Method To Reduce Quintic Equations using Linear Algebra

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Abstract - The usual method for the reduction of polynomial equations is the Tschirnhaus one. We developed a method of reduction using linear algebra. The method consist to associate to the polynomial an operator chosen to be element of an abelian Lie algebra. The reduction of the polynomial can be done by relating the operator to another whose characteristic polynomial is reduced. The two operators are related by a basis change.

1. INTRODUCTION

Raoelina Andriambololona has developed an unique method to obtain the n roots of a polynomial of any degree n by using a translation and circulating matrix, and has applied it to the case of 2,3,4 degree polynomials[1]. Based on this method, we developed a new one for the reduction of polynomial equations. According to the Galois theory, polynomial equations of degree superior than or equal to 5 are not generally solvable in radicals. However, one can express the solution of some reduced equations, for instance Bring- Jerard form for quintic, by using special functions (ultra-radicals, elliptic function, hypergeometric function,…). Therefore, reduction of general equation is needed. Here, we propose a method using linear algebra and properties of characteristic polynomials [2].

2. METHOD TO REDUCE QUINTIC EQUATIONS USING LINEAR ALGEBRA

In this method, the polynomial is associated to a diagonalizable linear operator which has a characteristic polynomial as the polynomial itself. The operator is considered as element of an abelian Lie algebra. And we connect this operator to another of the same algebra which has a reduced characteristic polynomial. The two operators is related by a basis change.

2.1. Use of abelian Lie algebra

We adopt the following convention developed in the reference [2].

Let E_5 be a vector space of dimension 5 over $\mathbb{C}$ and J the linear operator on E_5 which satisfies $J^5 = I$ where I is the identity operator. The set, denoted V_5, of all linear operators X on E_5 which can be written as a polynomial in J :

$$V_5 = \{X / \exists\, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 \in \mathbb{C} \quad X = \gamma^0 (J)^0 + \gamma^1 (J)^1 + \gamma^2 (J)^2 + \gamma^3 (J)^3 + \gamma^4 (J)^4\}$$

has a complex abelian Lie algebra structure. $(J)^0 = I = J_0$, $(J)^1 = J_1, (J)^2 = J_2, (J)^3 = J_3, (J)^4 = J_4$ form a basis, denoted $B_J = [J_0 \;\; J_1 \;\; J_2 \;\; J_3 \;\; J_4]$ of V_5. $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ and γ^4 are the components of X in the basis B_J .

$\forall X \in V_5$, $\forall k \in \mathbb{N}$, $(X)^k \in V_5$. If $(X)^0 = I$ and the first four power $(X)^1, (X)^2$, $(X)^3$ and $(X)^4$ of X are linearly independent then they form a basis of V_5 , for instance $X = J$. We put $(X)^0 = X_0$, $X_1 = X$, $X_2 = (X)^2$, $X_3 = (X)^3$, $X_4 = (X)^4$ and we note B_X the basis of V_5 such as: $B_X = [X_0 \;\; X_1 \;\; X_2 \;\; X_3 \;\; X_4]$

Let $A(x) = x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$ be the characteristic polynomial of X.

Consider the expression of the power $(X)^k$ of X in the basis B_X for $k \in \mathbb{N}$.

$(X)^k = \alpha_k^0 X_0 + \alpha_k^1 X_1 + \alpha_k^2 X_2 + \alpha_k^3 X_3 + \alpha_k^4 X_4$

It is shown that:

- for $k < 5$: $\alpha_k^l = \delta_k^l = \begin{cases} 1 \ if \ k = l \\ 0 \ if \ k \neq l \end{cases}$ (δ_k^l is the Kronecker symbol)
- for $k = 5$: $\alpha_5^0 = -a_5$ $\alpha_5^1 = -a_4$ $\alpha_5^2 = -a_3$ $\alpha_5^3 = -a_2$ $\alpha_5^4 = -a_1$
- for $k \geq 5$ we have the following recurrence relations :

$$\begin{cases} \alpha_{k+1}^0 = -\alpha_k^4 a_5 \\ \alpha_{k+1}^1 = (\alpha_k^0 - \alpha_k^4 a_4) \\ \alpha_{k+1}^2 = (\alpha_k^1 - \alpha_k^4 a_3) \\ \alpha_{k+1}^3 = (\alpha_k^2 - \alpha_k^4 a_2) \\ \alpha_{k+1}^4 = (\alpha_k^3 - \alpha_k^4 a_1) \end{cases}$$

Proof

For $k < 5$, it's obvious because $(X)^0 = X_0$, $(X)^1 = X_1$, $(X)^2 = X_2$, $(X)^3 = X_3$, $(X)^4 = X_4$ are the basis vector of B_X.

For $k = 5$, We use the Cayley – Hamilton theorem $A(X) = 0$

$$(X)^5 + a_1(X)^4 + a_2(X)^3 + a_3(X)^2 + a_4 X + a_5 I = 0$$

$$\begin{aligned} (X)^5 &= -a_5(X)^0 - a_4(X)^1 - a_3(X)^2 - a_2(X)^3 - a_1(X)^4 \\ &= -a_5 X_0 - a_4 X_1 - a_3 X_2 - a_2 X_1 - a_1 X_4 \\ &= \alpha_5^0 X_0 + \alpha_5^1 X_1 + \alpha_5^2 X_2 + \alpha_5^3 X_3 + \alpha_5^4 X_4 \end{aligned}$$

By identification, we deduce the components α_5^l $(l = 0,1,2,3,4)$ of $(X)^5$ in the basis B_X.

For $k \geq 5$

$$\begin{aligned} (X)^k &= \alpha_k^0 X_0 + \alpha_k^1 X_1 + \alpha_k^2 X_2 + \alpha_k^3 X_3 + \alpha_k^4 X_4 \\ &= \alpha_k^0 (X)^0 + \alpha_k^1 (X)^1 + \alpha_k^2 (X)^2 + \alpha_k^3 (X)^3 + \alpha_k^4 (X)^4 \end{aligned}$$

$$\begin{aligned} (X)^{k+1} &= (X)^k (X)^1 \\ &= \alpha_k^0 X + \alpha_k^1 (X)^2 + \alpha_k^2 (X)^3 + \alpha_k^3 (X)^4 + \alpha_k^4 (X)^5 \\ &= \alpha_k^0 (X)^1 + \alpha_k^1 (X)^2 + \alpha_k^2 (X)^3 + \alpha_k^3 (X)^4 + \alpha_k^4 (-a_5 (X)^0 - a_4 (X)^1 - a_3 (X)^2 - a_2 (X)^3 - a_1 (X)^4) \\ &= -\alpha_k^4 a_5 (X)^0 + (\alpha_k^0 - \alpha_k^4 a_4)(X)^1 + (\alpha_k^1 - \alpha_k^4 a_3)(X)^2 + (\alpha_k^2 - \alpha_k^4 a_2)(X)^3 + (\alpha_k^3 - \alpha_k^4 a_1)(X)^4 \\ &= -\alpha_k^4 a_5 X_0 + (\alpha_k^0 - \alpha_k^4 a_4) X_1 + (\alpha_k^1 - \alpha_k^4 a_3) X_2 + (\alpha_k^2 - \alpha_k^4 a_2) X_3 + (\alpha_k^3 - \alpha_k^4 a_1) X_4 \\ &= \alpha_{k+1}^0 X_0 + \alpha_{k+1}^1 X_1 + \alpha_{k+1}^2 X_2 + \alpha_{k+1}^3 X_3 + \alpha_{k+1}^4 X_4 \end{aligned}$$

By identification, we obtain the recurrence relation.

Let X be an element of V_5 which defines the basis $B_X = [X_0 \quad X_1 \quad X_2 \quad X_3 \quad X_4]$ and Y any element of V_5, such as :

$Y = p^0 X_0 + p^1 X_1 + p^2 X_2 + p^3 X_3 + p^4 X_4$ where $p^0, p^1, p^2, p^3 \ and \ p^4$ are the components of Y in B_X.

Consider the expression of the power $(Y)^k$ of Y in the basis B_X: $(Y)^k = p_k^0 X_0 + p_k^1 X_1 + p_k^2 X_2 + p_k^3 X_3 + p_k^4 X_4$

It is shown that:

- for $k = 0$: $p_0^l = \delta_0^l = \begin{cases} 1 \ if \ l = 0 \\ 0 \ if \ l \neq 0 \end{cases}$
- for $k = 1$: $p_1^0 = p^0$ $p_1^1 = p^1$ $p_1^2 = p^2$ $p_1^3 = p^3$ $p_1^4 = p^4$
- for $k \geq 0$ we have the following recurrence relations :

$$
\begin{cases}
p_{k+1}^0 = p^0 p_k^0 + \alpha_5^0(p^1 p_k^4 + p^2 p_k^3 + p^3 p_4^2 + p^4 p_k^1) \\
\qquad +\alpha_6^0(p^2 p_k^4 + p^3 p_k^3 + p^4 p_k^2) + \alpha_7^0(p^3 p_k^4 + p^4 p_k^3) + \alpha_8^0 p^4 p_k^4 \\
p_{k+1}^1 = (p^0 p_k^1 + p^1 p_k^0) + \alpha_5^1(p^1 p_k^4 + p^2 p_k^3 + p^3 p_4^2 + p^4 p_k^1) \\
\qquad +\alpha_6^1(p^2 p_k^4 + p^3 p_k^3 + p^4 p_k^2) + \alpha_7^1(p^3 p_k^4 + p^4 p_k^3) + \alpha_8^1 p_4 p_k^4 \\
p_{k+1}^2 = (p^0 p_k^2 + p^1 p_k^1 + p^2 p_k^0) + \alpha_5^2(p^1 p_k^4 + p^2 p_k^3 + p^3 p_k^2 + p^4 p_k^1) \\
\qquad +\alpha_6^2(p^2 p_k^4 + p^3 p_k^3 + p^4 p_k^2) + \alpha_7^2(p^3 p_k^4 + p^4 p_k^3) + p^4 p_k^4 \\
p_{k+1}^3 = (p^0 p_k^3 + p^1 p_k^2 + p^2 p_k^1 + p^3 p_k^0) + \alpha_5^3(p^1 p_k^4 + p^2 p_k^3 + p^3 p_k^2 + p^4 p_k^1) \\
\qquad +\alpha_6^3(p^2 p_k^4 + p^3 p_k^3 + p^4 p_k^2) + \alpha_7^3(p^3 p_k^4 + p^4 p_k^3) + \alpha_8^3 p^4 p_k^4 \\
p_{k+1}^4 = (p^0 p_k^4 + p^1 p_k^3 + p^2 p_k^2 + p^3 p_k^1 + p^4 p_k^0) + \alpha_5^4(p^1 p_k^4 + p^2 p_k^3 + p^3 p_4^2 + p^4 p_k^1) \\
\qquad +\alpha_6^4(p^2 p_k^4 + p^3 p_k^3 + p^4 p_k^2) + \alpha_7^4(p^3 p_k^4 + p^4 p_k^3) + \alpha_8^4 p^4 p_k^4
\end{cases}
$$

Proof

For $k = 0$ and $k = 1$:

$(Y)^0 = I = X_0$

$(Y)^1 = Y = p^0 X_0 + p^1 X_1 + p^2 X_2 + p^3 X_3 + p^4 X_4$

For $k \geq 0$

$$(Y)^k = p_k^0 X_0 + p_k^1 X_1 + p_k^2 X_2 + p_k^3 X_3 + p_k^4 X_4$$
$$= p_k^0 I + p_k^1 X + p_k^2 (X)^2 + p_k^3 (X)^3 + p_k^4 (X)^4$$

$$(Y)^{k+1} = (Y)^k Y$$
$$= [p_k^0 I + p_k^1 X + p_k^2 (X)^2 + p_k^3 (X)^3 + p_k^4 (X)^4][p_0 I + p_1 X + p_2 (X)^2 + p_3 (X)^3 + p_4 (X)^4]$$

Replacing $(X)^5, (X)^6, (X)^7, (X)^8$ by their expressions and doing identification with :

$(Y)^{k+1} = p_{k+1}^0 X_0 + p_{k+1}^1 X_1 + p_{k+1}^2 X_2 + p_{k+1}^3 X_3 + p_{k+1}^4 X_4$ the recurrence relation is obtained.

Let X and Y be two elements of V_5 which respectively define the two basis:

$B_X = [(X)^0 \ (X)^1 \ (X)^2 \ (X)^3 \ (X)^4] = [X_0 \ X_1 \ X_2 \ X_3 \ X_4]$

$B_Y = [(Y)^0 \ (Y)^1 \ (Y)^2 \ (Y)^3 \ (Y)^4] = [Y_0 \ Y_1 \ Y_2 \ Y_3 \ Y_4]$

Let $A(x) = x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$ be the characteristic polynomial of X.

And $(Y)^k = p_k^0 X_0 + p_k^1 X_1 + p_k^2 X_2 + p_k^3 X_3 + p_k^4 X_4$

Therefore the basis change matrix from B_X to B_Y is given by:

$$[P] = \begin{pmatrix} 1 & p_1^0 & p_2^0 & p_3^0 & p_4^0 \\ 0 & p_1^1 & p_2^1 & p_3^1 & p_4^1 \\ 0 & p_1^2 & p_2^2 & p_3^2 & p_4^2 \\ 0 & p_1^3 & p_2^3 & p_3^3 & p_4^3 \\ 0 & p_1^4 & p_2^4 & p_3^4 & p_4^4 \end{pmatrix}$$

$[(Y)^0 \ (Y)^1 \ (Y)^2 \ (Y)^3 \ (Y)^4] = [(X)^0 \ (X)^1 \ (X)^2 \ (X)^3 \ (X)^4][P]$

$[(X)^0 \ (X)^1 \ (X)^2 \ (X)^3 \ (X)^4] = [(Y)^0 \ (Y)^1 \ (Y)^2 \ (Y)^3 \ (Y)^4][P^{-1}]$

2.2. Characteristic polynomial of a linear operator and traces of its powers

Let X be a linear operator over $\mathbb{C}$-vector space of dimension 5 and $A(x) = x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$ its characteristic polynomial. Let x_0, x_1, x_2, x_3, x_4 be the roots of $A(x)$ which are also eigenvalues of X

It is shown that:

$Tr[(X)^0] = (x_0)^0 + (x_1)^0 + (x_2)^0 + (x_3)^0 + (x_4)^0 = Tr[I\ ] = 5$

$Tr[(X)^1] = (x_0)^1 + (x_1)^1 + (x_2)^1 + (x_3)^1 + (x_4)^1 = -a_1$

$Tr[(X)^2] = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = a_1{}^2 - 2a_2$

$Tr[(X)^3] = (x_0)^3 + (x_1)^3 + (x_2)^3 + (x_3)^3 + (x_4)^3 = -a_1{}^3 + 3a_1a_2 - 3a_3$

$Tr[(X)^4] = (x_0)^4 + (x_1)^4 + (x_2)^4 + (x_3)^4 + (x_4)^4 = a_1{}^4 - 4a_1{}^2a_2 + 2a_2{}^2 + 4a_1a_3 - 4a_4$

$Tr[(X)^5] = (x_0)^5 + (x_1)^5 + (x_2)^5 + (x_3)^5 + (x_4)^5 = -a_1{}^5 + 5(-a_1a_2{}^2 + a_1{}^3a_2 - a_1{}^2a_3 + a_1a_4 + a_2a_3 - a_5)$

2.3. Transformation to reduce a polynomial equation

Let $A(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x^1 + a_5$ a polynomial to reduced

$A(x)$ can be considered as the characteristic polynomial of an operator X of V_5. Let Y be another element of V_5.

We suppose that X and Y define bases B_X and B_Y of V_5 :

$B_X = [(X)^0\ \ (X)^1\ \ (X)^2\ \ (X)^3\ \ (X)^4\ ] = [X_0\ \ X_1\ \ X_2\ \ X_3\ \ X_4]$

$B_Y = [(Y)^0\ \ (Y)^1\ \ (Y)^2\ \ (Y)^3\ \ (Y)^4\ ] = [Y_0\ \ Y_1\ \ Y_2\ \ Y_3\ \ Y_4]$

Let $Y = p^0X_0 + p^1X_1 + p^2X_2 + p^3X_3 + p^4X_4$ and $(X)^k = \alpha_1^0X_0 + \alpha_k^1X_1 + \alpha_k^2X_2 + \alpha_k^3X_3 + \alpha_k^4X_4$

We determine $\alpha_k^l = \alpha_k^l(a_1, a_2, a_3, a_4, a_5)$ for $k = 0$ to 8

$(Y)^k = p_k^0X_0 + p_k^1X_1 + p_k^2X_2 + p_k^3X_3 + p_k^4X_4$

We determine $p_k^l = p_k^l(a_1, a_2, a_3, a_4, a_5, p^0, p^1, p^2, p^3, p^4)$ for $k = 0$ to 5

We apply the Trace operator to the expressions of Y, $(Y)^2$, $(Y)^3, (Y)^4$ and $(Y)^5$:

$$\begin{cases} Tr[(Y)^1] = p_1^0Tr[(X)^0] + p_1^1Tr[(X)^1] + p_1^2Tr[(X)^2] + p_1^3Tr[(X)^3] + p_1^4Tr[(X)^4] \\ Tr[(Y)^2] = p_2^0Tr[(X)^0] + p_2^1Tr[(X)^1] + p_2^2Tr[(X)^2] + p_2^3Tr[(X)^3] + p_2^4Tr[(X)^4] \\ Tr[(Y)^3] = p_3^0Tr[(X)^0] + p_3^1Tr[(X)^1] + p_3^2Tr[(X)^2] + p_3^3Tr[(X)^3] + p_3^4Tr[(X)^4] \\ Tr[(Y)^4] = p_4^0Tr[(X)^0] + p_4^1Tr[(X)^1] + p_4^2Tr[(X)^2] + p_4^3Tr[(X)^3] + p_4^4Tr[(X)^4] \\ Tr[(Y)^5] = p_5^0Tr[(X)^0] + p_5^1Tr[(X)^1] + p_5^2Tr[(X)^2] + p_5^3Tr[(X)^3] + p_5^4Tr[(X)^4] \end{cases}$$

Let $B(y) = y^5 + b_1y^4 + b_2y^3 + b_3y^2 + b_4y + b_5$ be the characteristic polynomial of $Y, Tr[(Y)^k]$ is function of $b_1, b_2, b_3, b_4\ , b_5$. Because of the above relations between X and Y, it remains five arbitrary parameters, among $b_1, b_2, b_3, b_4, b_5, p^0, p^1, p^2, p^3, p^4$ that can be fixed provided that a well basis change is defined: particularly, we can choose some b_k equal to 0 to get $B(y)$ as a reduced polynomial. If we know roots y of the reduced polynomial $B(y)$ wich are eigenvalues of Y, One obtain from the inverse of the basis change matrix the eigenvalues x of X i.e the solutions of $A(x) = 0$.

For the reduction:

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x^1 + a_5 \ \rightarrow\ y^5 + b_2y^3 + b_3y^2 + b_4y + b_5$$

$Y = p^0(X)^0 + (X)^1 = p^oX_0 + X_1$

$$\begin{cases} p^0 = \frac{a_1}{5} \\ b_2 = \frac{-2{a_1}^2}{5} + a_2 \\ b_3 = \frac{4{a_1}^3}{25} - \frac{3a_1a_2}{5} + a_3 \\ b_4 = \frac{-3{a_1}^4}{125} + \frac{3{a_1}^2a_2}{25} - \frac{2a_1a_3}{5} + a_4 \\ b_5 = \frac{4{a_1}^5}{3125} - \frac{{a_1}^3a_2}{125} + \frac{{a_1}^2a_3}{25} - \frac{a_1a_2}{5} + a_5 \end{cases} \qquad [P] = \begin{pmatrix} 1 & p^0 & (p^0)^2 & (p^0)^3 & (p^0)^4 \\ 0 & 1 & 2p^0 & 3(p^0)^2 & 4(p^0)^3 \\ 0 & 0 & 1 & 3p^0 & 6(p^0)^2 \\ 0 & 0 & 0 & 1 & 4p^0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For the reduction:

$$x^5 + a_2x^3 + a_3x^2 + a_4x^1 + a_5 \rightarrow y^5 + b_3y^2 + b_4y + b_5$$

$Y = p^0(X)^0 + p^1(X)^1 + (X)^2 = p^0X_0 + p^1X_1 + X_2$

$$\begin{cases} p_0 = \dfrac{2a_2}{5} \\ p_1 = \dfrac{-15a_3 \mp \sqrt{28(a_2)^3 + 225(a_3)^2 + 200a_4}}{10a_2} \end{cases}$$

$b_3,\ b_4,\ b_5$ can be deduced from: $b_3 = -\frac{Tr[(Y)^3]}{3}$, $b_4 = -\frac{Tr[(Y)^4]}{3}$, $b_5 = -\frac{Tr[(Y)^5]}{3}$

The coefficients of the basis change matrix $[P]$ can be obtained from p_0 and $p_1$using the associate recurrence relation. Finding the expression of X as linear combination of power of Y need only the calculation of $[P^{-1}]$.

3. CONCLUDING REMARKS

The method gives a systematic approach for the reduction of polynomial equations: The transformation are related with a basis change in an abelian Lie algebra. The method may explicit criteria on solvability in radicals, of a polynomial equation, by studying the possibility of existence of suitable transformation which reduce the equation to another solvable in radicals. We have considered the case of quintic polynomial. But the generalization to a polynomial of any degree is straight forward.

References

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