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Long-distance effects in charm mixing

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I review challenges in the understanding of long distance effects in theoretical calculations of mixing rates of charmed mesons in the standard model.

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1 Introduction

Flavor physics, especially physics of charmed mesons, offers incredibly rich opportunities not only to study soft Quantum Chromodynamics (QCD), but also search for glimpses of new physics (NP) [1, 2]. That search is only possible if the standard model (SM) predictions for experimental observables are known well, which means that uncertainties of theoretical predictions are understood and under control. The experimental observables, such as meson mixing parameters, rates and asymmetries in rare decays and/or CP-violating asymmetries, are designed to provide likely places where NP can be observed [1]. Among those, a steady improvement of precision of experimental observation of $D^0 - \overline{D}^0$ mixing rate, offers a great hope that possible NP contributions in the up-quark sector would be soon constrained or observed [3]. Unfortunately, quantitative theoretical understanding of $D^0 - \overline{D}^0$ mixing rate remains one of the most difficult problems in flavor physics.

The $\Delta C = 2$ interactions, generated either at one loop level in the SM or possibly by NP particles, mix a D^0 state into a $\overline{D}^0$ state, which results in physical (measurable) mass and lifetime differences between new mass eigenstates [4],

$$x_D \equiv \frac{m_2 - m_1}{\Gamma}, \quad y_D \equiv \frac{\Gamma_2 - \Gamma_1}{2\Gamma}, \tag{1}$$

where $m_{1,2}$ and $\Gamma_{1,2}$ are the masses and widths of $D_{1,2}$ and the mean width and mass are $\Gamma = (\Gamma_1 + \Gamma_2)/2$ and $m = (m_1 + m_2)/2$. The mass eigenstates themselves are usually defined as

$$|D_{\frac{1}{2}}\rangle = p|D^0\rangle \pm q|\overline{D}^0\rangle, \tag{2}$$

where the complex parameters p and q are obtained from diagonalizing the $D^0 - \overline{D}^0$ mass matrix. The mass and lifetime differences introduced above can be calculated as absorptive and dispersive parts of a certain correlation function,

$$
\begin{aligned}
x_D &= \frac{1}{M_D \Gamma_D} \mathrm{Re} \Bigg[2 \langle \overline{D}^0 | H_w^{|\Delta C|=2} | D^0 \rangle \\
&+ \langle \overline{D}^0 | i \int d^4 x \, \mathrm{T} \left\{ H_w^{|\Delta C|=1}(x) H_w^{|\Delta C|=1}(0) \right\} | D^0 \rangle \Bigg], \tag{3} \\
y_D &= \frac{1}{M_D \Gamma_D} \mathrm{Im} \left[\langle \overline{D}^0 | i \int d^4 x \, \mathrm{T} \left\{ H_w^{|\Delta C|=1}(x) H_w^{|\Delta C|=1}(0) \right\} | D^0 \rangle \right]. \tag{4}
\end{aligned}
$$

It is understood that only quarks whose masses are lighter than m_D can go on mass shell in Eq. (3) and provide nonzero value for the lifetime difference y_D.

Charm system is quite unique because x_D is *not* dominated by the contribution of the $\Delta C = 2$ operator that is local at the charm scale. This is very different from the case of B-mixing, where x is completely dominated by the top quark contribution. Since Glashow-Iliopoulos-Maiani (GIM) guarantees that the mixing amplitude

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is proportional to the power of intrinsic quark mass running in the box diagram, suppressions due to a combination of Cabibbo-Kobayash-Maskawa (CKM) greatly diminish the contribution due to b-quark, the only heavy quark intermediate state possible in $D^0 - \overline{D}^0$ mixing. Thus, it is absolutely important to calculate the contribution due to the correlation functions in Eq. (3) with light intermediate s and d quarks.

The hardest problem in charm mixing is to properly evaluate the integrals in the above equations. This can be done in several ways, depending on whether one considers the decaying particle as heavy or light compared to the QCD's scale Λ_{QCD}. Since $m_c \simeq 1.3$ GeV, both approaches are possible for D-decays and mixing calculations.

If the decaying particle is heavy, it is possible to show [6] that the integrals in Eq. (3) are dominated by short distances, so a short-distance operator product expansion (OPE) can be used to evaluate the products of $|\Delta C| = 1$ Hamiltonians. Similar approaches worked very well for the calculations of lifetime differences of B_s mesons [7]. If the decaying particle is considered light, no short-distance expansion of operator products is possible, as the integrals are dominated by the long distances. However, only a few open channels are available for such light particles, so the calculations can be done by explicitly summing over the contributions from each of the channels. This approach worked well for kaon physics*. The number of available decay channels is quite large, but some predictions can nevertheless be made.

2   Expectations for x_D and y_D

Before proceeding to the calculation of $D^0 - \overline{D}^0$ mixing amplitude, let us understand the underlying flavor symmetry structure. GIM mechanism implies that meson mixing amplitudes must be proportional to mass factors of quarks propagating in the loops providing $\Delta C = 2$ interactions. Neglecting for a moment the third generation, only s and d quarks give contribution to x_D and y_D in the standard model. This means that GIM mechanism implies that $D^0 - \overline{D}^0$ mixing is an $SU(3)_F$-breaking effect, and predicting the Standard Model values of x_D and y_D depends crucially on estimating the size of $SU(3)_F$ breaking. The question is at what order in $SU(3)_F$-breaking parameter m_s does the effect become non-zero?

To answer that, let us look at the group-theoretical structure of mixing matrix elements $\langle \overline{D}^0 | H_w H_w | D^0 \rangle$ that define x_D and y_D. Here H_w denote the $\Delta C = -1$ part of the weak Hamiltonian. Let D be the field operator that creates a D^0 meson and annihilates a $\overline{D}^0$. Then the matrix element, whose $SU(3)$ flavor group theory

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*It is important to remember that this statement only refers to the bilocal part of the expressions for x and y. The mass difference in kaons is dominated by the contribution from heavy t and c quarks, i.e. by the $H_w^{|\Delta C|=2}$.

properties we will study, may be written as

$$\langle 0 | \, D \, H_w H_w \, D \, | 0 \rangle \, . \tag{5}$$

Since the operator D is of the form $\overline{c}u$, it transforms in the fundamental representation of $SU(3)_F$, which we will represent with a lower index, D_i. We use a convention in which the correspondence between matrix indexes and quark flavors is $(1, 2, 3) = (u, d, s)$. The only nonzero element of D_i is $D_1 = 1$. The $\Delta C = -1$ part of the weak Hamiltonian has the flavor structure $(\overline{q}_i c)(\overline{q}_j q_k)$, so its matrix representation is written with a fundamental index and two antifundamentals, H_k^{ij}. This operator is a sum of irreps contained in the product $3 \times \overline{3} \times \overline{3} = \overline{15} + 6 + \overline{3} + \overline{3}$. In the limit in which the third generation is neglected, H_k^{ij} is traceless, so only the $\overline{15}$ and 6 representations appear. That is, the $\Delta C = -1$ part of H_w may be decomposed as $\frac{1}{2}(O_{\overline{15}} + O_6)$.

Since we are interested in $SU(3)_F$ breaking, let is introduce it through the quark mass operator M, whose matrix representation is $M_j^i = \mathrm{diag}(m_u, m_d, m_s)$. We set $m_u = m_d = 0$ and let $m_s \neq 0$ be the only $SU(3)$ violating parameter. All nonzero matrix elements built out of D_i, H_k^{ij} and M_j^i must be $SU(3)_F$ singlets.

We now prove that $D^0 - \overline{D}^0$ mixing arises only at second order in $SU(3)$ violation, by which we mean second order in m_s. First, we note that the pair of D operators is symmetric, and so the product $D_i D_j$ transforms as a 6 under $SU(3)$. Second, the pair of H_w's is also symmetric, and the product $H_k^{ij} H_n^{lm}$ is in one of the reps which appears in the product

$$\left[(\overline{15} + 6) \times (\overline{15} + 6) \right]_S = (\overline{15} \times \overline{15})_S + (\overline{15} \times 6) + (6 \times 6)_S \tag{6}$$
$$= (\overline{60} + \overline{24} + 15 + 15' + \overline{6}) + (42 + 24 + 15 + \overline{6} + 3) + (15' + \overline{6}) \, .$$

A direct computation shows that only three of these representations actually appear in the decomposition of $H_w H_w$. They are the $\overline{60}$, the 42, and the 15'

$$DD = D_6 \, , \qquad H_w H_w = O_{\overline{60}} + O_{42} + O_{15'} \, , \tag{7}$$

where subscripts denote the representation of $SU(3)_F$. Since there is no $\overline{6}$ in the decomposition of $H_w H_w$, there is no $SU(3)$ singlet which can be made with D_6, and no $SU(3)$ invariant matrix element of the form (5) can be formed. This is the well known result that $D^0 - \overline{D}^0$ mixing is *prohibited by $SU(3)$ symmetry*. Now consider a single insertion of the $SU(3)$ violating spurion M. The combination $D_6 M$ transforms as $6 \times 8 = 24 + \overline{15} + 6 + \overline{3}$. There is still no invariant to be made with $H_w H_w$, thus $D^0 - \overline{D}^0$ mixing is *not induced at first order in $SU(3)_F$ breaking*. With two insertions of M, it becomes possible to make an $SU(3)_F$ invariant. The decomposition of DMM is

$$6 \times (8 \times 8)_S \; = \; 6 \times (27 + 8 + 1) \tag{8}$$
$$= \; (60 + \overline{42} + 24 + \overline{15} + \overline{15}' + 6) + (24 + \overline{15} + 6 + \overline{3}) + 6 \, .$$

There are three elements of the 6×27 part which can give invariants with $H_w H_w$. Each invariant yields a contribution to $D^0 - \overline{D}^0$ mixing proportional to $s_1^2 m_s^2$. Thus, $D^0 - \overline{D}^0$ mixing arises only at *second order* in the $SU(3)$ violating parameter m_s [9], in the Standard Model x and y are generated only at second order in $SU(3)_F$ breaking,

$$x \, , \, y \sim \sin^2 \theta_C \times [SU(3) \text{ breaking}]^2 \, , \tag{9}$$

where θ_C is the Cabibbo angle. This result should be reproduced in all explicit calculations of $D^0 - \overline{D}^0$ mixing parameters.

The use of the OPE relies on local quark-hadron duality, and on expansion parameter Λ/m_c being small enough to allow truncation of the series after the first few terms. Let us see what one can expect at leading order in $1/m_c$ expansion, i.e. assuming that the integrals in Eq. (3) are dominated by the short distances. The leading-order result is then generated by calculating the usual box diagram with intermediate s and d quarks.

Unitarity of the CKM matrix assures that the leading-order, mass-independent contribution due to s-quark is completely cancelled by the corresponding contribution due to a d-quark. A non-zero contribution can be obtained if the mass insertions are added on each quark line in the box diagram. However, adding only one mass insertion flips the chirality of the propagating quarks, from being left-handed to right-handed. This does not give a contribution to the resulting amplitude, as right-handed quarks do not participate in weak interaction. Thus, a second mass insertion is needed on each quark line. Neglecting m_d compared to m_s we see that the resulting contribution to x_D is $O(m_s^2 \times m_s^2) \sim O(m_s^4)$! It is easy to convince yourself that y_D has additional m_s^2 suppression due to on-shell propagation of left-handed quarks emitted from a spin-zero meson, which brings total suppression of y_D to $O(m_s^6)$! An explicit calculation of the leading order mixing amplitude, as well as perturbative QCD corrections to it [8] agrees precisely with the hand-waving arguments above. Clearly, leading order contribution in $1/m_c$ gives "too much" of $SU(3)_F$ suppression compared to the theorem that was proven above [9].

Somewhat surprisingly, the resolution of this paradox follows from considerations of higher-order corrections in $1/m_c$ [10]. Among many higher-dimensional operators that encode $1/m_c$ corrections to the leading four-fermion operator contribution, there exists a class of operators that result from chirality-flipping interactions with background quark condensates. These interactions do not bring additional powers of light quark mass, but are suppressed by powers of Λ_{QCD}/m_c, which is not a very small number. The leading $O(m_s^2)$ order of $SU(3)_F$ breaking is obtained from matrix elements of dimension twelve operators that are suppressed by $(\Lambda_{QCD}/m_c)^6$ compared to the parametrically-leading contribution in $1/m_c$ expansion [10]! As usual in OPE calculation, proliferation of the number of operators at higher orders (over 20) makes it difficult to pinpoint the precise value of the effect.

3 Threshold effects in OPE and exclusive approaches to calculation of mixing parameters

There are several concerns that one need to deal with when calculating $D^0 - \overline{D}^0$ mixing using OPE-based methods. First, the numerically leading order effect comes from dimension twelve operators. Quark-hadron duality was never checked for such case [11]. Second, the number of matrix elements of operators is very large. It is not clear how to properly combine uncertainties associated with computations of those matrix elements. Third concern, which is also related to the issue of quark-hadron duality, regards the proper way of dealing with hadronic thresholds in OPE framework.

Let us concentrate on calculation of y_D. In order to illustrate the issue of hadronic thresholds, one needs to recall that heavy quark operator expansion is really an expansion in the energy released in the process of the decay. In D-decays this energy is not always large: for example, for KKK intermediate state the energy released in the decay is $E_r \sim m_D - 3m_K \sim O(\Lambda_{QCD})$, which is by no means large. In the limit $m_c \to \infty$ one immediately sees that $m_c \gg M_{\text{intermediate state}}$, so $E_r \sim m_c$. This is why this approach works very well for B-decays, but might have issues for charm. It is clear that more fork is needed to understand range of applicability of OPE methods to $D^0 - \overline{D}^0$ mixing [12].

It is possible to calculate $D^0 - \overline{D}^0$ mixing rates by dealing explicitly with hadronic intermediate states which result from every common decay product of D^0 and $\overline{D}^0$ [13]. In the $SU(3)_F$ limit, these contributions cancel when one sums over complete $SU(3)$ multiplets in the final state. The cancellations depend on $SU(3)_F$ symmetry both in the decay matrix elements and in the final state phase space. While there are $SU(3)$ violating corrections to both of these, it is difficult to compute the $SU(3)_F$ violation in the matrix elements in a model independent manner. As experimental data on nonleptonic decay rates becomes better and better, it is possible to use it to calculate y_D by directly inputing it into Eq. (3),

$$y_D = \frac{1}{\Gamma} \sum_n \int [\text{P.S.}]_n \langle \overline{D}^0 | H_w | n \rangle \langle n | H_w | D^0 \rangle, \tag{10}$$

where the sum is over distinct final states n and the integral is over the phase space for state n. Alternatively, with some mild assumptions about the momentum dependence of the matrix elements, the $SU(3)_F$ violation in the phase space depends only on the final particle masses and can be computed [9]. It was shown that this source of $SU(3)_F$ violation can generate y_D and x_D of the order of a few percent. The calculation of x_D relies on further model-dependent assumptions about off-shell behavior of decay form-factors [9]. Restricting the sum over all final states to final states F which

transform within a single $SU(3)_F$ multiplet R, the result is

$$y_D = \frac{1}{2\Gamma} \langle \overline{D}^0 | H_w \Big\{ \eta_{CP}(F_R) \sum_{n \in F_R} |n\rangle \rho_n \langle n| \Big\} H_w |D^0 \rangle \,, \tag{11}$$

where ρ_n is the phase space available to the state n, $\eta_{CP} = \pm 1$ [9]. In the $SU(3)_F$ limit, all the ρ_n are the same for $n \in F_R$, and the quantity in braces above is an $SU(3)_F$ singlet. Since the ρ_n depend only on the known masses of the particles in the state n, incorporating the true values of ρ_n in the sum is a calculable source of $SU(3)_F$ breaking.

This method does not lead directly to a calculable contribution to y, because the matrix elements $\langle n|H_w|D^0 \rangle$ and $\langle \overline{D}^0|H_w|n \rangle$ are not known. However, CP symmetry, which in the Standard Model and almost all scenarios of new physics is to an excellent approximation conserved in D decays, relates $\langle \overline{D}^0|H_w|n \rangle$ to $\langle D^0|H_w|\overline{n} \rangle$. Since $|n\rangle$ and $|\overline{n}\rangle$ are in a common $SU(3)_F$ multiplet, they are determined by a single effective Hamiltonian. Hence the ratio

$$
\begin{aligned}
y_{F,R} &= \frac{\sum_{n \in F_R} \langle \overline{D}^0 | H_w |n\rangle \rho_n \langle n|H_w |D^0 \rangle}{\sum_{n \in F_R} \langle D^0 | H_w |n\rangle \rho_n \langle n|H_w |D^0 \rangle} \\
&= \frac{\sum_{n \in F_R} \langle \overline{D}^0 | H_w |n\rangle \rho_n \langle n|H_w |D^0 \rangle}{\sum_{n \in F_R} \Gamma(D^0 \to n)}
\end{aligned} \tag{12}
$$

is calculable, and represents the value which y_D would take if elements of F_R were the only channel open for D^0 decay. To get a true contribution to y_D, one must scale $y_{F,R}$ to the total branching ratio to all the states in F_R. This is not trivial, since a given physical final state typically decomposes into a sum over more than one multiplet F_R. The numerator of $y_{F,R}$ is of order s_1^2 while the denominator is of order 1, so with large $SU(3)_F$ breaking in the phase space the natural size of $y_{F,R}$ is 5%. Indeed, there are other $SU(3)_F$ violating effects, such as in matrix elements and final state interaction phases. Here we assume that there is no cancellation with other sources of $SU(3)_F$ breaking, or between the various multiplets which occur in D decay, that would reduce our result for y by an order of magnitude. This is equivalent to assuming that the D meson is not heavy enough for duality to enforce such cancellations. Performing the computations of $y_{F,R}$, we see [9] that effects at the level of a few percent are quite generic. Our results are summarized in Table 1. Then, y_D can be formally constructed from the individual $y_{F,R}$ by weighting them by their D^0 branching ratios,

$$y_D = \frac{1}{\Gamma} \sum_{F,R} y_{F,R} \Big[\sum_{n \in F_R} \Gamma(D^0 \to n) \Big]. \tag{13}$$

However, the data on D decays are neither abundant nor precise enough to disentangle the decays to the various $SU(3)_F$ multiplets, especially for the three- and four-body

Final state representation		$y_{F,R}/s_1^2$	$y_{F,R}$ (%)
PP	8	-0.0038	-0.018
	27	-0.00071	-0.0034
PV	8_A	0.032	0.15
	8_S	0.031	0.15
	10	0.020	0.10
	$\overline{10}$	0.016	0.08
	27	0.04	0.19
$(VV)_{s\text{-wave}}$	8	-0.081	-0.39
	27	-0.061	-0.30
$(VV)_{p\text{-wave}}$	8	-0.10	-0.48
	27	-0.14	-0.70
$(VV)_{d\text{-wave}}$	8	0.51	2.5
	27	0.57	2.8
$(3P)_{s\text{-wave}}$	8	-0.48	-2.3
	27	-0.11	-0.54
$(3P)_{p\text{-wave}}$	8	-1.13	-5.5
	27	-0.07	-0.36
$(3P)_{\text{form-factor}}$	8	-0.44	-2.1
	27	-0.13	-0.64
$4P$	8	3.3	16
	27	2.2	11
	$27'$	1.9	9.2

Table 1: Values of $y_{F,R}$ for some two-, three-, and four-body final states.

final states. Nor have we computed $y_{F,R}$ for all or even most of the available representations. Instead, we can only estimate individual contributions to y by assuming that the representations for which we know $y_{F,R}$ to be typical for final states with a given multiplicity, and then to scale to the total branching ratio to those final states. The total branching ratios of D^0 to two-, three- and four-body final states can be extracted from the Review of Particle Physics. Rounding to the nearest 5% to emphasize the uncertainties in these numbers, we conclude that the branching fractions for PP, $(VV)_{s\text{-wave}}$, $(VV)_{d\text{-wave}}$ and $3P$ approximately amount to 5%, while the branching ratios for PV and $4P$ are of the order of 10% [9].

It can be easily seen that there are terms in Eq. (13), like nonresonant $4P$, which could make contributions to y_D at the level of a percent or larger. There, the rest masses of the final state particles take up most of the available energy, so phase space differences are very important. One can see that y_D on the order of a few

percent is completely natural, and that anything an order of magnitude smaller would require significant cancellations which do not appear naturally in this framework. The normalized mass difference, x_D, can then be calculated via a dispersion relation

$$x_D = -\frac{1}{\pi} \; \mathrm{P} \int_{2m_\pi}^\infty dE \frac{y_D(E)}{E - m_D} \tag{14}$$

that additionally contain guesses on the off-shell behavior of hadronic form-factors in $y_D(E)$ [9]. Here P denotes principal value. The result of the calculation yields $x_D \sim O(1\%)$ [9].

Since experimental data on nonleptonic decays of charmed mesons improved significantly in the past several years, it can be used to estimate some contributions [14]. For example, concentrating on the $\pi\pi$, KK, and πK intermediate states,

$$\begin{aligned} y_{2D} &=& \mathrm{Br}(D^0 \to K^+K^-) + \mathrm{Br}(D^0 \to \pi^+\pi^-) \\ &-& 2\cos\delta\sqrt{\mathrm{Br}(D^0 \to K^+\pi^-)\mathrm{Br}(D^0 \to \pi^+K^-)} \end{aligned} \tag{15}$$

The PDG values
citeBeringer:1900zz for the branching ratios above are known quite well for the purpose of calculation of y_{2D},

$$\begin{aligned} \mathrm{Br}(D^0 \to K^+K^-) &=& (3.96 \pm 0.00) \times 10^{-3}, \\ \mathrm{Br}\,(D^0 \to \pi^+\pi^-) &=& (1.401 \pm 0.027) \times 10^{-3}, \\ \mathrm{Br}(D^0 \to K^+\pi^-) &=& (3.88 \pm 0.05) \times 10^{-2}, \\ \mathrm{Br}(D^0 \to \pi^+K^-) &=& (1.31 \pm 0.08) \times 10^{-4}. \end{aligned} \tag{16}$$

Notice that $\cos\delta$ is not known well. Its value however is very important for numerical calculation of Eq. (15), as large cancellations (between the first and the second lines of that equation) are expected. Taking the U-spin limit $\cos\delta = 1$ [16], one arrives at the contribution $y_{2D} = (0.85 \pm 0.17) \times 10^{-3}$. Unfortunately, other branching ratios, especially for three or four body decays, are not known that well. Therefore, saturating Eq. (12) with experimental data would only be sensitive to the values of experimental uncertainties of measurements of branching ratios, not to the true size of the effect.

It must be pointed out that similar calculations of y_D have been recently carried out using the simpler language of U-spin with consistent results [17].

4   Outlook

The calculation of $D^0 - \overline{D}^0$ mixing is a challenging theoretical exercise. It is not clear if brute-force improvements of the calculations would result in much more precise results.

However, a glimpse of hope for yet another approach have recently been identified. Probably not surprisingly, it came from lattice QCD calculations, which usually shined away from the calculations of non-leptonic decay amplitudes. It remains to be seen if multichannel generalizations of Lellouch-Luscher approaches [18] to calculations of weak matrix elements will be successful in calculating non-leptonic decay rates of charm mesons [19]. Yet, this approach will certainly have impact on charm physics and, in particular, on calculations of $D^0 - \overline{D}^0$ mixing rate.

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