implying that left-right operator contributions to ε_K can be neglected. In this case there is a strong correlation between NP contributions to ε_K and $K \to \pi \nu \bar{\nu}$ and the ε_K constraint implies the blue branch structure shown in figure 2. On the horizontal branch the NP contribution to $K \to \pi \nu \bar{\nu}$ is real and therefore vanishes in the case of $K_L \to \pi^0 \nu \bar{\nu}$. On the second branch the NP contribution is purely imaginary and this branch is parallel to the Grossman-Nir (GN) bound [71]. In practice, due to uncertainties in ε_K , there are moderate deviations from this structure which is characteristic for the LHT model [80], or Z or Z' FCNC scenarios with either pure LH or RH couplings [52, 56].

3. If left-right operators give a significant contribution to ε_K or generally if the correlation between ε_K and $K \to \pi \nu \bar{\nu}$ is weak or absent, the two branch structure is also absent. Dependent on the values of ξ or θ , any value of $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ and $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})$ is in principle possible. The red region in figure 2 shows the resulting structure for a fixed value of ξ and $0 \leq \theta \leq 2\pi$. Randall-Sundrum models with custodial protection belong to this class of models [81]. However, it should be kept in mind that usually the removal of the correlation with ε_K requires subtle cancellations between different contributions to ε_K and consequently some tuning of the parameters [52, 81]. This presentation was rather general. We will return to this correlation in explicit models in Section 6.



Figure 2: Illustrations of common correlations in the $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ versus $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})$ plane. The expanding red region illustrates the lack of correlation for models with general LH and RH NP couplings. The green region shows the correlation present in models obeying CMFV. The blue region shows the correlation induced by the constraint from ε_K if only LH or RH couplings are present. From [76].

Unfortunately, on the basis of only these two branching ratios alone it is not possible to find out how important the contributions of right-handed currents are, as their effects are hidden in a single function X_{eff} . In this sense the decays $K_{L,S} \to \mu^+ \mu^-$, $B \to K(K^*) \nu \bar{\nu}$, as well as $B_{s,d} \to \mu^+ \mu^-$ are complementary, and the correlation between $K \to \pi \nu \bar{\nu}$ decays and the latter ones can help in identifying the presence or absence of right-handed currents.

Another important issue addressed two years ago in [45, 46] and analyzed in detail very recently in [53] is the impact of scalar currents on $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ decays. In particular such contributions signal the Majorana character of neutrinos. In the latter paper an anatomy of the impact of scalar currents on the correlation between $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ branching ratios has been made. However as emphasized in the latter paper on the basis of branching ratios only it is not possible to distinguish between Dirac and Majorana neutrinos. To this end the distributions in q^2 , the invariant-mass squared of the neutrino pair, are required and we will return to this issue briefly in Section 4.

$K_L ightarrow \mu^+ \mu^-$ 2.2

Only the so-called SD part of a dispersive contribution to $K_L \to \mu^+ \mu^-$ can be reliably calculated. Despite this limitation, this contribution puts important bounds on certain NP scenarios. In contrast to $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ now instead of the vector current the axial-vector current contributes. Relating the relevant matrix element $\langle 0|\bar{s}\gamma_{\mu}P_{L}d|K_{L}\rangle$ to the branching ratio $\mathcal{B}(K^+ \to \mu^+ \nu_{\mu})$ one finds $(\lambda = 0.2252)$

$$\mathcal{B}(K_L \to \mu^+ \mu^-)_{\rm SD} = \kappa_\mu \left(\frac{\operatorname{Re} Y_{\rm eff}}{\lambda^5} + \frac{\operatorname{Re} \lambda_c}{\lambda} P_c(Y)\right)^2, \qquad (16)$$

where

$$\kappa_{\mu} = 2.01 \cdot 10^{-9}, \qquad P_c(Y) = 0.115 \pm 0.017$$
(17)

with $P_c(Y)$ representing the charm contribution at NNLO [40].

Similar to (2)

$$Y_{\rm eff} = V_{ts}^* V_{td} \left(Y_L(K) - Y_R(K) \right), \tag{18}$$

except for the explicit minus sign that has been introduced to emphasize that this decay is governed by axial-vector currents as opposed to $K \to \pi \nu \bar{\nu}$ decays governed by vector-currents.

Here

$$Y_L(K) = Y(x_t) + \Delta Y_L(K), \qquad Y(x_t) = 0.942,$$
(19)

represent contributions of left-handed currents with $\Delta Y_L(K)$ denoting BSM contributions and $Y(x_t)$ being the SM contribution at the NNLO [82]. $Y_R(K)$ represents the contributions of right-handed currents. Examples of the BSM contributions for Z' scenarios can be found in [56].

We find then

$$\mathcal{B}(K_L \to \mu^+ \mu^-)_{\rm SD}^{\rm SM} \approx (0.8 \pm 0.1) \cdot 10^{-9}$$
. (20)

The extraction of the short distance part from data is subject to considerable uncertainties. Here the important issue is the sign of the interference of the SD dispersive part $\chi_{\rm SD}$ of the decay amplitude of $K_L \to \mu \bar{\mu}$ with the corresponding LD parts. Allowing for both signs implies a conservative bound $|\chi_{\rm SD}| \leq 3.1$ [10]. This gives then the known upper bound [10]

$$\mathcal{B}(K_L \to \mu^+ \mu^-)_{\rm SD} \le 2.5 \cdot 10^{-9},$$
 (21)

roughly three times as large as the SM value. This bound is also obtained for the sign favoured in [16,17] that implies $-1.7 \le \chi_{SD} \le 3.1$.

On the other hand the opposite sign is favoured in [18], giving $-3.1 \leq \chi_{SD} \leq 1.7$ and therefore approximately

$$\mathcal{B}(K_L \to \mu^+ \mu^-)_{\rm SD} \le \mathcal{B}(K_L \to \mu^+ \mu^-)_{\rm SD}^{\rm SM} \,. \tag{22}$$

The implications of these bounds will be discussed in Section 6. We will find there that they do not allow large enhancements of $\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})$ for models with NP governed by left-handed currents but are much less important if right-handed currents dominate NP contributions.

More recently, it has been pointed out in [11] that the SD parameters of the decay $K \rightarrow \mu^+\mu^-$ can be cleanly extracted from a measurement of the $K_L - K_S$ interference term in the time-dependent rate and consequently direct CP-violation can be measured in this decay. This brings us to the next even more interesting decay.

$2.3 \quad K_S ightarrow \mu^+ \mu^-$

The decay $K_S \to \mu \bar{\mu}$ provides a sensitive probe of imaginary parts of short-distance couplings. Its branching fraction receives LD and SD contributions, which are added incoherently in the total rate [10,83]. This is in contrast to the decay $K_L \to \mu \bar{\mu}$, where LD and SD amplitudes interfere and moreover $\mathcal{B}(K_L \to \mu \bar{\mu})$ is sensitive to the real parts of couplings. The SD part of $\mathcal{B}(K_S \to \mu \bar{\mu})$ is given as

$$\mathcal{B}(K_S \to \mu\bar{\mu})_{\rm SD} = \tau_{K_S} \frac{G_F^2 \alpha^2}{8\pi^3 \sin^4 \theta_W} m_K F_K^2 \sqrt{1 - \frac{m_\mu^2}{m_K^2}} m_\mu^2 \left[\text{Im} Y_{\rm eff} \right]^2 = 1.04 \times 10^{-5} \left[\text{Im} Y_{\rm eff} \right]^2, \quad (23)$$

with Y_{eff} given in (18).

In 2017 the LHCb collaboration improved the upper bound on $K_S \rightarrow \mu \bar{\mu}$ by one order of magnitude [84]

$$\mathcal{B}(K_S \to \mu \bar{\mu})_{\text{LHCb}} < 0.8 \,(1.0) \times 10^{-9}$$
 at 90% (95%) C.L. (24)

to be compared with the SM prediction [10, 11]

$$\mathcal{B}(K_S \to \mu\bar{\mu})_{\rm SM} = (4.99_{\rm LD} + 0.19_{\rm SD}) \times 10^{-12} = (5.2 \pm 1.5) \times 10^{-12}.$$
 (25)

There are good future prospects to improve this bound, LHCb expects [2] with 23 fb⁻¹ sensitivity to regions $\mathcal{B}(K_S \to \mu \bar{\mu}) \in [4, 200] \times 10^{-12}$, close to the SM prediction. As already mentioned previously it has been demonstrated in [15] that the short distance contribution to $K_S \to \mu^+ \mu^-$ can be extracted from data, making it another precision observable.

This is important because this decay being dominated by direct CP-violation in models with axial-vector currents is very sensitive to NP contributions as recently analysed in [85] but also earlier. See in particular analyses of Z' models [56], leptoquark models [86] and several models reviewed in [6].

$2.4 \quad K_L ightarrow \pi^0 \ell^+ \ell^-$

The rare decays $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ are dominated by CP-violating contributions. In the SM the main contribution comes from the indirect (mixing-induced) CP violation and its interference with the direct CP-violating contribution [9,87–89]. The direct CP-violating contribution to the branching ratio is within the SM in the ballpark of $4 \cdot 10^{-12}$, while the CP conserving contribution is at most $3 \cdot 10^{-12}$. Among the rare K meson decays, the decays in question belong to the theoretically cleanest, but certainly cannot compete with the $K \to \pi \nu \bar{\nu}$ decays. Moreover, the dominant indirect CP-violating contributions are practically determined by the measured decays $K_S \to \pi^0 \ell^+ \ell^-$ and the parameter ε_K . Consequently they are not as sensitive as the $K_L \to \pi^0 \nu \bar{\nu}$ decay to NP contributions, present only in the subleading direct CP violation. However, in the presence of large new CP-violating phases, the direct CP-violating contribution can become the dominant contribution and the branching ratios for $K_L \to \pi^0 \ell^+ \ell^-$ can be enhanced significantly, with a stronger effect in the case of $K_L \to \pi^0 \mu^+ \mu^-$ as already analyzed in [12, 88, 89]. But what is even more important are the correlations of these decays with $K_L \to \pi^0 \nu \bar{\nu}$ and the ratio ε'/ε which we will encounter in Section 6.

The expressions for the branching ratios are now more complicated because more operators enter the analysis and it is better to use the Wilson coefficients of involved operators than generalizing the one loop functions to include NP contributions.

We follow here [86] where the formulae in [9, 12, 88, 89] have been generalized to include NP contributions.

The rare decays in question are described by the general $\Delta F = 1$ Hamiltonian of the semi-leptonic FCNC transition of down-type quarks into leptons below the electroweak (EW) scale $\mu_{\rm EW}$

$$\mathcal{H}_{d \to d(\ell\ell,\nu\nu)} = -\frac{4G_F}{\sqrt{2}} \lambda_t^{ji} \frac{\alpha_e}{4\pi} \sum_k C_k^{baji} Q_k^{baji} + \text{h.c.}$$
(26)

with a, b being lepton indices and i, j down-quark indices. There are eight semi-leptonic operators relevant for $d_i \ell_a \rightarrow d_j \ell_b$ when considering UV completions that give rise to the SMEFT above the electroweak scale [90]

$$Q_{9(9')}^{baji} = [\bar{d}_j \gamma_\mu P_{L(R)} d_i] [\bar{\ell}_b \gamma^\mu \ell_a], \qquad Q_{10(10')}^{baji} = [\bar{d}_j \gamma_\mu P_{L(R)} d_i] [\bar{\ell}_b \gamma^\mu \gamma_5 \ell_a], Q_{S(S')}^{baji} = [\bar{d}_j P_{R(L)} d_i] [\bar{\ell}_b \ell_a], \qquad Q_{P(P')}^{baji} = [\bar{d}_j P_{R(L)} d_i] [\bar{\ell}_b \gamma_5 \ell_a].$$
(27)

The SM contribution to these Wilson coefficients is lepton-flavour diagonal

$$C_k^{baji} = C_{k,\text{SM}} \,\delta_{ba} + \frac{\pi}{\alpha_e} \frac{v^2}{\lambda_t^{ji}} \,C_{k,\text{NP}}^{baji} \,, \tag{28}$$

where $v = 246 \,\text{GeV}$ and a normalisation factor has been introduced for the NP contribution that proves convenient for matching calculations in the SMEFT. The non-vanishing SM contributions

$$C_{9,\rm SM} = \frac{Y(x_t)}{s_W^2} - 4Z(x_t), \qquad C_{10,\rm SM} = -\frac{Y(x_t)}{s_W^2}, \qquad C_{L,\rm SM} = -\frac{X(x_t)}{s_W^2}, \qquad (29)$$

are given by the gauge-independent functions $X(x_t)$, $Y(x_t)$, which we encountered in previous decays and $Z(x_t)$ is an additional one-loop function that has to be included due to the presence of QED penguins that do not contribute to the previous decays [91]. Here $s_W \equiv \sin \theta_W$.

Generalising in particular the formulae in [12] to include NP contributions and adapting them to our notations one finds, dropping scalar and pseudoscalar contributions [86]

$$\mathcal{B}(K_L \to \pi^0 \ell \bar{\ell}) = \left(C_{\rm dir}^\ell \pm C_{\rm int}^\ell \left| a_s \right| + C_{\rm mix}^\ell \left| a_s \right|^2 + C_{\rm CPC}^\ell \right) \times 10^{-12} \,. \tag{30}$$

Here [12]

$$C_{\rm dir}^{e} = (4.62 \pm 0.24) [(\omega_{7V}^{e})^{2} + (\omega_{7A}^{e})^{2}], \qquad C_{\rm int}^{e} = (11.3 \pm 0.3) \,\omega_{7V}^{e}, C_{\rm dir}^{\mu} = (1.09 \pm 0.05) [(\omega_{7V}^{\mu})^{2} + 2.32(\omega_{7A}^{\mu})^{2}], \qquad C_{\rm int}^{\mu} = (2.63 \pm 0.06) \,\omega_{7V}^{\mu},$$
(31)

and

$$C_{\text{mix}}^{e} = 14.5 \pm 0.05, \qquad C_{\text{CPC}}^{e} \simeq 0, \qquad |a_{s}| = 1.2 \pm 0.2, C_{\text{mix}}^{\mu} = 3.36 \pm 0.20, \qquad C_{\text{CPC}}^{\mu} = 5.2 \pm 1.6.$$
(32)

The SM and NP contributions enter through

$$\omega_{7V}^{\ell} = \frac{1}{2\pi} \left(P_0 + C_{9,\text{SM}} \right) \left[\frac{\text{Im}\lambda_t^{sd}}{1.407 \times 10^{-4}} \right] + \frac{1}{\alpha_e} \frac{v^2}{2} \frac{\text{Im} \left[C_{9,\text{NP}}^{\ell\ell sd} + C_{9',\text{NP}}^{\ell\ell sd} \right]}{1.407 \times 10^{-4}} \,, \tag{33}$$

$$\omega_{7A}^{\ell} = \frac{1}{2\pi} C_{10,\text{SM}} \left[\frac{\text{Im}\lambda_t^{sd}}{1.407 \times 10^{-4}} \right] + \frac{1}{\alpha_e} \frac{v^2}{2} \frac{\text{Im} \left[C_{10,\text{NP}}^{\ell\ell sd} + C_{10',\text{NP}}^{\ell\ell sd} \right]}{1.407 \times 10^{-4}}, \tag{34}$$

where $P_0 = 2.88 \pm 0.06$ [31] includes NLO QCD corrections and ℓ either e or μ .

NP contributions do not depend on λ_t^{sd} but the factor 1.407×10^{-4} is present because it has been used in [12] to obtain the numbers in (31) and (32).

The effect of NP contributions with vector and axial-vector currents is mainly felt in ω_{7A} , as the corresponding contributions in ω_{7V} cancel each other to a large extent. The case of scalar and pseudoscalar contributions is different and is briefly mentioned below.

The present experimental bounds

$$\mathcal{B}(K_L \to \pi^0 e^+ e^-) < 28 \cdot 10^{-11} \quad [92], \qquad \mathcal{B}(K_L \to \pi^0 \mu^+ \mu^-) < 38 \cdot 10^{-11} \quad [93] \tag{35}$$

are still by one order of magnitude larger than the SM predictions [12]

$$\mathcal{B}(K_L \to \pi^0 e^+ e^-)_{\rm SM} = 3.54^{+0.98}_{-0.85} \left(1.56^{+0.62}_{-0.49} \right) \cdot 10^{-11} \,, \tag{36}$$

$$\mathcal{B}(K_L \to \pi^0 \mu^+ \mu^-)_{\rm SM} = 1.41^{+0.28}_{-0.26} \left(0.95^{+0.22}_{-0.21} \right) \cdot 10^{-11}, \tag{37}$$

with the values in parentheses corresponding to the "-" sign in (30), that is the destructive interference between direct and indirect CP-violating contributions. The last discussion of the theoretical status of this interference sign can be found in [94], where the results of [88,89,95] are critically analysed. From this discussion, constructive interference seems to be favoured though more work is necessary. In view of significant uncertainties in the SM prediction it is common to use these decays to test whether the correlations of them with $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$ decays in various NP scenarios can have an impact on the latter decays. In any case there is still much room for NP contributions in these decays. For a recent theoretical study of long distance aspects of these decays including $K^{\pm} \to \pi^{\pm} \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$ see [13, 14].

Detailed numerical analyses of these formulae have been presented in [12] and in a number of papers listed in Section 5. In particular the correlation between branching ratios for the $\pi^0 e^+ e^-$ and $\pi^0 \mu^+ \mu^-$ channels can be found in Figs. 2 and 4 of that paper.

As far as scalar and pseudoscalar operators are concerned they do not play any role for $\pi^0 e^+ e^-$ but are larger for $\pi^0 \mu^+ \mu^-$. Yet, the general conclusion of [12] is that the effects of these operators are expected to be much smaller than from vector and axial-vector operators. This finding has been confirmed much later in [96]. As the measurements of both branching ratios will begin to be of interest only in the second half of this decade we only show in Fig. 4 and 6 the correlation of both branching ratios with each other and with branching ratios for $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ including only vector and axial-vector operators.

3 $|V_{cb}|$ -Independent Ratios

Already many years ago suggestions have been put forward to eliminate at least approximately the dependence on $|V_{cb}|$ from phenomenology of rare decays in the SM due to strong dependence of branching ratios on $|V_{cb}|$, in particular in K decays but also B decays [19,97]. With improved experiments and theory these ideas could be applied in B physics in [98] and generalized to many rare B and K decays in [24]. In the latter reference 16 $|V_{cb}|$ -independent ratios have been presented and summarized in Table 4 of that paper. These relations are not only independent of $|V_{cb}|$ but are theoretically very clean and depend generally only on the angles β and γ of the unitarity triangle. In certain cases they depend only on β or on γ and sometimes they are CKM independent. Table 4 in [24] summarizes these dependences. The angle β is known already with good precision and γ should be measured with a precision of 1° by LHCb and Belle II in the coming years. As these 16 relations are specific for the SM a pattern of possible future violations of these relations will give us very useful hints for the type of NP that influences rare K and B decays as well as quark mixing. Here we will concentrate on those relations that deal entirely with the Kaon system.

To this end it is useful to define the "reduced" branching ratios [19]

$$B_1 = \frac{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})}{\kappa_+ (1 + \Delta_{\rm EM})}, \qquad B_2 = \frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{\kappa_L}.$$
(38)

We find then

$$B_1 = B_2 \left[1 + \frac{1}{\sigma^2} (\cot\beta + \frac{\sqrt{\sigma}P_c(X)}{\sqrt{B_2}})^2 \right], \qquad \sigma = \left(\frac{1}{1 - \frac{\lambda^2}{2}}\right)^2.$$
(39)

This relation summarizes analytically the correlation in Fig. 1. We stress again that to an excellent approximation this relation is independent of $|V_{cb}|$, γ and m_t . Therefore it can be used in principle to determine the angle β , the sole parameter in this formula [19].

Alternatively one can derive a more transparent formula [24].

$$\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu}) = (7.92 \pm 0.30) \times 10^{-11} \left[\frac{\sin 22.2^{\circ}}{\sin \beta} \right]^{1.4} \left[\frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{2.61 \times 10^{-11}} \right]^{0.7}.$$
 (40)

This formula reproduces (39) with an accuracy in the ballpark of 4%. Consequently the ratio

$$R_0 = \frac{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})}{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})^{0.7}},\tag{41}$$

is approximately $|V_{cb}|$ -independent. Restricting the value of β to the PDG value from $S_{\psi K_S} = 22.2(7)^{\circ}$, and including all other uncertainties one finds [24]

$$(R_0)_{\rm SM} = (2.03 \pm 0.11) \times 10^{-3}$$
. (42)

Of particular interest are also the following relations [24]

$$R_{11} = \frac{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})}{|\varepsilon_K|^{0.82}} = (1.31 \pm 0.05) \times 10^{-8} \left(\frac{\sin 22.2^\circ}{\sin \beta}\right)^{0.71} \left(\frac{\sin \gamma}{\sin 67^\circ}\right)^{0.015}, \tag{43}$$

$$R_{12} = \frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{|\varepsilon_K|^{1.18}} = (3.87 \pm 0.06) \times 10^{-8} \left(\frac{\sin\beta}{\sin 22.2^\circ}\right)^{0.98} \left(\frac{\sin\gamma}{\sin 67^\circ}\right)^{0.03}, \quad (44)$$

and

$$\frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})} = (2.95 \pm 0.12) \, |\varepsilon_K|^{0.36} \left(\frac{\sin\beta}{\sin 22.2^\circ}\right)^{1.69} \left(\frac{\sin\gamma}{\sin 67^\circ}\right)^{0.015}.$$
 (45)

The first two of these formulae express explicitly the fact that combining on the one hand $K^+ \to \pi^+ \nu \bar{\nu}$ and ε_K and on the other hand $K_L \to \pi^0 \nu \bar{\nu}$ and ε_K allows within the SM to determine to a very good approximation the angle β independently of the value of $|V_{cb}|$ and γ . The last one just follows from them. Indeed the dependence on γ is very weak. An important test will be whether these two determinations of β will agree with each other.

In obtaining these formulae it was important to use the results from [65] where the significant QCD uncertainty from the pure charm contribution to ε_K has been practically removed through a clever but simple trick by using CKM unitarity differently than done until now in the literature. Moreover, the inclusion of the two-loop electroweak effects in the top contribution further increases the precision in evaluating ε_K [99].

Assuming then no NP in ε_K allows to find the most accurate SM predictions for $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ branching ratios to date [24]

$$\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})_{\rm SM} = (8.60 \pm 0.42) \times 10^{-11}, \quad \mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})_{\rm SM} = (2.94 \pm 0.15) \times 10^{-11}.$$
(46)

It should be emphasized that these predictions are $|V_{cb}|$ -independent and to an excellent accuracy γ -independent. But they obviously depend on the assumption that NP contributions to ε_K are absent.

Comparing with (7) and (11) we note a very strong reduction of the error, in particular to obtain (7) and (11) values of $|V_{cb}|$ in the ballpark of inclusive one have been used, which are controversial these days.

Similar one finds [24]

$$R_{\rm SL} = \frac{\mathcal{B}(K_S \to \mu\bar{\mu})_{\rm SD}}{\mathcal{B}(K_L \to \pi^0 \nu\bar{\nu})} = 1.55 \times 10^{-2} \left[\frac{\lambda}{0.225}\right]^2 \left[\frac{Y(x_t)}{X(x_t)}\right]^2 , \qquad (47)$$

to be independent of any SM parameter except for m_t and λ which are both precisely known. Consequently using (46) one finds

$$\mathcal{B}(K_S \to \mu \bar{\mu})_{\rm SD} = (1.85 \pm 0.10) \times 10^{-13}$$
. (48)

It should be emphasized that in obtaining the results in (46) and (48) only the absence of NP contributions to ε_K and the mixing-induced CP-asymmetry $S_{\psi K_S}$ has been assumed. No assumption about the absence of NP in *B* decays and in mass differences $\Delta M_{s,d}$ has been made and importantly no global fit to obtain these results was necessary. As stressed by the authors of [24] this allows to avoid pollution from NP and larger theoretical uncertainties in a number of *B* decays that are necessarily included in a global fit.

4 q^2 distributions

Of particular importance are differential q^2 distributions with q^2 being the invariant masssquared of the neutrino system in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ and of the $\ell^+ \ell^$ system in the case of the $K_L \to \pi^0 \ell^+ \ell^-$ decays. They are crucial for the separation of NP vector current contributions from the scalar ones which cannot be done on the basis of the branching ratios alone.

4.1 $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$

The first studies of q^2 distributions in $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ have been performed in [45, 46] and analyzed in great detail very recently in [53]. Our presentation here, based on the latter paper, is very brief and is intended only to give an of idea how these distributions look like. The interest in these distributions originates from the fact that they can distinguish between *vector* current and *scalar* current contributions. As pointed out in [45, 46] the presence of scalar current contributions would be a hint for neutrinos being Majorana particles while the usual vector contributions represent Dirac neutrinos.

Beginning with the vector current contributions, assuming neutrino flavour universality, summing over neutrino flavour and adjusting the notation in [45, 46] and [53] to ours we find for vector-currents

$$\left[\frac{d\Gamma(K^+ \to \pi^+ \nu \bar{\nu})}{dq^2}\right]_V = \frac{r^2}{2^9 \pi^3 m_{K^+}^3} \left|\lambda_c X_{\rm NNL} + X_{\rm eff}(K)\right|^2 \lambda^{3/2} (q^2, m_{K^+}^2, m_{\pi^+}^2) |f_+^{K^+}(q^2)|^2, \quad (49)$$

$$\left[\frac{d\Gamma(K_L \to \pi^0 \nu \bar{\nu})}{dq^2}\right]_V = \frac{r^2}{2^9 \pi^3 m_{K_L}^3} \left(\text{Im} X_{\text{eff}}(K)\right)^2 \lambda^{3/2} (q^2, m_{K_L}^2, m_{\pi^0}^2) |f_+^{K_L}(q^2)|^2,$$
(50)

with $X_{\text{eff}}(K)$ defined in (2), $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac)$ and

$$r = \frac{G_F}{\sqrt{2}} \frac{2\alpha}{\pi \sin^2 \theta_W}, \qquad X_{\rm NNL} = \lambda^4 P_c(X).$$
(51)

In order to be consistent with (46) for the SM contribution we use the values in (4) and

$$V_{cs}^* V_{cd} = -0.219, \qquad V_{ts}^* V_{td} = -0.20 |V_{cb}|^2 \exp(-i23^\circ), \qquad |V_{cb}| = 41.8 \times 10^{-3}.$$
 (52)

The form factor arising from the quark vector current is given by [8],

$$f_{+}^{K}(q^{2}) = f_{+}^{K}(0) \left(1 + \lambda_{+}^{\prime} \frac{q^{2}}{m_{\pi}^{2}} + \lambda_{+}^{\prime\prime} \frac{q^{4}}{2m_{\pi}^{4}} \right),$$
(53)

with

$$f_{+}^{K^{+}}(0) = 0.9778, \quad f_{+}^{K_{L}}(0) = 0.9544,$$
(54)

and $\lambda'_{+} = 24.82 \times 10^{-3}$, $\lambda''_{+} = 1.64 \times 10^{-3}$. In Fig. 3 we show the distributions (49) and (50) for the SM. They are represented by blue dashed lines. We have checked that integrating over q^2 reproduces the SM branching ratios in (46). For scalar current contributions we find using the formalism in [45,46] and [53]

$$\left[\frac{d\Gamma(K^+ \to \pi^+ \nu \bar{\nu})}{dq^2}\right]_S = |N_S^+|^2 q^2 \lambda^{1/2} (q^2, m_{K^+}^2, m_{\pi^+}^2) |f_0^{K^+}(q^2)|^2 , \qquad (55)$$

$$\left[\frac{d\Gamma(K_L \to \pi^0 \nu \bar{\nu})}{dq^2}\right]_S = |N_S^L|^2 q^2 \lambda^{1/2} (q^2, m_{K_L}^2, m_{\pi^0}^2) |f_0^{K_L}(q^2)|^2 , \qquad (56)$$

with scalar formfactors given by [8]

$$f_0^{K^+}(q^2) = f_+^{K_+}(0) \left(1 + \lambda_0 \frac{q^2}{m_{\pi^+}^2}\right), \qquad f_0^{K_L}(q^2) = f_+^{K_L}(0) \left(1 + \lambda_0 \frac{q^2}{m_{\pi^0}^2}\right), \tag{57}$$

 $f_{+}^{K^{+}}(0)$ and $f_{+}^{K_{L}}(0)$ given in (54) and $\lambda_{0} = 13.38 \times 10^{-3}$. In the left panel of Fig. 3 we compare the scalar and vector contributions for $K^{+} \to \pi^{+}\nu\bar{\nu}$ (left) and for $K_{L} \to \pi^{0}\nu\bar{\nu}$ (right). The coefficients N_{S}^{+} and N_{S}^{L} have been chosen such that the scalar and vector distributions are of comparable size. Note that we have presented our results in the center of mass frame of the Kaon for both $d\Gamma/dq^{2}$ as well as Γ_{tot} , in contrast with Fig. 6 of [46] where the former quantity is in the Lab frame. After accounting for the boost factor, for $K^{+} \to \pi^{+}\nu\bar{\nu}$ our results agree with Fig. 6 of [46].

The plots in Fig. 3 demonstrate in an impressive manner the usefulness of q^2 distributions in the distinctions between vector and scalar contributions. More details can be found in [45,46] and in particular in [53].

$4.2 \quad K_L o \pi^0 \ell^+ \ell^-$

The differential $z = q^2/m_K^2$ distribution for these decays, taking into account only potentially interesting terms, can be written as [88]

$$\frac{d\Gamma}{dz} = \frac{d\Gamma_{\rm CPC}}{dz} + \frac{d\Gamma_{\rm CPV}}{dz} , \qquad (58)$$



Figure 3: The q^2 distributions for $K^+ \to \pi^+ \nu \bar{\nu}$ (left panel) and for $K_L \to \pi^0 \nu \bar{\nu}$ (right panel) normalized to the total decay rate of the corresponding Kaon are shown. The scalar contributions are given in black whereas the vector contributions are represented by the blue, red and green lines for $X_{\rm eff} = X_{\rm SM}$, $0.5X_{\rm SM}$, $1.5X_{\rm SM}$ respectively, where $X_{\rm SM} = 1.462V_{ts}^*V_{td}$. The factors N_S^+ and N_S^L have been chose to be 10^{-12} , such that the scalar and vector contributions are of comparable size. Furthermore, the results are scaled by a factor of 10^{10} for better readability.

where

$$\frac{d\Gamma_{\rm CPC}}{dz} = \frac{\alpha^2 G_F^2 m_K^5 \beta_\pi(z) \beta_\ell(z)}{2^{11} \pi^5} \left\{ r_\ell^2 \beta_\ell^2(z) z |S_0(z)|^2 \right\},$$

$$\frac{d\Gamma_{\rm CPV}}{dz} = \frac{\alpha^2 G_F^2 m_K^5 \beta_\pi(z) \beta_\ell(z)}{2^{11} \pi^5} \left\{ r_\ell^2 z |P_0(z)|^2 + \frac{2}{3} \beta_\pi^2(z) \left(1 + \frac{2r_\ell^2}{z} \right) |V_0(z)|^2 + \left[\frac{2}{3} \beta_\pi^2(z) \left(1 + \frac{2r_\ell^2}{z} \right) + 4r_\ell^2(2 + 2r_\pi^2 - z) \right] |A_0(z)|^2 + 4r_\ell^2(1 - r_\pi^2) \operatorname{Re}\left[A_0(z)^* P_0(z) \right] \right\}.$$
(59)

Here

$$r_i = \frac{m_i}{m_K} , \quad \beta_\ell(z) = \left(1 - \frac{4r_\ell^2}{z}\right)^{1/2}, \quad \beta_\pi(z) = \left(1 + r_\pi^4 + z^2 - 2z - 2r_\pi^2 - 2zr_\pi^2\right)^{1/2}, \quad (61)$$

and the kinematical range for z reads

$$4r_{\ell}^2 \le z \le (1 - r_{\pi})^2 .$$
(62)

Moreover one has the following CPV distribution

$$\frac{dA_{\rm FB}}{dz} = \frac{\alpha^2 G_F^2 m_K^5 \beta_\pi^2(z) \beta_\ell(z)}{2^{12} \pi^5} \operatorname{Re} \left[V_0(z)^* \left(4r_\ell^2 \beta_\ell(z) S_0(z) \right) \right] .$$
(63)

With a proper normalization $A_{\rm FB}$ can be identified with the forward-backward or the energy asymmetry of the two leptons. The formfactors S_0 , P_0 , V_0 and A_0 can be found in [88]. A detailed numerical analysis of these formulae has been presented in [88] and in particular in [12] and we will not repeat it here.

5 Searching for New Physics with Rare K Decays

While precise measurements of all the channels discussed by us constitute important tests of the SM, their particular role in the coming years will be in the context of the search for NP through deviations from SM predictions. An extendable list of possible avenues for the exploration of these decays is as follows:

- To study correlations among these decays with constraints from ε_K , as already investigated in the past in particular in [19, 76, 78] but also with ε'/ε and ΔM_K investigated recently [100, 101]. Not only an efficient search for new CP-violating phases can be made in this manner but also getting a handle on right-handed currents is possible.
- Of importance are also correlations between these decays and rare *B*-meson decays, in particular with $B \to K(K^*)\nu\bar{\nu}$ [77, 102–104] but also with $B_{s,d} \to \mu^+\mu^-$.
- A classification of correlations between rare K decays and rare B decays as well as quark mixing and ε'/ε can be found in Chapter 19 of [6]. See also DNA-charts in [49], Table 10 in [56] in the context of Z' models and models with induced FCNCs mediated by Z as well as Table 1 in [81] in the case of Randall-Sundrum models.
- These decays have been analyzed in numerous SM extensions. Among the analyses performed in the previous decade of particular interest are selected models with Z' exchanges [56, 105, 106], induced FCNCs mediated by the Z boson [107, 108] as well as the analyses of these decays in the context of lepton flavour universality violation (LFUV). The latter ones are within models with vector-like quarks [58], leptoquark models [86, 109, 110] and also in $K \to \pi \nu \bar{\nu}$ decays [111] through the presence of the 3rd generation neutrinos. The tests of LFUV can also be made through $K \to \pi \ell \ell'$ and $K \to \ell \ell'$ [112].
- An important issue is the pattern of correlations within the SMEFT that are influenced by top Yukawa couplings. These effects have been left out in many papers in the past. We will return to this important issue in Section 6.
- Finally the presence of lepton number violating interactions in $K \to \pi\nu\nu$ decays would signal the Majorana character of neutrinos. As mentioned already above and analysed first in [45,46] and very recently in [53], the cases of Dirac and Majorana neutrinos can be distinguished through kinematical distributions. They are sensitive to scalar currents representing Majorana neutrinos and to NP scales as high as 20 TeV. Although neutrinoless double β decay probes higher scales, it is limited to first generations of leptons and quarks, while the rare Kaon decays in question open up a window to different quark and neutrino flavours.

NP Scenario	References	Decays
Z-FCNC	[100, 101, 107, 108, 113]	$K^+ \to \pi^+ \nu \bar{\nu}, \ K_L \to \pi^0 \nu \bar{\nu}, \ \varepsilon' / \varepsilon$
Z'	[56, 100, 101, 105, 106],	$K^+ \to \pi^+ \nu \bar{\nu}, \ K_L \to \pi^0 \nu \bar{\nu}, \ \Delta M_K, \ \varepsilon' / \varepsilon$
Simplified Models	[76]	$K_L \to \pi^0 \nu \bar{\nu}, \varepsilon' / \varepsilon$
LHT	[114-116]	All K decays
331 Models	[105]	Small effects in $K \to \pi \nu \bar{\nu}$
Vector-Like Quarks	[58]	$K^+ \to \pi^+ \nu \bar{\nu}, \ K_L \to \pi^0 \nu \bar{\nu} \text{ and } \Delta M_K$
Supersymmetry	[117-120], [121-125]	$K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$
2HDM	[126, 127]	$K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$
Universal Extra Dimensions	[128, 129]	$K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$
Randall-Sundrum models	[81, 130 - 133]	All rare K decays
Leptoquarks	[86, 109, 110]	all rare K decays
SMEFT	[101, 134]	several processes in K and B system
SU(8)	[135]	$b \to s \ell^+ \ell^-, \ K^+ \to \pi^+ \nu \bar{\nu}, \ K_L \to \pi^0 \nu \bar{\nu}$
Diquarks	[136, 137]	$\varepsilon_K, K^+ \to \pi^+ \nu \bar{\nu}, K_L \to \pi^0 \nu \bar{\nu}$
Vectorlike compositeness	[138]	$R(K^{(*)}), R(D^{(*)}), \varepsilon_K, K^+ \to \pi^+ \nu \bar{\nu}, K_L \to \pi^0 \nu \bar{\nu}$

Table 2: Papers studying rare decays including other processes.

A selected list of analyses of rare K decays in specific models can be found in Table 2. Once the experimental data improve it will be interesting to look at these papers again possibly updating the parameters and including new bounds from collider data at the LHC which should also improve in this decade. In this context one should also mention the analysis in [52] in which it has been demonstrated that rare K decays could provide some information about scales as short as 10^{-21} m, corresponding to scales in the ballpark of 100 TeV. While even much higher scales can be probed by $K^0 - \bar{K}^0$ mixing, rare K decays, considered simultaneously can give us a better insight into the Dirac structure of new interactions.

6 The Impact of BSM Physics

In [101] a SMEFT analysis involving the rare Kaon decays $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi^0 \nu \bar{\nu}$, $K_L \to \pi^0 \ell^+ \ell^-$, $K_S \to \mu^+ \mu^-$ and their correlation with the observables ΔM_K , ε_K and in particular with the ratio ε'/ε have been analysed in the context of a Z' and a flavour-violating Z model. In this section we review the findings of this comprehensive BSM study and present an update, taking into account the most recent experimental and theoretical results. While this study involved specific models, it illustrates well the general structure of correlations between various K physics observables.

6.1 SMEFT

In this subsection we briefly review the concepts of the SMEFT, which nowadays is one of the most common ways to describe NP effects. It is governed by the following Lagrangian:

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}}^{(4)} + \sum_{d=5}^{\infty} \sum_{i} \frac{C_{i}^{(d)}}{\Lambda^{d-4}} O_{i}^{(d)} , \qquad (64)$$

which contains the four-dimensional SM Lagrangian, $\mathcal{L}_{SM}^{(4)}$, as well as higher dimensional operators $O_i^{(d\geq 5)}$, weighted by their Wilson coefficients and suppressed by the NP scale Λ , at which NP effects are expected to become relevant. A complete set of non-redundant SMEFT operators up to d = 6 has first been presented in [139]. BSM contributions are now parameterized in terms of the Wilson coefficients $C_i^{(d)}$, which depend on the particular NP model. The renormalization group (RG) evolution of the SMEFT Wilson coefficients from the NP scale Λ down to the EW scale μ_W is known at the one-loop level [140–142]. At μ_W the SMEFT is commonly matched onto the Weak Effective Theory (WET), by integrating out the W, Zand Higgs boson as well as the top quark. This matching is known at the tree-level [143] and one-loop [144, 145]. The QCD and QED RG evolution of the WET at one-loop for the complete WET Lagrangian has been known already for some time [146, 147]. Recently the QCD RG evolution in WET has been extended to NLO for $\Delta F = 2$ and $\Delta F = 1$ non-leptonic decays in [148] and [149, 150], respectively. The QCD evolution in the SMEFT for $\Delta F = 1$ decays are known only at the LO but for non-leptonic $\Delta F = 2$ transitions an NLO analysis has just been completed [151].

There are several public codes on the market which deal with one or several aspects of the SMEFT and WET. A recent review can be found in [152]. In the following we will make use of the matchrunner Python package wilson [153], as well as the Mathematica package DsixTools [154,155]. Finally, we will adopt the WCxf [156] convention for the Wilson coefficients.

6.2 Setup

As a first step, we will allow for NP contributions to the observables ε'/ε and ε_K , writing

$$\frac{\varepsilon'}{\varepsilon} = \left(\frac{\varepsilon'}{\varepsilon}\right)^{\text{SM}} + \left(\frac{\varepsilon'}{\varepsilon}\right)^{\text{BSM}}, \qquad \varepsilon \equiv \varepsilon_K = e^{i\phi_\epsilon} \left[\varepsilon_K^{\text{SM}} + \varepsilon_K^{\text{BSM}}\right].$$
(65)

A non-vanishing BSM contribution to ε'/ε is motivated by the fact that the SM prediction for this ratio is presently very uncertain, which is mostly due to hadronic uncertainties. As summarized in [157], taking the present estimates from the LQCD RBC-UKQCD collaboration [158], ChPT [159] and Dual QCD (DQCD) [160, 161] into account together with isospin breaking effects, that are included only in ChPT and DQCD, a rather broad range

$$3 \times 10^{-4} \le (\varepsilon'/\varepsilon)_{\rm SM} \le 18 \times 10^{-4} \tag{66}$$

is still allowed. Compared with the experimental world average from the NA48 [162] and KTeV [163, 164] collaborations

$$(\varepsilon'/\varepsilon)_{\rm exp} = (16.6 \pm 2.3) \times 10^{-4},$$
 (67)

one is then motivated to parameterize BSM contributions to ε'/ε as follows [100]

$$\left(\frac{\varepsilon'}{\varepsilon}\right)^{\text{BSM}} = \kappa_{\varepsilon'} \cdot 10^{-3}, \qquad 0 \le \kappa_{\varepsilon'} \le 1.5.$$
(68)

We will also allow for some modest NP contribution to ε_K ,

$$(\varepsilon_K)^{\text{BSM}} = \kappa_{\varepsilon} \cdot 10^{-3}, \quad -0.2 \le \kappa_{\varepsilon} \le 0.2.$$
 (69)

The range of κ_{ε} is consistent with [65, 165, 166], but depends on whether inclusive or exclusive determinations of the CKM elements $|V_{ub}|$ and $|V_{cb}|$ are used.

Anticipating then an ε'/ε anomaly as hinted within the DQCD approach [160, 161], our main goal will be to present its implications for rare K decays within the SMEFT framework. To this end we will proceed as follows. In view of the large uncertainty of $\kappa_{\varepsilon'}$ several SM parameters will be set to their central values. We choose for instance for the CKM factors and the CKM phase δ :

$$\operatorname{Re}\lambda_t = -3.3 \cdot 10^{-4}, \qquad \operatorname{Im}\lambda_t = 1.40 \cdot 10^{-4}, \qquad \delta = 1.15,$$
(70)

being in good agreement with estimates obtained by the UTfit [165] and CKMfitter [166] collaborations.

6.3 Z': A case study

Next, in order to illustrate NP effects in a concrete NP scenario, in addition to the SM field content we will assume a new heavy Z' boson which is governed by the following interactions with the SM fermions:

$$\mathcal{L}_{Z'} = -g_q^{ij} (\bar{q}^i \gamma^{\mu} q^j) Z'_{\mu} - g_u^{ij} (\bar{u}^i \gamma^{\mu} u^j) Z'_{\mu} - g_d^{ij} (\bar{d}^i \gamma^{\mu} d^j) Z'_{\mu}$$

$$-g_\ell^{ij} (\bar{\ell}^i \gamma^{\mu} \ell^j) Z'_{\mu} - g_e^{ij} (\bar{e}^i \gamma^{\mu} e^j) Z'_{\mu} + \text{h.c.}.$$
(71)

Having this setup at hand, we will study ε'/ε , rare K decays and ΔM_K in three different scenarios of the heavy Z' boson. The left-handed scenario (LHS), in which we allow for a flavour violating coupling to the left-handed fermions, the right-handed scenario (RHS), in which flavour violation results from coupling to right-handed fermions, and thirdly the leftright scenario (LR), being a mixture of the first two scenarios. The non-zero Z' couplings to the SM fermions in these three cases read:

LHS:
$$g_q^{11,21}$$
, g_u^{11} , g_d^{11} , g_l^{11} , g_l^{12} , (72)

RHS:
$$g_q^{11}$$
, g_u^{11} , $g_d^{11,21}$, g_l^{11} , g_l^{22} , (73)

LR:
$$g_q^{11,21}$$
, g_u^{11} , $g_d^{11,21}$, g_l^{11} , g_l^{22} . (74)

Assuming these three different scenarios, the BSM contributions to the branching ratios for $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi^0 \nu \bar{\nu}$, $K_{L,S} \to \mu^+ \mu^-$ and $K_L \to \pi^0 \ell^+ \ell^-$ and to ΔM_K will be discussed in the next subsections. For this purpose we will introduce the following quantities:

$$R_{\Delta M_K} = \frac{\Delta M_K^{\rm BSM}}{\Delta M_K^{exp}}, \quad R_{\nu\bar{\nu}}^+ = \frac{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})}{\mathcal{B}(K^+ \to \pi^+ \nu \bar{\nu})_{\rm SM}}, \quad R_{\nu\bar{\nu}}^0 = \frac{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})}{\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu})_{\rm SM}}, \quad (75)$$
$$R_{\mu^+\mu^-}^S = \frac{\mathcal{B}(K_S \to \mu^+ \mu^-)}{\mathcal{B}(K_S \to \mu^+ \mu^-)_{\rm SM}}, \quad R_{\pi\ell^+\ell^-}^0 = \frac{\mathcal{B}(K_L \to \pi^0 \ell^+ \ell^-)}{\mathcal{B}(K_L \to \pi^0 \ell^+ \ell^-)_{\rm SM}}.$$

Now we discuss the different Z' scenarios defined in (72)-(74), namely the LHS, the RHS and the LR scenario. In the following we will assume a heavy Z' boson with a mass of $M_{Z'} = 10 \text{ TeV}$ to evade constraints from direct searches [167, 168].

6.3.1 LHS scenario

We start our discussion with the so called electroweak penguin (EWP) scenario defined as follows:

$$g_q^{21} \neq 0$$
, $g_u^{11} = -2g_d^{11}$, $g_\ell^{11,22} \neq 0$ (EWP scenario). (76)

This choice of parameters generates after matching and running to the EW scale the EWP operator

$$Q_8 = 6 \left(\bar{s}^{\alpha} \gamma_{\mu} P_L d^{\beta} \right) \sum_q Q_q (\bar{q}^{\beta} \gamma^{\mu} P_R q^{\alpha}) , \qquad (77)$$

with the electric charge Q_q of the quark q.



Figure 4: The EWP scenario for a Z' of 10 TeV. The correlation between the ratios for the process $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi \nu \bar{\nu}$ defined in (75) is plotted (left). The blue (orange) lines are allowed by κ_{ε} (κ_{ε} and $R_{\Delta M_K}$) constraints and the black line represents the GN bound. The correlations between the ratio for $K_L \to \pi^0 \nu \bar{\nu}$ and the ones for $K \to \pi \ell^+ \ell^-$ and $K_S \to \mu^+ \mu^-$ after imposing κ_{ε} and $R_{\Delta M_K}$ constraints are shown in the right panel.

The results are shown in Figs. 4 and 5. They are self-explanatory but let us mention several of observations:

• The ε_K constraint forces in this scenario the correlation between the branching ratios for $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ decays to take place, as seen in the left panel of Fig. 4, on the two solid blue lines [78]. But as pointed out in [101] when also the ΔM_K constraint is taken into account the horizontal line is excluded.



Figure 5: The EWP scenario for a Z' of 10 TeV. The ratios for ΔM_K and for the process $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi \nu \bar{\nu}$ defined in (75) are plotted against $\kappa_{\varepsilon'}$ (left panel) and the predictions for the ratios of the decays $K_S \to \mu^+ \mu^-$, $K_L \to \pi^0 \mu^+ \mu^-$ and $K_L \to \pi^0 e^+ e^-$ defined in eq. (75) are plotted against $\kappa_{\varepsilon'}$ (right panel).

• As seen in Fig. 5 the ε'/ε anomaly, if confirmed, will have a large impact on all decays except $K^+ \to \pi^+ \nu \bar{\nu}$.

6.3.2 RHS scenario

In this section we discuss the right-handed QCD penguin (QCDP) scenario, defined by the following choice of parameters:

$$g_q^{11} \neq 0, \qquad g_d^{21} \neq 0, \qquad (\text{RH-QCDP scenario})$$
(78)

which generates the QCP operator

$$Q_6 = 4 \left(\bar{s}^{\alpha} \gamma_{\mu} P_L d^{\beta} \right) \sum_q \left(\bar{q}^{\beta} \gamma^{\mu} P_R q^{\alpha} \right), \tag{79}$$

at the EW scale. However, this scenario is excluded due to κ_{ϵ} , originating from RG running effects from the NP scale down to the EW scale [101]. Further details to this so-called backrotation effect are discussed in [169].

6.3.3 LR scenario

Finally we discuss a combination of left-handed and right-handed flavour changing couplings, represented by the LR-EWP scenario:

$$g_q^{21}, g_d^{21} \neq 0, \qquad g_u^{11} = -2g_d^{11}.$$
 (LR-EWP scenario) (80)

In such a scenario not only the two branches like in the LHS but the full parameter space can be reached, by allowing for right-handed couplings. We show a particular example in Fig. 6. It shows the GN bound in black, together with the two-branch system in blue, which is also given in Fig. 4 (left). Imposing the constraint from ΔM_K only allows for points on the tilted Monika-Blanke branch (orange), as discussed above and also in [78]. The LR model shown in green allows to populate all parameter space, which is allowed by the GN bound. This fact is also illustrated by the red region in Fig. 2.



Figure 6: The $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi \nu \bar{\nu}$ ratios defined in (75) are shown. The LR scenario is shown in green, the LH-EWP scenario in blue and red with $\varepsilon_K \in [-0.2, 0.2]$ and [-0.5, 0.5] for a Z' of 3 TeV. Points satisfying the $R_{\Delta M_K} \in [-1.0, 0]$ constraint are shown in orange. The black line represents the GN bound.

6.4 Z from Z'

In this section an example is shown, where flavour violating effects result from modified Zcouplings through RG mixing. Choosing the following parameters:

$$g_q^{21} \neq 0, \quad g_u^{11} = g_d^{11} = 0, \quad g_q^{33} = 2,$$
(81)

induces through large Yukawa running effects flavour-violating Z - s - d couplings. The effects of such new couplings are shown for the different observables in Fig. 7.

The scenario has the largest impact on the observable ΔM_K , which is reduced for positive NP contributions to ε'/ε . Also $K_L \to \pi^0 \nu \bar{\nu}$ becomes smaller for a constructive contribution to ε'/ε . All the other rare Kaon decays and in particular $K^+ \to \pi^+ \nu \bar{\nu}$ are to a good approximation not affected in this scenario.



Figure 7: Shown are the Z-contributions to ε'/ε and other Kaon observables, generated from a Z' with purely left-handed quark couplings through RG running.

7 Outlook

In our paper we have concentrated on rare K decays that will be measured in this decade at various laboratories. In the coming years the main role will be played by the measurements of $K^+ \to \pi^+ \nu \bar{\nu}$ by the NA62 collaboration at CERN SPS [1], and for $K_L \to \pi^0 \nu \bar{\nu}$ by KOTO [4] at J-PARC and later by KLEVER [5] at CERN SPS. Already these measurements will allow for a deep insight into possible NP at short distances, in particular if also q^2 distributions will be measured. Only in the second half of this decade we will be able to benefit from the measurements of $K_S \to \mu \bar{\mu}$ and $K_L \to \pi^0 \ell^+ \ell^-$ decays.

But it should be emphasized, as done in particular in [6, 49], that an important role in the search for NP is played by correlations between decays considered by us and other observables. Therefore this decade should be very exciting for flavour phenomenology, not only because of the decays considered here but also due to *B* decays explored at Belle II [170] and LHCb [2,3]. Moreover also ATLAS and CMS will contribute in an important manner [2,171] and generally BSM searches beyond colliders at CERN [172]. Moreover, with the advances in LQCD it will be possible to make clear cut conclusions about the presence of NP in processes in which hadronic effects play an important role [2,173–175].

Acknowledgements

We thank Gino Isidori, Julia Harz and Elena Venturini for the discussions. J. A. acknowledges financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement 833280 (FLAY), and from the Swiss National Science Foundation (SNF) under contract 200020-204428. A.J.B acknowledges financial support from the Excellence Cluster ORIGINS, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2094 – 390783311. J.K. is financially supported by the Alexander von Humboldt Foundation's postdoctoral research fellowship.

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