Wilson-loop correlators on the lattice and asymptotic behaviour of hadronic total cross sections

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We show how universal, Froissart-like hadronic total cross sections can be obtained in QCD in the functional-integral approach to soft high-energy scattering, and we discuss indications of this behaviour obtained from lattice simulations.

1 Introduction

The recent measurements of hadronic total cross sections at the LHC [1] have revived the interest in trying to understand their behaviour at large energy from the theoretical point of view. Experimental data support the “Froissart-like” rising behaviour

\[ \sigma_{\text{tot}}(s) \sim B \log^2 s \]

at large energy, with a universal prefactor \( B \), i.e., independent of the type of hadrons involved in the scattering process [2]. This behaviour is consistent with (and named after) the well-known Froissart-Lukaszuk-Martin bound, which states that

\[ \sigma_{\text{tot}}(s) \lesssim (\pi/m_{\pi}^2) \log^2 (s/s_0) \]

for \( s \rightarrow \infty \), where \( m_{\pi} \) is the pion mass and \( s_0 \) is an unspecified scale. In principle, it should be possible to predict the “Froissart-like” behaviour and its universality from QCD, which we believe to be the microscopic theory of strong interactions, but a satisfactory derivation is still lacking.

The main reason why it is so difficult to obtain predictions for \( \sigma_{\text{tot}}(s) \) from first principles is that this requires a better understanding of the nonperturbative (NP) dynamics of QCD, which is known to be a hard task. Indeed, total cross sections are part of the more general problem of soft high-energy scattering, characterised by a large total center-of-mass energy squared \( s \) and a small transferred momentum, \( |t| \ll 1 \text{GeV}^2 \ll s \). In this energy regime, perturbation theory is not fully reliable, and one has to attack the problem with NP methods. In this context, a functional-integral approach in the framework of QCD has been proposed in [4] and further developed in [5], which we now briefly recall, focussing for simplicity on the case of the elastic scattering of two mesons of equal mass \( m \).

The elastic meson-meson scattering amplitude \( M_{(hh)} \) is reconstructed from the scattering amplitude \( M_{(dd)} \) of two dipoles of fixed transverse sizes \( \vec{r}_{1,2} \), with fixed longitudinal momentum fractions \( f_{1,2} \) of the quarks, after folding with appropriate squared wave functions \( \rho_{1,2} = |\psi_{1,2}|^2 \) describing the interacting hadrons [6],

\[ M_{(hh)}(s,t) = \int d^2 \nu_1 \rho_{1}(\nu_1) \rho_{2}(\nu_2) M_{(dd)}(s,t;\nu_1,\nu_2) \equiv \langle \langle M_{(dd)}(s,t;\nu_1,\nu_2) \rangle \rangle, \tag{1} \]

where \( \nu_i = (\vec{r}_{i,1}, f_i) \) denotes collectively the dipole variables, \( d^2 \nu = d\nu_1 d\nu_2 \), \( d\nu_i = \int d^2 \vec{r}_{i,1} \int_0^1 df_i \), and \( \int d\nu_i \rho_i(\nu_i) = 1 \). In turn, the dipole-dipole (dd) scattering amplitude in impact-parameter
space is given by the (properly normalised) correlation function (CF) of two Wilson loops (WL) in the fundamental representation, running along the classical paths described by the quark and antiquark in each dipole, thus forming a hyperbolic angle $\chi \approx \log(s/m^2)$ in the longitudinal plane, and properly closed by straight-line “links” in the transverse plane in order to ensure gauge invariance. Eventually, one has to take loops of infinite longitudinal extension. The relevant Minkowskian CF $C_M(\chi; \vec{z}_\perp; \nu_1, \nu_2)$, where $\vec{z}_\perp$ is the impact parameter, i.e., the transverse distance between the dipoles, can be reconstructed by means of analytic continuation from the Euclidean CF of two Euclidean WL, $C_E(\theta; \vec{z}_\perp; \nu_1, \nu_2) \equiv \lim_{T \to \infty} \langle W_{1}^{(T)} W_{2}^{(T)} \rangle / \langle (W_{1}^{(T)}) (W_{2}^{(T)}) \rangle - 1$, where $\langle \ldots \rangle$ is the average in the sense of the Euclidean QCD functional integral \cite{6,7,8}. The relevant Euclidean WL form an angle $\theta$ in the longitudinal plane, while having the same very same configuration in the transverse plane as in Minkowski space\cite{10}. The $dd$ scattering amplitude is then obtained from $C_E(\theta; \ldots)$ [with $\theta \in (0, \pi)$] by means of analytic continuation as $(t = -|q_\perp|^2)$

$$M_{(dd)}(s, t; \nu_1, \nu_2) = -i 2s \int d^2\vec{z}_\perp e^{iq_\perp \cdot \vec{z}} C_E(\theta \rightarrow -i \chi; \vec{z}_\perp; \nu_1, \nu_2).$$

(2)

2 Lattice results and total cross sections

In Euclidean space one can compute $C_E$ exploiting the available NP techniques, including the Stochastic Vacuum Model (SVM) \cite{10}, the Instanton Liquid Model (ILM) \cite{11,12}, the AdS/CFT correspondence for planar $\mathcal{N} = 4$ SYM \cite{13}, and in particular Lattice Gauge Theory (LGT), which allows to obtain by means of Monte Carlo simulations the true QCD prediction for the CF $C_E$ (within the errors). In Refs. \cite{13,12}, $C_E$ was computed numerically in quenched QCD on a $16^4$ hypercubic lattice at lattice spacing $a \approx 0.1$ fm, for loops of transverse size $a$ and with $f_{1,2} = 1/2$ (which causes no loss of generality \cite{12}), at distances $|z_\perp|/a = 0, 1, 2$, for several angles $\theta$ and different configurations in the transverse plane. These included the one relevant to meson-meson scattering, where the orientation of dipoles is averaged over (“ave”). The comparison of the numerical results with the analytic results obtained in QCD-related models (SVM and ILM) showed a poor agreement, both quantitatively (comparing with the numerical predictions of the models) and qualitatively (fitting the data with the model functions) \cite{13,12}. Moreover, these models do not lead to a “Froissart-like” asymptotic behaviour of $\sigma_{tot}^{(hh)}$: SVM and ILM lead to constant $\sigma_{tot}^{(hh)}$, while the AdS/CFT expression leads to power-like $\sigma_{tot}^{(hh)}$.\cite{15}.

In \cite{16} we introduced and partially justified a class of parameterisations of the lattice data that lead to Froissart-like and universal $\sigma_{tot}^{(hh)}$, which allow to improve the best fits. These parameterisations are of the general form $C_E = \exp\{K_E\} - 1$ (with $K_E$ real), with $K_E$ decaying exponentially at large $|\vec{z}_\perp|$, and such that after analytic continuation $K_E(\theta \rightarrow -i \chi) \rightarrow i \beta(\nu_1, \nu_2) e^{\eta(\chi)} e^{-\eta |\vec{z}_\perp|}$ at large $\chi$ and large $|\vec{z}_\perp|$, with $\eta \geq 0$, and $\eta$ a real function such that $\eta \rightarrow \infty$ as $\chi \rightarrow \infty$. The exponential form of the correlator is rather well justified: it is satisfied at large $N_c$, where $C_E \sim O(1/N_c^2)$; all the known analytical models satisfy it; the lattice data of Refs. \cite{13,12} confirm it. The exponential decay of $K_E \sim e^{-\eta |\vec{z}_\perp|}$ at large impact-parameter is natural in a confining theory like QCD, with the relevant mass scale $\mu$ being related to the masses of particles (including, possibly, also glueballs) exchanged between the two WL. Finally, the request $\text{Im} \beta \geq 0$ corresponds to a stronger version of the unitarity constraint on the impact parameter amplitude $A(s, |\vec{z}_\perp|) = \langle \langle C_M(\chi; \vec{z}_\perp; \nu_1, \nu_2) \rangle \rangle$. It is known that $|A + 1| \leq 1$; as this

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1 More precisely, the relevant paths are obtained by connecting the quark $[q]$-antiquark $[\bar{q}]$ straight-line paths, $C_i : X_i^{[q]}(\tau) = z_i + \frac{R_i}{2} \tau + f_i^{[q]}(\tau), i = 1, 2$, with $\tau \in [-T,T]$, by means of straight-line paths in the transverse plane at $\tau = \pm T$. Here $p_{1,2} = m(\pm \sin \frac{\theta}{2}, \vec{0}_\perp, \cos \frac{\theta}{2})$, $r_i = (0, \vec{r}_\perp, 0)$, $z_i = \delta_{i1}(0, \vec{z}_\perp, 0)$, and $f_i^0 \equiv 1 - f_i, f_i^0 \equiv -f_i$. 
constraint has to be satisfied for all physical choices of $\rho_{1,2}$ in Eq. [1], it is natural to assume the strongest constraint $|\mathcal{C}_i| \geq 1 \forall \mathcal{C}_i$, $i_1, i_2$ at large $\chi$. In [16] we showed that our parameterisations lead to $\sigma_{\text{tot}}^{(hh)} \sim 2\pi n^2/\mu^2$ at large $\chi$. Taking $e^{\eta} = e^{2n_k}$, one obtains the universal result $\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$, with $B = 2\pi n^2/\mu^2$ independently of the mesons involved in the process.

The analysis of the lattice data was performed using $C_{1,2}^{\text{ave}}$, which is closer to the physical amplitude (the analysis above can be repeated for $C_{i,2}^{\text{ave}}$ without altering any conclusion). Our best parameterisations are $C_{i,2}^{\text{ave}} = \exp[K_{i,2}^{(i)}] - 1$, $i = 1, 2, 3$, with $K_{i,2}^{(i)} = K_{i,2}^{(1)} + K_{2,2} \cot^2 \theta + K_{3,2} \cos \theta \cot \theta$ and $K_{i,2}^{(3)} = K_{i,2}^{(1)} + K_{2,2} (\frac{\pi}{2} - \theta) \cot \theta + K_{3,2} \cos \theta \cot \theta$, which are essentially two proper modifications of the AdS/CFT result (taking into account that $C_{1,2}^{\text{ave}}$ is symmetric under crossing $\{\mathcal{C}_i\}$, $i = 1, 2, 3$, independently of the mesons involved in the process. In the three cases, the unitarity condition is satisfied if $K_2 \geq 0$: this is actually the case for our best fits (within the errors). The value of $B = 2\pi/\mu^2$, obtained through a fit of the coefficient of the leading term with an exponential function, is found to be compatible with the experimental result (within the large errors) in all the three cases (see Table III). However, this must be taken only as an estimate, as our lattice data are quenched, and available only for rather small $|\tilde{z}_\perp|$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\mu$ (GeV)</th>
<th>$B = \frac{\pi}{\mu^2}$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.64(2.38)</td>
<td>$0.113^{+0.384}_{-0.247}$</td>
</tr>
<tr>
<td>2</td>
<td>3.79(1.46)</td>
<td>$0.17^{+0.277}_{-0.061}$</td>
</tr>
<tr>
<td>3</td>
<td>3.18(98)</td>
<td>$0.245^{+0.261}_{-0.106}$</td>
</tr>
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Table III: Mass-scale $\mu$ and the coefficient $B$ obtained with our parameterisations $C_{i,2}^{\text{ave}}$.

3 Acknowledgments

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References