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# Resolving the Inflationary Power Spectrum

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Recently there have been differing viewpoints on how to evaluate the curvature power spectrum generated during inflation. In a series of papers by Parker and collaborators it has been argued that the renormalization scheme adopted for the inflaton field  $\varphi(x)$  to make  $\langle\varphi^2(x)\rangle$  finite should also be applied to  $|\varphi_k|^2$ . But this then modifies the curvature power spectrum in a non-trivial way. On the other hand, others (Durrer, Marozzi and Rinaldi) have criticized this approach and suggested alternatives, which have been further countered by Parker et al. We discuss these differing viewpoints and indicate inconsistencies in both approaches. We then resolve the issue by showing why the standard expression, without any non-trivial regularization, is still valid.

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# 1 Introduction

In the standard inflationary Universe quantum fluctuations of the inflaton field give rise to a curvature perturbation that is constant for modes outside the horizon. This curvature perturbation is then the seed for structure formation in the Universe. For the inflaton field  $\varphi$  given by

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k [a_k \varphi_k(t) e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger \varphi_k^*(t) e^{-i\vec{k}\cdot\vec{x}}] \quad (1)$$

the gauge invariant curvature perturbation generated during inflation on superhorizon scales is given (in the spatially flat gauge) by

$$\zeta_k = \frac{1}{3} \frac{\delta\rho(k)}{\rho + p} \quad (2)$$

$$= \frac{1}{3} \frac{V'(\varphi_0) \delta\varphi(k)}{\dot{\varphi}_0^2} \quad (3)$$

where  $\delta\varphi(k)^2 = [k^3/(2\pi^2)] |\varphi_k|^2$  and  $\varphi_0$  represents the classical homogeneous background. For a very flat inflaton potential the inflaton can be taken to be massless and  $\delta\varphi(k) = H/(2\pi)$  where  $H$  is the Hubble parameter during inflation. The curvature power spectrum is defined as  $|\zeta_k|^2$ .

In a series of papers [1, 2, 3, 4, 5, 6, 7, 8], it has been argued that the regularisation and renormalisation scheme adopted to make  $\langle\varphi^2(x)\rangle = 1/(2\pi)^3 \int d^3k |\varphi_k|^2$  finite should also be applied when considering  $\delta\varphi(k)^2$ . Then in adiabatic regularisation the subtraction scheme applied to the integrand in  $\langle\varphi^2\rangle$  should be retained while obtaining  $\delta\varphi(k)^2$ , and so  $\delta\varphi(k)^2 = k^3/(2\pi^2) [|\varphi_k|^2 - |\varphi_K|^2]$ , where  $\varphi_K$  is the adiabatic solution to the second order. This then modifies the power spectrum since the subtraction scheme which cancels the contribution of high momentum modes in  $\langle\varphi^2(x)\rangle$  also modifies the contribution of the superhorizon low momentum modes. As argued in Ref. [2, 3, 4], this reduces the amplitude of the power spectrum for a massive inflaton, retains the scale free nature of the spectrum, modifies the tensor-scalar ratio  $r$ , and allows for the compatibility of quartic chaotic inflation with data.

However, it was argued in Ref. [9] that while the fluctuation mode functions are constant outside the horizon the adiabatic solution is not and so the power spectrum then depends on the time after horizon crossing at which the power spectrum is evaluated. It was also argued that different adiabatic subtraction schemes gave different results. It was therefore concluded that one should carry out adiabatic subtraction only for high momentum modes.

The above result was countered in Ref. [5] by arguing that adiabatic regularisation required subtracting the adiabatic solution for all modes, not just high momentum modes. The authors further argued that their adiabatic subtraction scheme differed

from that in Ref. [9], and that their scheme agreed with de Witt-Schwinger renormalisation (in momentum space in the massless limit) which identifies counterterms without invoking any adiabatic condition.

Ref. [10] then argued that the de Witt-Schwinger expansion is relevant for large momentum modes but is not valid for superhorizon modes that leave the horizon. This was further countered by Ref. [8] wherein it was re-emphasised that adiabatic subtraction must be applied to all modes, that the energy momentum tensor and  $\langle\varphi^2\rangle$  require mode subtractions at the 4th and 2nd order respectively, and that the adiabatic solution at the appropriate order need not approximate the solution for all momenta.

Since the curvature power spectrum is an essential ingredient in the process of extracting early Universe parameters from current observations, it is important that the above issues be resolved and that there is clarity on what is the appropriate expression for the power spectrum. We comment on some issues related to both viewpoints on obtaining the power spectrum and then present arguments as to why the standard expression in the literature is appropriate [11].

## 2 The power spectrum

The argument in Refs. [9, 10] on applying the subtraction scheme only to high momentum modes is equivalent to introducing a time dependent cutoff such as  $\Theta(k - aH)$  to subtract only high momentum modes while calculating  $\langle\varphi^2(x)\rangle$ . (Refs. [9, 10] actually calculate  $\langle Q^2(x)\rangle$ , where  $Q$  is the Mukhanov-Sasaki variable.) Now for a rigid spacetime ignoring metric perturbations the equation of motion for  $\varphi_k$  implies

$$\dot{\rho}_k = -3H(\rho_k + p_k). \quad (4)$$

Integrating over all  $k$  modes then gives

$$\dot{\rho}_\varphi = -3H(\rho_\varphi + p_\varphi). \quad (5)$$

But if we replace  $\rho_\varphi$  and  $p_\varphi$  by renormalised quantities  $\rho_{\text{ren}}$  and  $p_{\text{ren}}$  with the contribution of high momentum modes cut off at  $k = a(t)H$ , then this time dependent cutoff spoils the equality above because the time derivative on the left hand side of Eq. (5) acts on the cutoff too.

$$\dot{\rho}_{\text{ren}} = \frac{d}{dt} \int \frac{d^3k}{(2\pi)^{3/2}} [\rho_k - \Theta(k - aH)\rho_K] e^{i\vec{k}\cdot\vec{x}} \quad (6)$$

and

$$\begin{aligned} -3H(\rho_{\text{ren}} + p_{\text{ren}}) = & -3H \int \frac{d^3k}{(2\pi)^{3/2}} [\rho_k - \Theta(k - aH)\rho_K \\ & + p_k - \Theta(k - aH)p_K] e^{i\vec{k}\cdot\vec{x}} \end{aligned} \quad (7)$$

where the subscript  $K$  refers to the adiabatic solution. With the adiabatic solution cancelling (to the relevant adiabatic order) the high momentum contribution we then get

$$\begin{aligned}
\dot{\rho}_{\text{ren}} &= \frac{d}{dt} \int_0^{a(t)H} \frac{d^3k}{(2\pi)^{3/2}} \rho_k e^{i\vec{k}\cdot\vec{x}} \\
&\neq -3H(\rho_{\text{ren}} + p_{\text{ren}}) \\
&= -3H \int_0^{a(t)H} \frac{d^3k}{(2\pi)^{3/2}} [\rho_k + p_k] e^{i\vec{k}\cdot\vec{x}} \tag{8}
\end{aligned}$$

because of the contribution of the time derivative acting on the upper limit of the first integral. This suggests that a regularisation prescription, as proposed by Refs. [9, 10], that only subtracts the high momentum modes is not appropriate.

One may now question whether regularisation and renormalisation itself are relevant for the power spectrum, as insisted on by Refs. [1, 2, 3, 4, 5, 6, 7, 8]. After all, the curvature power spectrum depends on  $\delta\varphi(k)$  and not  $\langle\varphi^2(x)\rangle$ , and it is the latter that involves the divergent integral over  $k$ . This issue can be resolved by identifying the quantity that enters in physical observables or in expressions derived from physical observables. Let us consider the cosmic microwaved background (CMB) temperature anisotropy variable

$$C_l = \frac{1}{4\pi} \int d^2\hat{n} d^2\hat{n}' P_l(\hat{n}\cdot\hat{n}') \langle\Delta T(\hat{n})\Delta T(\hat{n}')\rangle \tag{9}$$

where  $\Delta T(\hat{n}) = T(\hat{n}) - T_0$  represents the difference in temperature of the CMB in a direction  $\hat{n}$  from the mean temperature  $T_0$ .  $\langle\Delta T(\hat{n})\Delta T(\hat{n}')\rangle$  above is obtained from observations. \* Then using

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)_{SW} = \frac{1}{3} \delta\phi(\hat{n}r_L), \tag{10}$$

where  $r_L$  is the distance to the surface of last scattering,  $\delta\phi$  is the perturbation in the gravitational potential and  $SW$  refers to the Sachs-Wolfe effect, we get

$$\begin{aligned}
C_l &\sim \dots \langle\delta\phi(\hat{n}r_L)\delta\phi(\hat{n}'r_L)\rangle \\
&\sim \dots \int d^3q d^3q' e^{i\vec{q}\cdot\hat{n}r_L} e^{i\vec{q}'\cdot\hat{n}'r_L} \langle\delta\phi_{\vec{q}}\delta\phi_{\vec{q}'}\rangle \\
&\sim \dots \int d^3q d^3q' e^{i\vec{q}\cdot\hat{n}r_L} e^{i\vec{q}'\cdot\hat{n}'r_L} P_\phi(q) \delta(\vec{q} + \vec{q}') \\
&\sim \dots \int d^3q e^{i\vec{q}\cdot\hat{n}r_L} e^{-i\vec{q}\cdot\hat{n}'r_L} P_\phi(q), \tag{11}
\end{aligned}$$

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\*More precisely, we measure  $\Delta T(\hat{n})\Delta T(\hat{n}')$ . The difference gives rise to cosmic variance [12], which we ignore here.

where  $\langle \delta\phi_{\vec{q}} \delta\phi_{\vec{q}'} \rangle = P_\phi(q) \delta(\vec{q} + \vec{q}')$  and  $P_\phi(q)$  is the power spectrum associated with  $\delta\phi$ . Thus we see that it is the coordinate space correlation function of the gravitational potential perturbation that is primary. Since the gravitational potential perturbation is related to quantum fluctuations of the inflaton we would argue that the relevant quantity for physical observables is the inflaton correlation function in coordinate space, and this must be renormalised and finite. Then, as argued in Refs. [1, 2, 3, 4, 5, 6, 7, 8], the power spectrum should reflect the renormalisation prescription for the coordinate space correlation function for the inflaton field.

But Eq. (11) indicates that  $C_l$  actually depends on the correlation function of the inflaton at two different points in space. So we would argue that  $\langle \varphi(\vec{x}, t) \varphi(\vec{y}, t) \rangle$  is the relevant quantity to be used to obtain the power spectrum for  $\varphi$ , and so the power spectrum should reflect the renormalisation prescription, if any, for  $\langle \varphi(\vec{x}, t) \varphi(\vec{y}, t) \rangle$ , rather than for  $\langle \varphi^2(x) \rangle$ . Now

$$\langle \varphi(\vec{x}, t) \varphi(\vec{y}, t) \rangle = \frac{1}{2\pi^2} \int dk k^2 \left[ \frac{1}{2ka^2} + \frac{H^2}{2k^3} \right] \frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|} \quad (12)$$

This quantity does not require renormalisation as the sine function makes the integral ultraviolet finite. <sup>†</sup> Therefore there will be no need of adiabatic subtraction and hence no modification of the integrand. Then associating  $\delta\varphi(k)$  with the expression in brackets in Eq. (12) we get the standard expression for  $\delta\varphi(k) = H/(2\pi)$ , and thus for the primordial curvature power spectrum. Note that if we define the power spectrum using Eq. (12) then both the terms in the brackets will be included but only the second term contributes in the large wavelength limit,  $k \ll aH$ .

We believe that the above prescription might be the appropriate way of obtaining the power spectrum generated during inflation. The power spectrum is also not time dependent as in the prescription of Refs. [1, 2, 3, 4, 5, 6, 7, 8].

In the literature different authors define the power spectrum using either the integrand of  $\langle \varphi^2(x) \rangle$  or of  $\langle \varphi(\vec{x}, t) \varphi(\vec{y}, t) \rangle$ . If one is ignoring renormalisation of these quantities then both approaches give the same momentum space power spectrum. But, as we clarify above, the spatial correlation function enters in the expression for physical observables like  $C_l$  and so one must consider renormalised quantities in coordinate space, and hence in momentum space too, as argued in Refs. [1, 2, 3, 4, 5, 6, 7, 8]. However, the spatial correlation function that is relevant is  $\langle \varphi(\vec{x}, t) \varphi(\vec{y}, t) \rangle$ , not the divergent  $\langle \varphi^2(x) \rangle$  which is considered in Refs. [1, 2, 3, 4, 5, 6, 7, 8], and the former quantity does not require regularisation. We thus get the standard expression for the power spectrum.

We must add here that we have only temporarily set aside the necessity of renormalisation of  $\langle \varphi^2(x) \rangle$ . In an interacting theory, our prescription above is relevant for

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<sup>†</sup> Note that the first term is present even in flat spacetime, and is finite and equal to the equal time Feynman propagator for a massless scalar field, namely  $i/(4\pi^2|\vec{x} - \vec{y}|^2)$ , in Minkowski spacetime [13].

calculating the power spectrum only to lowest order. For example, in a  $\lambda\varphi^4$  theory

$$\langle\varphi(x)\varphi(y)\rangle_{\text{int}} = \langle\varphi(x)\varphi(y)\rangle + i\lambda \int d^4z \langle\varphi(x)\varphi(y)\varphi^4(z)\rangle \quad (13)$$

and the second term will be proportional to  $\langle\varphi^2(z)\rangle$ , which will require renormalisation. (Note, however, that a cubic self interaction will not require such renormalisation of the correlation function.) Renormalisation of  $\langle\varphi^2(x)\rangle$  will also be needed for obtaining the renormalised energy momentum tensor in free and interacting field theories.

We mention in passing that the expression for  $\langle\varphi(\vec{x}, t)\varphi(\vec{y}, t)\rangle$  has an infrared divergence just like  $\langle\varphi^2(x)\rangle$ . However for realistic inflation models the inflaton may have a mass, albeit small, or the mass may even be generated non-perturbatively [14], or inflation may be preceded by a radiation dominated era, which should remove the infrared divergence.

### 3 Conclusion

In conclusion, in this talk we have discussed two differing viewpoints on obtaining the power spectrum generated during inflation. We point out that subtracting only the contribution of high momentum modes to  $\langle\varphi^2\rangle$ , as suggested by Refs. [9, 10], may not be appropriate as one does not obtain the standard energy equation for renormalised quantities. However we also point out that, keeping in mind physical observables, it is more relevant to obtain the power spectrum from  $\langle\varphi(\vec{x}, t)\varphi(\vec{y}, t)\rangle$ , rather than from  $\langle\varphi^2(x)\rangle_{\text{ren}}$  as suggested in Refs. [1, 2, 3, 4, 5, 6, 7, 8]. Our prescription then gives the standard expression for the power spectrum and thereby validates the results in the literature based on this expression.

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