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Supersymmetric Lattice Theories: Contribution to Snowmass 2022

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Abstract: In this white paper we summarise the construction and applications of lattice theories possessing exact supersymmetry focusing, in particular, on $\mathcal{N} = 4$ Yang-Mills theory. Lattice formulations of this theory allow for numerical simulation of the theory at strong coupling and hence give a window on non-perturbative physics away from the planar limit. This has important applications to our understanding of holographic approaches to quantum gravity and conformal field theories.

Contents

1 Executive summary

In this white paper we summarise the construction and applications of lattice theories possessing exact supersymmetry focusing, in particular, on $\mathcal{N} = 4$ Yang-Mills theory. Lattice formulations of this theory allow for numerical simulation of the theory at strong coupling and hence give a window on non-perturbative physics away from the planar limit. This has important applications to our understanding of holographic approaches to quantum gravity and conformal field theories. In particular:

- We find that quantities scale with the 't Hooft coupling λ in a way that is consistent with holography. In particular, Wilson loops scale as $\exp(-c\sqrt{\lambda})$, where c is some constant.
- Success in this regime opens the door to other interesting studies at strong coupling and away from the planar limit including tests of S-duality, computations of the dimension of the Konishi operator and calculations of string loop corrections to classical supergravity.
- Such calculations can also help bridge to other theoretical efforts such as the scattering amplitudes and conformal bootstrap programs.

2 Review

In recent years a new approach to the problem of formulating supersymmetric lattice theories has been developed with the result that a certain class of supersymmetric theory can be discretized while preserving one or more supercharges at non-zero lattice spacing. These theories can be derived in two independent ways; by exploiting orbifold and deconstruction techniques or by careful discretization of a topologically twisted formulation of the target supersymetric theory [1] [1].

[1] Actually the orbifold methods only yield Yang-Mills theories while the topological constructions are also capable of describing Wess-Zumino models

In the case of $\mathcal{N}=4$ Yang-Mills the resultant lattice action is

$$S = \frac{N}{4\lambda}\mathcal{Q}\sum_x \text{Tr}\,\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a + \frac{1}{2}\eta d + \kappa\,\eta\,\left(\text{Re}\det\left[\mathcal{U}_a(x)\right]-1\right)\right) + S_{\text{closed}} \tag{2.1}$$

where the lattice field strength

$$\mathcal{F}_{ab}(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+\hat{a}) - \mathcal{U}_b(x)\mathcal{U}_a(x+\hat{b}) \tag{2.2}$$

where $\mathcal{U}_a(x)$ denotes a *complexified* gauge field living on the lattice link running from $x \to x+\hat{a}$ and where $\hat{a}$ denotes one of the five basis vectors of an underlying A_4^* lattice. Similarly

$$\overline{\mathcal{D}}_a\mathcal{U}_a = \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-\hat{a})\mathcal{U}_a(x-\hat{a}). \tag{2.3}$$

The five fermion fields ψ_a, being superpartners of the gauge fields, live on the corresponding links, while the ten fermion fields $\chi_{ab}(x)$ are associated with new face links running from $x+\hat{a}+\hat{b} \to x$. The scalar fermion $\eta(x)$ lives on the lattice site x and is associated with a conserved supercharge $\mathcal{Q}$ which acts on the fields in the following way[2]

$$\begin{aligned}
\mathcal{Q}\,\mathcal{U}_a &\to \psi_a \\
\mathcal{Q}\,\psi_a &\to 0 \\
\mathcal{Q}\,\eta &\to d \\
\mathcal{Q}\,d &\to 0 \\
\mathcal{Q}\,\chi_{ab} &\to \overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\,\overline{\mathcal{U}}_a &\to 0
\end{aligned} \tag{2.4}$$

Notice that $\mathcal{Q}^2 = 0$ which guarantees the supersymmetric invariance of the first part of the lattice action. The auxiliary site field $d(x)$ is needed for nilpotency of $\mathcal{Q}$ offshell. The second term S_{closed} is given by

$$S_{\text{closed}} = -\frac{N}{16\lambda}\sum_x \text{Tr}\,\epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \tag{2.5}$$

where the covariant difference operator acting on the fermion field χ_{de} takes the form

$$\overline{\mathcal{D}}_c\chi_{de}(x) = \overline{\mathcal{U}}_c(x-\hat{c})\chi_{de}(x+\hat{a}+\hat{b}) - \chi_{de}(x-\hat{d}-\hat{e})\overline{\mathcal{U}}_c(x+\hat{a}+\hat{b}) \tag{2.6}$$

To retain exact supersymmetry all fields reside in the algebra of the gauge group – taking their values in the adjoint representation of $U(N)$: $f(x) = \sum_{A=1}^{N^2} T^A f^A(x)$ with $\text{Tr}\,(T^A T^B) = -\delta^{AB}$. The latter term can be shown to be supersymmetric via an exact lattice Bianchi identity $\epsilon_{abcde}\overline{\mathcal{D}}_c\chi_{de} = 0$. This action is invariant under $\mathcal{Q}$, $SU(N)$ lattice gauge invariance and the S^5 point group symmetry of the A_4^* lattice.[3] Carrying

[2] One of the things that is learned from the orbifold construction is that the number conserved supercharges is equal to the the number of site fermions.

[3] Notice that there are five lattice vectors, $\hat{a} = \hat{1}, \ldots, \hat{5}$, corresponding to the nearest-neighbor links of the A_4^* lattice, and the fact that we have five complexified "gauge fields." The A_4^* lattice is four-dimensional, in spite of having five primitive vectors.

out the $\mathcal{Q}$ variation and integrating out the auxiliary field d we obtain the supersymmetric lattice action $S = S_b + S_f$ where

$$S_b = \frac{N}{4\lambda} \sum_x \text{Tr} \left(\mathcal{F}_{ab}\overline{\mathcal{F}}_{ab}\right) + \frac{1}{2}\text{Tr} \left(\overline{\mathcal{D}}_a\mathcal{U}_a - \kappa \left[\text{Re}\det\left[\mathcal{U}_a(x)\right] - 1\right]^2\right) \tag{2.7}$$

and

$$S_f = -\frac{N}{4\lambda} \sum_x \left(\text{Tr}\ \chi_{ab}\mathcal{D}_{[a}\psi_{b]} + \eta\overline{\mathcal{D}}_a\psi_a - \frac{\kappa}{2}\text{Tr}\ (\eta)\det\left(\mathcal{U}_a(x)\right)\text{Tr}\ \left(\mathcal{U}_a^{-1}(x)\psi_a(x)\right)\right) + S_{\text{closed}} \tag{2.8}$$

In the continuum this action can be obtained by discretization of the Marcus or GL twist of $\mathcal{N} = 4$ Yang-Mills but in flat space is completely equivalent to it. In the continuum the twist is done as a prelude to the construction of a topological quantum field theory but in the context of lattice supersymmetry it is merely used as a change of variables that allows for discretization while preserving a single exact supersymmetry. The twisting removes the spinors from the theory replacing them by the antisymmetric tensor fields η, ψ_a, χ_{ab} which appears as components of a Kähler-Dirac field. The latter is equivalent at zero coupling to a (reduced) staggered field and hence describes four physical Majorana fermions in the continuum limit - as required for $\mathcal{N} = 4$ Yang-Mills. The twisting procedure also packs the six scalar fields of the continuum theory together with the four gauge fields into five complex gauge fields corresponding to the lattice fields $\mathcal{U}_a$. The coupling κ is needed to project the theory from $U(N)$ to $SU(N)$ and thereby evade instability issues that otherwise would arise at strong coupling.

General arguments have been put forward that the theory should approach the continuum $\mathcal{N} = 4$ theory after tuning a single marginal operator [2]. The theory can be simulated using the same algorithms that are employed for lattice QCD [3–5]. [4] It has also been used to explore the physics of black holes and gauge-gravity duality in lower dimensions [7–16]. There is one final wrinkle that needs to be mentioned. To regulate the flat directions of the theory to do simulations it is necessary to add a soft supersymmetry breaking term of the form

$$S_{\text{mass}} = \mu^2 \sum_x \text{Tr} \left(\overline{\mathcal{U}}_a(x)\mathcal{U}_a(x) - I\right)^2 \tag{2.9}$$

While this breaks the exact supersymmetry softly all counter terms induced by this breaking will have couplings that are multiplicative in μ^2 and hence vanishing as $\mu^2 \to 0$.

3 Conformal invariance and holography

$\mathcal{N} = 4$ Yang-Mills is thought to be a non-trivial conformal field theory for any value of the 't Hooft coupling. Simulations are consistent with this and show a single phase theory with vanishing string tension. Furthermore, the theory can be solved in the planar limit $N \to \infty$ and exhibits a non-trivial dependence on the 't Hooft coupling λ. Specifically circular

[4]The theory does not appear to suffer from a sign problem although the exact reasons for this are not well understood [6].

supersymmetric Wilson loops $W_{\rm susy}$ in the planar strong coupling limit are independent of size and depend only on $\sqrt{\lambda}$ [17, 18].

$$\ln W_{\rm susy} = {\rm const}\sqrt{\lambda}$$

This result was first derived by exploiting holography to relate this Yang-Mills theory to classical supergravity in five dimensional AdS space.

The characteristic $\sqrt{\lambda}$ dependence can also be seen in the results of numerical simulations at strong coupling *even for small numbers of colors* - see fig. 1 which plots the logarithm of the square lattice Wilson loop constructed from $\mathcal{U}_a$ as a function of $\sqrt{\lambda}$ for $N = 2$. The dependence on loop size R reflects the presence of a constant perimeter term

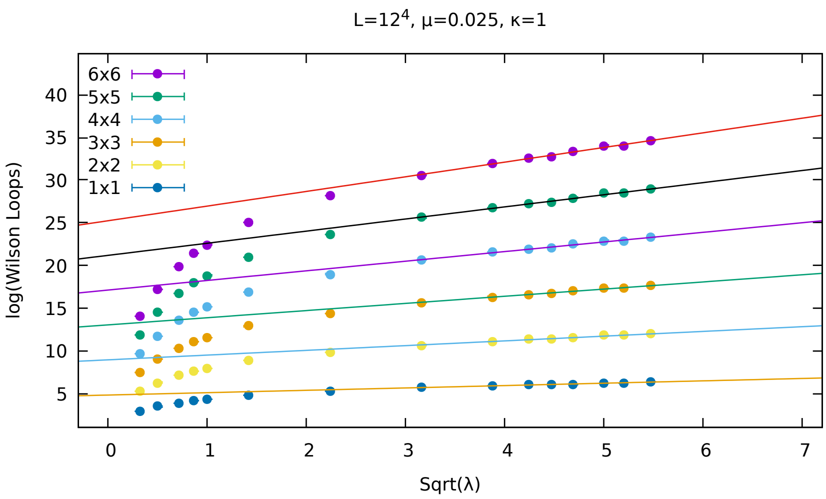

Figure 1. Supersymmetric $n \times n$ Wilson loops on 12^4 lattice at $\mu = 0.025$

in the static potential arising from the (static) quark mass [19]. Indeed if this is subtracted out by normalizing the Wilson loops by appropriate powers of the Polyakov line one obtains the plot in fig. 2 which exhibits both an insensitivity to loop size and also the $\sqrt{\lambda}$ behavior expected from holography. The strange $\sqrt{\lambda}$ dependence *cannot* be seen in perturbation theory and this result is a very non-trivial test of the correctness of the lattice approach in a non-perturbative regime.

4 Future Directions - executive summary

Supersymmetric lattice actions can be formulated which conserve one or more continuum supersymmetries and flow to the continuum theory with minimal tuning as the lattice spacing is sent to zero. One of the most interesting examples that has been studied is $\mathcal{N} = 4$ super Yang-Mills. Results that have been obtained so far are consistent with a single conformally invariant phase for any value of the 't Hooft coupling and agree with holographic predictions for Wilson loops even for small numbers of colors – an unexpected and non-trivial result. Future work will focus on a variety of outstanding issues

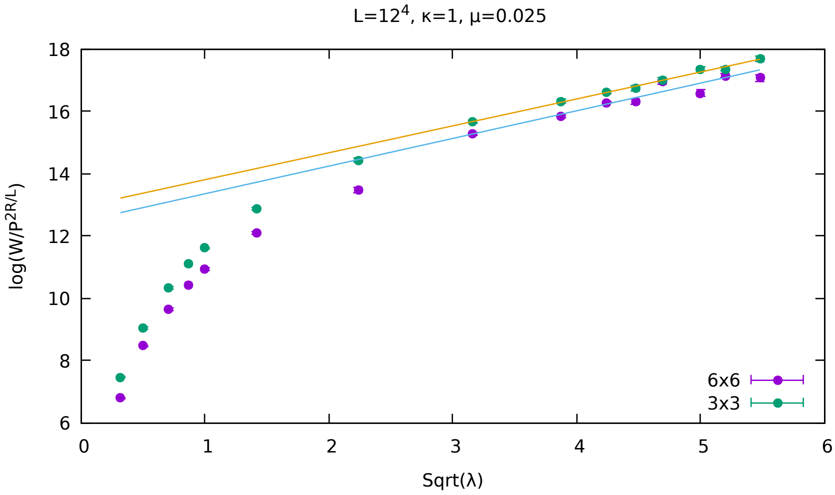

Figure 2. Renormalized supersymmetric 6×6 and 3×3 Wilson loops on 12^4 lattice at $\mu = 0.025$

- Look for precise numerical agreement of the lattice and continuum results for supersymmetric Wilson loops in the planar limit at strong coupling.
- Explore whether fine tuning is indeed needed to restore the remaining supersymmetries in the continuum limit.
- Compute the Konishi operator and supergravity operator scaling dimensions that characterize the conformal behavior of the theory for arbitary numbers of colors comparing with bootstrap and planar calculations. Here it is important to take into account the impact of discretization on the $SU(4)_R \simeq SO(6)$ flavor symmetry.
- Search for evidence of S-duality in the lattice theory by measuring gauge boson and monopole masses in the Coulomb phase of the lattice theory. Here one has a precise, BPS-protected formula to compare to.

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